Weak convergence to the Rosenblatt sheet

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(based on joint works with Litan Yan and Dongjin Zhu)

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Motivation

- Donsker Theorem (Billingsley New York: Chapman Hall 1968): Consider a sequence of i.i.d random variables $\{\xi_i^{(n)}, i = 1, 2, ...\}$ with $E\xi_i^{(n)} = 0, E(\xi_i^{(n)})^2 = 1$. The Donsker Theorem says that the sequence of processes

$$W_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \xi_i^{(n)}, \quad t \in [0, T], \quad n = 1, 2, ...$$

converges weakly, in the Skorohod topology, to a standard Brownian motion

Motivation

- $$B_t^H = \int_0^t K_H(t,s) dW_s, t \geq 0$$

  where

  $$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} \, \mathrm{d}u \ \text{ for } t > s, H > \frac{1}{2}$$

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- F.Biagini, Y. Hu, B. Øksendal and T. Zhang, *Stochastic calculus for fBm and applications*, Probability and its application, Springer, Berlin (2008).

Motivation

- Define

$$K_H^n(t,s) = n\int_{s-\frac{1}{n}}^{s} K_H(\frac{[nt]}{n}, u)du, n \geq 1,$$

and let

$$B_t^n = \int_0^t K_H^n(t,s)dW_s^{(n)} = \sum_{i=1}^{[nt]} n\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_H(\frac{[nt]}{n}, s)ds\frac{\xi_i^{(n)}}{\sqrt{n}}, n \geq 1.$$

Sottinen (Finance and Stochastics 2001) proved that the perturbed random walk B^n converges weakly to the fractional Brownian motion.

Motivation

- Fractional Brownian sheet

$$B^{\alpha,\beta}(t,s) = \int_0^t \int_0^s K_\alpha(t,v) K_\beta(s,u) B(dv,du),$$

where $(t,s) \in [0,T] \times [0,S]$, B is a Brownian sheet.

Motivation

- Wang, Yan and Yu (Electron. Commun. Probab. 2013) extend this result (Sottinen Finance and Stochastics 2001) to fractional Brownian sheet.

Motivation

- Torres and Tudor (Stoch. Anal. Appl 2009) proved that the family of stochastic processes

$$Z_t^n = \sum_{i,j=1,i\neq j}^{[nt]} n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} Q_H(\frac{[nt]}{n}, u, v) dvdu \frac{\xi_i^{(n)}}{\sqrt{n}} \frac{\xi_j^{(n)}}{\sqrt{n}}, t \in [0, T]$$

converges weakly, in the Skorohod topology, to the Rosenblatt process.

Hermite Processes

Hermite process is the limits of the *Non-Central Limit Theorem* studied in Dobrushin and Major (1979), Taqqu (1979). Let us briefly recall the general context.

Hermite Processes

- $\{\xi_n,\ n \geq 0\}$: a stationary Gaussian sequence with mean zero and variance 1 such that correlation function

$$r(n) := E(\xi_0 \xi_n) = n^{\frac{2H-2}{k}} L(n) \tag{1.1}$$

with $k \geq 1$ integer and $H \in (\frac{1}{2}, 1)$, where L is a slowly varying function at infinity;

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- $H_m(x)$: the Hermite polynomial of degree m;

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with $k \geq 1$ integer and $H \in (\frac{1}{2}, 1)$, where L is a slowly varying function at infinity;
- $H_m(x)$: the Hermite polynomial of degree m;
- g : a function satisfying $E(g(\xi_0)) = 0$ and $E(g(\xi_0)^2) < \infty$;
- k : Hermite rank of g, that is, if
$$g(x) = \sum_{j \geq 0} c_j H_j(x), \qquad c_j = E(g(\xi_0 H_j(\xi_0))),$$
then $k = \min\{j\ ;\ c_j \neq 0\} \geq 1$.

Hermite Processes

- Then, the *Non Central Limit Theorem* (see Taqqu (1975) says that the sequence of stochastic processes, as $n \to \infty$

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\xi_j)$$

converges to the Hermite process $Z_H^k(t)$ in the sense of finite dimensional distributions.

