Recurrent properties of regime-switching diffusions

Jinghai Shao

Beijing Normal University

Aug. 16, 2014

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What is the switching diffusion?

It is a two-component process $(X(t), \Lambda(t))$, where $(X(t))$ describes the continuous dynamics, and $(\Lambda(t))$ describes the random switching device.

- The first component $(X(t))$ satisfies the following SDE

$$dX(t) = \sigma(X(t), \Lambda(t))\mathrm{d}B(t) + b(X(t), \Lambda(t))\mathrm{d}t, \qquad (1)$$

  with $X(0) = x \in \mathbb{R}^d$.

- the second component $(\Lambda(t))$ is a Markov chain with state space $S := \{1, 2, \dots, N\}$, $2 \le N \le \infty$, such that

$$\mathbb{P}\{\Lambda(t+\delta) = l | \Lambda(t) = k\} = \begin{cases} q_{kl}\delta + o(\delta), & \text{if } k \neq l, \\ 1 + q_{kk}\delta + o(\delta), & \text{if } k = l \end{cases} \qquad (2)$$

  provided $\delta \downarrow 0$. The Q-matrix (q_{ij}) is irreducible and conservative. provided $\delta \downarrow 0$.

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Diffusion process in a fixed environment

For $k \in S$, let $(X^{(k)}(t))$ be a process satisfying the SDE:

$$\mathrm{d}X^{(k)}(t) = \sigma(X^{(k)}(t), k)\mathrm{d}B(t) + b(X^{(k)}(t), k)\mathrm{d}t,$$

with $X^{(k)}(0) = x \in \mathbb{R}^d$. Then $(X^{(k)}(t))$ is called the corresponding diffusion of $(X(t), \Lambda(t))$ in the fixed environment k.

- The recurrent property of the process $(X(t), \Lambda(t))$ is obviously connected with the recurrent property of $(X^{(k)}(t))$, $k \in S$.
- Some important phenomena occur when the environment is random.

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Ornstein-Uhlenbeck process in random environment

Consider the following switching diffusion

$$\mathrm{d}X_t = b_{\Lambda_t} X_t \mathrm{d}t + \sigma_{\Lambda_t} \mathrm{d}B_t, \quad X_0 = x \in \mathbb{R},$$

where (Λ_t) is a Markov chain in $S = \{1, 2, \ldots, N\}$, $N < \infty$, with Q-matrix (q_{ij}). Let (π_i) be the **invariant measure** of (q_{ij}).

X. Guyon, S. Iovleff and Jian-Feng Yao (2004):

- When $\sum_{i \in S} \pi_i b_i < 0$, then there exists a probability measure ν such that the distribution of X_t converges weakly to ν.

Let P_t be the semigroup corresponding to the process (X_t, Λ_t) on the space $\mathbb{R} \times S$. We have

Theorem (Shao, 2014)

(i) *If* $\sum_i \pi_i b_i < 0$, *then there exists a probability measure* ν *on* $\mathbb{R} \times S$ *and constants* $C, c > 0$ *such that*

$$\|P_t - \nu\|_{\text{var}} \leq Ce^{-ct}.$$

(ii) *If* $\sum_i \pi_i b_i > 0$, *then the process* (X_t, Λ_t) *is transient.*

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Some notation

- Let (X_t, Λ_t) be defined by (1) and (2) with $N < \infty$. $(X_t^{(i)})$ is the corresponding diffusion of (X_t) in the fixed environment i.
- For a diffusion process in $\mathbb{R}^d$ with generator

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i},$$

we write $L \sim (a(x), b(x))$ for simplicity, where $a(x) = (a_{ij}(x))$, $b(x) = (b_i(x))$.

- The generator of $(X_t^{(i)})$ is $L^{(i)} \sim (a^{(i)}(x), b^{(i)}(x))$, where $a^{(i)}(x) = \sigma(x,i)\sigma(x,i)^*$, $b^{(i)}(x) = b(x,i)$.

Some notation

Let B be a matrix or vector.

1. $B \geq 0$ means: all elements of B are non-negative.
2. $B > 0$ means: $B \geq 0$ and at least one element of B is positive.
3. $B \gg 0$ means: all elements of B are positive.

Definition (M-matrix)

A square matrix $A = (a_{ij})_{n\times n}$ is called an M-Matrix if A can be expressed in the form $A = sI - B$ with some $B \geq 0$ and $s \geq \mathrm{Ria}(B)$, where I is the $n \times n$ identity matrix, and $\mathrm{Ria}(B)$ the spectral radius of B. When $s > \mathrm{Ria}(B)$, A is called a nonsingular M-matrix.