Hermite Processes

- Hermite process of order k with index H

$$Z_H^k(t) = c_{H,k} \int_{\mathbb{R}^k} \int_0^t (\prod_{j=1}^k (s-y_j)^{-(\frac{1}{2}+\frac{1-H}{k})} 1_{\{s>y_j\}}) ds dW(y_1) \cdots dW(y_k),$$

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- $k=1$, Hemite process : fractional Brownian motion;
- $k=2$, Hemite process : Rosenblatt process.

Rosenblatt Processes

- Rosenblatt process on time interval $[0, T] (H > \frac{1}{2})$

$$Z_H(t) = d_H \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right] dB(y_1) dB(y_2), \tag{1.2}$$

where $K^H(t, s)$ is given by

$$K^H(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - 3/2} u^{H - 1/2} \, \mathrm{d}u \quad \text{for } t > s, \tag{1.3}$$

Rosenblatt Processes

with $c_H = \sqrt{\frac{H(2H-1)}{\Gamma(2-2H, H-\frac{1}{2})}}$, $H' = \frac{H+1}{2}$ and $d(H) = \frac{1}{H+1}\sqrt{\frac{H}{2(2H-1)}}$
For simplification, we denote

$$Q_H(t, y_1, y_2) = d_H \int_{y_1 \vee y_2}^{t} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du.$$

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(iv) the covariance function is

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- Hermite processes($k \geq 2$): 不是Gaussian, 不是Markov过程, 不是半鞅。

Hermite Processes

- 如果误差是具有长程相依性的线性过程的非线性变换，那么在误差的unit root testing问题中，其渐进分布是Hermite过程的泛函(Wu Econ. Theory 2005.);

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- 具有奇异非Gaussian数据的热传导方程的"parabolically rescaled solution"的极限分布有类似于Rosenblatt分布的结构(Leonenko and Woyczynski J. Stat. Phys. 2001);

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- 具有奇异非Gaussian数据的热传导方程的"parabolically rescaled solution"的极限分布有类似于Rosenblatt分布的结构(Leonenko and Woyczynski J. Stat. Phys. 2001);
- Rosenblatt分布也是与假设检验的半参数bootstrap方法相关的估计量的渐近分布(Hardle Stat. Infer. Stoch. Process. 2001)或长程相依性参数估计量的渐近分布(Kettani and Gubner Proc. 28th IEEE LCN03 2003)。

Rosenblatt Processes

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- Tudor(ESAIM Probab. Statist. 2008) Analysis of the Rosenblatt process;

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Rosenblatt Processes

- Shieh and Xiao (Bernoulli, 2010) studied the Hausdorff and packing dimensions of the image sets of the Rosenblatt sheet;

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- Shieh and Xiao (Bernoulli, 2010) studied the Hausdorff and packing dimensions of the image sets of the Rosenblatt sheet;
- Maejima and Tudor (Statist. Probab. Lett. 2013), On the distribution of the Rosenblatt process;
- Garzón, Torres and Tudor (J. Math. Anal. Appl. 2012), A strong convergence to the Rosenblatt process

.

random walks

Recall that a Rosenblatt sheet with parameters $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$ admits an integral representation of the form(Tudor (2014)), for $s, t \in [0, T]$

$$\begin{aligned} Z^{\alpha,\beta}(t,s) &= \int_0^t \int_0^s \int_0^t \int_0^s Q_\alpha(t, y_1, y_2) Q_\beta(s, u_1, u_2) B(dy_1, du_1) B(dy_2, du_2) \\ &= d_\alpha d_\beta \int_0^t \int_0^s \int_0^t \int_0^s \int_{y_1 \vee y_2}^t \frac{\partial K^{\alpha'}}{\partial m}(m, y_1) \frac{\partial K^{\alpha'}}{\partial m}(m, y_2) dm \\ &\quad \cdot \int_{u_1 \vee u_2}^s \frac{\partial K^{\beta'}}{\partial n}(n, u_1) \frac{\partial K^{\beta'}}{\partial n}(n, u_2) dn B(dy_1, du_1) B(dy_2, du_2), \end{aligned}$$

random walks

- Let $\{\xi_{i,j}^{(n)}, i, j = 1, 2, ...\}$ be an independent family of identically distribution and centered random variables with $E(\xi_{i,j}^{(n)}) = 1$.
  For $n \geq 1$, $(t, s) \in [0, T] \times [0, S]$, define

$$B_n(t, s) = \frac{1}{n} \sum_{i=1}^{[nt]} \sum_{j=1}^{[ns]} \xi_{i,j}^{(n)},$$

random walks

- Let

$$
\begin{aligned}
&Z_n(t,s)\\
&= \int_0^t \int_0^s \int_0^t \int_0^s Q_\alpha^{(n)}(t, y_1, y_2)\\
&\qquad Q_\beta^{(n)}(s, u_1, u_2) B_n(dy_1, du_1) B_n(dy_2, du_2)\\
&= n^2 \sum_{i=1}^{[nt]} \sum_{j=1}^{[ns]} \sum_{k=1, k\neq i}^{[nt]} \sum_{l=1, l\neq j}^{[ns]} \xi_{i,j}^{(n)} \xi_{k,l}^{(n)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{l-1}{n}}^{\frac{l}{n}}\\
&\qquad Q_\alpha(\frac{[nt]}{n}, y_1, y_2) Q_\beta(\frac{[ns]}{n}, u_1, u_2) dy_1 du_1 dy_2 du_2,
\end{aligned}
\tag{2.1}
$$

random walks

- where

$$Q_H^{(n)}(t,u,v) = n^2 \int_{\frac{u-1}{n}}^{\frac{u}{n}} \int_{\frac{v-1}{n}}^{\frac{v}{n}} Q_H(\frac{[nt]}{n}, r, p) dr dp, \quad n = 1, 2, \ldots.$$

main results

Theorem (Shen and Zhu (2014))

Let $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$. Then the family of process $Z_n(t,s)$ converges weakly in the Skorohod space $\mathcal{D}$, as n tends to infinity, to the Rosenblatt sheet $Z^{\alpha,\beta}(t,s)$ in the plane.

Poisson process

Recall that an $\mathcal{F}_{s,t}$ Poisson process is an adapted, cadlag process $N = \{N(s,t), (s,t) \in \mathbb{R}_+^2\}$, such that, $N(s,0) = N(0,t) = 0 \quad a.s.$, for all $(s,t) \le (s',t')$ the increment $\triangle_{s,t} N(s',t')$ is independent of $\mathcal{F}_{\infty,t} \vee \mathcal{F}_{s,\infty}$ and has a Poisson law of parameter $(s'-s)(t'-t)$. Here, we denote $\mathcal{F}_{\infty,t} := \vee_{s>0} \mathcal{F}_{s,t}$ and $\mathcal{F}_{s,\infty} := \vee_{t>0} \mathcal{F}_{s,t}$.

Poisson process

- Stroock (1982) studied the following relationship between the standard one-parament Poisson process and the standard Brownian motion: the family of process

$$y_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t (-1)^{N(s/\varepsilon)} ds,$$

where $\{N(t), t \geq 0\}$ is a standard Poisson process, converges in law in the space of continuous functions $\mathcal{C}([0,1])$, as ε tends to zero, to the standard Brownian motion $\{B(t), t \geq 0\}$.

Poisson process

- Bardina and Jolis (Bernoulli 2000) proved that the family of process

$$y_\varepsilon(s,t) = \int_0^t \int_0^s \frac{1}{\varepsilon^2} \sqrt{xy}(-1)^{N(x/\varepsilon, y/\varepsilon)} dxdy, \quad \varepsilon > 0,$$

where $\{N(x,y), (x,y) \in \mathbb{R}_+^2\}$ is a stand poisson process in the plane, converges in law in the space $\mathcal{C}([0,1]^2)$, as ε tends to zero, to the ordinary Brownian sheet.

Poisson process

- Bardina *et al.* (Statist. Probab. Lett. 2003)extend this result to fractional Brownian sheet.

$$y_\varepsilon(s,t) = \int_0^t \int_0^s K_H(s,u) K_H(t,v) \frac{1}{\varepsilon^2} \sqrt{uv} (-1)^{N(u/\varepsilon, v/\varepsilon)} du dv, \varepsilon > 0$$

converges in law in the space $\mathcal{C}([0,1]^2)$, as ε tends to zero, to the fractional Brownian sheet.