A is a nonsingular M-matrix $\Longleftrightarrow$ every real eigenvalue of A is positive $\Longleftrightarrow$ all the principal minors of A are positive.

Criterion via nonsingular M-matrix

A function $V \in C^2(\mathbb{R}^d)$ is said to satisfy the condition (H1) if there exist constants $r_0 > 0$, $\beta_i \in \mathbb{R}$, $i \in S$ such that

$$V(x) > 0, \quad L^{(i)}V(x) \le \beta_i V(x), \text{ for } |x| > r_0.$$

Theorem 1. Suppose that there exists a function $V \in C^2(\mathbb{R}^d)$ satisfying condition (H1), and the matrix $-(Q + \operatorname{diag}(\beta_1, \dots, \beta_N))$ is a nonsingular M-matrix.

- If $\lim_{|x|\to\infty} V(x) = 0$, then (X_t, Λ_t) is transient.
- If $\lim_{|x|\to\infty} V(x) = \infty$, then (X_t, Λ_t) is positive recurrent.

Criterion via Perron-Frobenius theorem

Theorem 2. Suppose that there exists a function $V \in C^2(\mathbb{R}^d)$ satisfying condition (H1). Let (μ_i) be the invariant probability measure of (Λ_t). It holds

$$\sum_{i=1}^{N} \mu_i \beta_i < 0.$$

Then (X_t, Λ_t) is transient if $\lim_{|x|\to\infty} V(x) = 0$, and is positive recurrent if $\lim_{|x|\to\infty} V(x) = \infty$.

Criterion via Fredholm alternative

Theorem 3. Suppose that there exist $g, h \in C^2(\mathbb{R}^d)$ such that $\exists\, r_0 > 0, \beta_i \in \mathbb{R}$,

$$h(x),\ g(x) > 0,\ L^{(i)}h(x) \le \beta_i g(x),\ |x| > r_0, \qquad \text{(H2)}$$

$$\lim_{|x|\to\infty} \frac{g(x)}{h(x)} = 0, \quad \lim_{|x|\to\infty} \frac{L^{(i)}g(x)}{g(x)} = 0.$$

Let (μ_i) be the invariant probability measure of (Λ_t). Assume

$$\sum_{i=1}^{N} \mu_i \beta_i < 0.$$

Then (X_t, Λ_t) is recurrent if $\lim_{|x|\to\infty} h(x) = \infty$, and is transient if $\lim_{|x|\to\infty} h(x) = 0$.

Application

Corollary

Let (X_t, Λ_t) be a regime-switching diffusion on $[0, \infty)$ with reflecting boundary at 0. (X_t) satisfies

$$\mathrm{d}X_t = b_{\Lambda_t} X_t^{\delta} \mathrm{d}t + \sigma_{\Lambda_t} \mathrm{d}B_t, \quad \delta \in [-1, 1),$$

where b_i, σ_i are constants for i in a finite set S. Then (X_t, Λ_t) is recurrent if and only if $\sum_{i \in S} \mu_i b_i \le 0$.

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In this part, we consider the situation that S is an infinite countable space. So the process (Λ_t) itself could be transient or recurrent. Its Q-matrix is still irreducible.

We provide two methods:

1. Finite partition method based on the criterion of nonsingular M-matrix.
2. Principal eigenvalue of a bilinear form.

Finite partition method

- Assume that the Q-matrix of (Λ_t) is bounded, i.e. $(q_i(x))_{i\in S, x\in\mathbb{R}^d}$ is bounded.
- Assume there exists a function $V \in C^2(\mathbb{R}^d)$ such that $\exists\, r_0 > 0,\ \beta_i \in \mathbb{R},\ i \in S,$

$$V(x) > 0,\ L^{(i)}V(x) \le \beta_i V(x), \quad |x| > r_0.$$

- Assume $(\beta_i)_{i\in S}$ is upper bounded, i.e. $M = \sup_{i\in S} \beta_i < \infty$.

Finite partition method

Let $\Gamma = \{-\infty = k_0 < k_1 < \cdots < k_{m-1} < k_m = M\}$ be a finite partition of $(-\infty, M]$. Corresponding to Γ, define a finite partition $F = \{F_1, \ldots, F_m\}$ of S by setting

$$F_i = \{j \in S;\ \beta_j \in (k_{i-1}, k_i]\}, \quad i = 1, 2, \ldots, m.$$

$$\beta_i^F = \sup_{j \in F_i} \beta_j, \quad q_{ii}^F = -\sum_{k \neq i} q_{ik}^F,$$

$$q_{ik}^F = \begin{cases} \sup_{x \in \mathbb{R}^d} \sup_{r \in F_i} \sum_{j \in F_k} q_{rj}(x), & \text{for } k < i, \\ \inf_{x \in \mathbb{R}^d} \inf_{r \in F_i} \sum_{j \in F_k} q_{rj}(x), & \text{for } k > i. \end{cases}$$