Poisson process

- We define for any $\varepsilon > 0$,

$$Z_\varepsilon^{\alpha,\beta}(t,s) = \int_0^1\int_0^1\int_0^1\int_0^1 Q_\alpha(t,y_1,y_2)Q_\beta(s,u_1,u_2)\frac{1}{\varepsilon^4}\sqrt{y_1y_2u_1u_2} \times (-1)^{N(y_1/\varepsilon,u_1/\varepsilon)+N(y_2/\varepsilon,u_2/\varepsilon)}dy_1dy_2du_1du_2. \tag{2.2}$$

Poisson process

- To simplify, put $n = \frac{1}{\varepsilon^2}$, $N_n(x,y) := N(x/\varepsilon, y/\varepsilon)$, then $N_n(x,y)$ is a Poisson process with intensity n, denote

$$\theta_n(x,y,u,v) = n^2\sqrt{xyuv}(-1)^{N_n(x,u)+N_n(y,v)}.$$

Thus, (2.2) can be rewritten as

$$Z_n^{\alpha,\beta}(t,s) = \int_{[0,1]^4} Q_\alpha(t,y_1,y_2)Q_\beta(s,u_1,u_2)\theta_n(y_1,y_2,u_1,u_2)dy_1dy_2du_1du_2. \quad (2.3)$$

main results

Theorem (Shen and Zhu (2014))

Let $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$. Then the family of process $Z_n^{\alpha,\beta}(t,s)$ given by (2.3) *converges weakly in the space $\mathcal{C}([0,1]^2)$, as n tends to infinity, to the Rosenblatt sheet $Z^{\alpha,\beta}(t,s)$ in the plane.*

main results

Lemma (Shen and Zhu (2014))

For any $f, g \in L^2([0,1] \times [0,1])$, There exits a constant $C > 0$, such that

$$E\left[\int_{[0,1]^4} f(x,y)g(u,v)\theta_n(x,y,u,v)dxdydudv\right]^2 \leq C\int_{[0,1]^4} f^2(x,y)g^2(u,v)dxdydudv.$$

Martingale difference

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\{\mathcal{F}_{s,t}; (s,t) \in [0,S] \times [0,T]\}$ be a family of sub-σ-fields of $\mathcal{F}$ such that:

(i) $\mathcal{F}_{s,t} \subseteq \mathcal{F}_{s',t'}$ for any $s \leq s', t \leq t'$;

(ii)$\mathcal{F}_{0,0}$ contains all null sets of $\mathcal{F}$;

(iii) for each $z \in [0,S] \times [0,T], \mathcal{F}_z = \bigcap_{z<z'} \mathcal{F}_{z'}$, where $z = (s,t) < z' = (s',t')$ denotes the partial order on $[0,S] \times [0,T]$, meaning that $s < s', t < t'$.

martingale differences

Denote $\mathcal{G}_{i,j}^{(n)} := \mathcal{F}_{i,n}^{(n)} \bigvee \mathcal{F}_{n,j}^{(n)}$, where $\mathcal{F}_{i,n}^{(n)}, \mathcal{F}_{n,j}^{(n)}$ denote the σ-fields generated by $\xi_{i,n}^{(n)}$ and $\xi_{n,j}^{(n)}$ respectively for $i, j = 1, 2, .., n$ and $n \geq 1$. Let $\{\xi^{(n)}\}_{n\geq 1} := \{\xi_{i,j}^{(n)}, \mathcal{G}_{i,j}^{(n)}\}_{n\geq 1}, i, j = 1, 2, ..., n$ be a sequence such that

$$E[\xi_{i+1,j+1}^{(n)} | \mathcal{G}_{i,j}^{(n)}] = 0$$

for all $n \geq 1$. Then we will call it a martingale differences sequence.

Poisson process

- Nieminen (Statist. Probab. Lett. 2004) Fractional Brownian motion and martingale-differences.

Poisson process

- Nieminen (Statist. Probab. Lett. 2004) Fractional Brownian motion and martingale-differences.
- Wang, Yan and Yu (Statist. Probab. Lett. 2014) Weak approximation of the fractional Brownian sheet using martingale differences

martingale differences

Morkvenas (Liet. Mat. Rink. 1984) if the martingale differences sequence $\xi^{(n)}$ satisfies the following condition

$$\sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]}(\xi_{i,j}^{(n)})^2 \to ts$$

in the sense of L^1, then

$$B_n(t,s) = \sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]}\xi_{i,j}^{(n)},$$

converges weakly to the Brownian sheet $B(t,s)$ in $\mathcal{D}$ as n tends to infinity.

martingale differences

Define

$$Z_n(t,s)$$

$$= \int_0^t\int_0^s\int_0^t\int_0^s Q_\alpha^{(n)}(t,y_1,y_2)Q_\beta^{(n)}(s,u_1,u_2)B_n(dy_1,du_1)B_n(dy_2,du_2)$$

$$= n^4\sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]}\sum_{k=1,k\neq i}^{[nt]}\sum_{l=1,l\neq j}^{[ns]}\xi_{i,j}^{(n)}\xi_{k,l}^{(n)}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{l-1}{n}}^{\frac{l}{n}} Q_\alpha(\frac{[nt]}{n},y_1,y_2)$$

$$\cdot Q_\beta(\frac{[ns]}{n},u_1,u_2)dy_1du_1dy_2du_2, \tag{2.4}$$

$$Q_H^{(n)}(t,u,v) = n^2\int_{\frac{u-1}{n}}^{\frac{u}{n}}\int_{\frac{v-1}{n}}^{\frac{v}{n}} Q_H(\frac{[nt]}{n},r,p)drdp, \quad n=1,2,\ldots.$$

main results

Theorem (Shen and Yan (2014))

Let $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$, *and* $\{\xi_{i,j}^{(n)}, i,j = 1,2,...,n\}$ *be a square integrable martingale differences sequence such that for all* $1 \leq i,j \leq n$

$$\lim_{n\to\infty} n\xi_{i,j}^{(n)} = 1 \quad a.s. \tag{2.5}$$

and

$$\max_{1\leq i,j\leq n} |\xi_{i,j}^{(n)}| \leq \frac{C}{n} \quad a.s. \tag{2.6}$$

for some $C \geq 1$. *Then,* $\{Z_n\}$ *converges weakly to the Rosenblatt sheet* $Z^{\alpha,\beta}$ *in the Skorohod space* $\mathcal{D}$ *as n tends to infinity.*

main results

Lemma (Shen and Yan (2014))

Let $Z_n(t,s)$ be the family of processes defined by (2.4). Then for any $(t,s) < (t',s')$, there exists a constant C such that

$$\sup_n E[(\triangle_{t,s} Z_n(t',s'))^2] \le C(t'-t)^{2\alpha}(s'-s)^{2\beta}.$$

main results

Lemma (Shen and Yan (2014))

Let $1/2 < \alpha, \beta < 1$, $(t_k, s_k), (t_l, s_l) \in [0,T] \times [0,S]$, *and* $\{\xi_{i,j}^{(n)}, i,j = 1,2,...,n\}$ *be a martingale differences sequence satisify* (2.5) *and* (2.6). *Then we have*

$$n^8 \sum_{i=1}^{[nT]} \sum_{j=1}^{[nS]} \sum_{k=1,k\neq i}^{[nT]} \sum_{l=1,l\neq j}^{[nS]} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{l-1}{n}}^{\frac{l}{n}} Q_\alpha(\frac{[nt_k]}{n}, y_1, y_2)$$

$$Q_\beta(\frac{[ns_k]}{n}, u_1, u_2) du_2 dy_2 du_1 dy_1 \times \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{l-1}{n}}^{\frac{l}{n}}$$

$$Q_\alpha(\frac{[nt_l]}{n}, y_1, y_2) Q_\beta(\frac{[ns_l]}{n}, u_1, u_2) du_2 dy_2 du_1 dy_1 (\xi_{i,j}^{(n)} \xi_{k,l}^{(n)})^2$$

converges to

$$\int_0^T \int_0^T Q_\alpha(t_k, y_1, y_2) Q_\alpha(t_l, y_1, y_2) dy_1 dy_2 \int_0^S \int_0^S Q_\beta(s_k, u_1, u_2) Q_\beta(t_l, u_1, u_2) du_1 du_2$$

as n tends to infinity.

- Approximation of multidimensional parameter Rosenblatt sheet in Skorohord space. Preprint.

Thank You!