Then

$$\beta_j \leq \beta_i^F,\ \forall\, j \in F_i,\ \text{and}\ \beta_{i-1}^F < \beta_i^F,\ i = 2, \ldots, m.$$

Finite partition method

Theorem

Using the notation defined above, if the matrix $-\left(\mathrm{diag}(\beta_1^F, \ldots, \beta_m^F) + Q^F\right) H_m$ *is a nonsingular M-matrix, where*

$$H_m = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{m \times m} .$$

Then (X_t, Λ_t) *is recurrent if* $\lim_{|x| \to \infty} V(x) = \infty$, *and is transient if* $\lim_{|x| \to \infty} V(x) = 0$.

Example:

Let (Λ_t) be a birth-death process on $S = \{1, 2, \ldots\}$ with $b_i \equiv b > 0$ and $a_i \equiv a > 0$. Let (X_t) be a random diffusion process on $[0, \infty)$ with reflecting boundary at 0 and satisfies

$$\mathrm{d}X_t = \beta_{\Lambda_t} X_t \mathrm{d}t + \sqrt{2}\mathrm{d}B_t,$$

where $\beta_i = \kappa - i^{-1}$ for $i \geq 1$.

- If $\kappa < b + 1, \quad \kappa^2 - (b + a + 1)\kappa + a > 0$, then (X_t, Λ_t) is recurrent.
- If $\kappa + b - 1 > 0, \quad \kappa^2 + (b + a - 1)\kappa - a > 0$, then (X_t, Λ_t) is transient.

Remark: The process (Λ_t) and (X_t, Λ_t) may own very different recurrent property. Precisely, take $b = 2$, $a = 1$, then (Λ_t) is transient; but take $\kappa < 2 - \sqrt{3}$, then (X_t, Λ_t) is recurrent. If take $b = 1$, $a = 2$, then (Λ_t) is exponentially ergodic, but take $\kappa > \sqrt{3} - 1$, (X_t, Λ_t) is transient.

Principal eigenvalue method

Let $V \in C^2(\mathbb{R}^d)$ satisfying

$$V(x) > 0,\ L^{(i)}V(x) \le \beta_i V(x), \quad |x| > r_0, \quad \beta. \not\equiv 0.$$

No boundedness assumption on Q-matrix and β_i, but (q_{ij}) is state-independent.

Assume that (Λ_t) is reversible w.r.t. a probability measure (π_i). Define

$$D(f) = \frac{1}{2}\sum_{ij}^{N} \pi_i q_{ij}(f_j - f_i)^2 - \sum_{i}^{N} \pi_i \beta_i f_i^2, \quad f \in L^2(\pi).$$

$$\mathscr{D}(D) = \{f \in L^2(\pi);\ D(f) < \infty\}.$$

Define the principal eigenvalue of $D(f)$ by

$$\lambda_0 = \inf\{D(f);\ f \in \mathscr{D}(D), \|f\|_{L^2(\pi)} = 1\}.$$

Principal eigenvalue method

Theorem

1. When $N < \infty$. Assume $\lambda_0 > 0$, then (X_t, Λ_t) is positive recurrent if $\lim_{|x|\to\infty} V(x) = \infty$, and is transient if $\lim_{|x|\to\infty} V(x) = 0$.
2. When $N = \infty$. Assume $\lambda_0 > 0$ and $\exists\, g \in L^2(\pi)$ such that $D(g) = \lambda_0 \|g\|^2_{L^2(\pi)}$. Then (X_t, Λ_t) is transient if $\lim_{|x|\to\infty} V(x) = 0$. Assume further that $\liminf_{i\to\infty} g_i \neq 0$, then (X_t, Λ_t) is recurrent if $\lim_{|x|\to\infty} V(x) = \infty$.

Principal eigenvalue method

Example:

Let (X_t) satisfy the following SDE:

$$dX_t = \mu_{\Lambda_t} X_t dt + dB_t, \quad X_0 = x_0 \in \mathbb{R},$$

where (Λ_t) is a birth-death process on $S = \{0, 1, 2, \ldots\}$ with $b_i = b(i+1)$, $a_i = a(i+1)$, and $a > b > 0$. Assume $\mu_0 = c$, $\mu_i = \gamma$ for $i \geq 1$. Then (X_t, Λ_t) is recurrent if $c - b > 0$ and $a - b - \gamma > 0$.

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Thank You !