Limit Theorems for Some Critical Superprocesses

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Outline

References

This talk is based on the following paper:

[1]. Y.-X. Ren, R. Song and R. Zhang: Limit theorems for some critical superprocesses, arXiv:1403.1342.

References

For related works see the following joint papers with R. Song and R. Zhang:

[2]. Central limit theorems for super Ornstein-Uhlenbeck processes. Acta Appl. Math. **130** (2014), 9–49.

[3]. Central limit theorems for supercritical branching Markov processes. J. Funct. Anal. **266** (2014), 1716-1756.

[4]. Central limit theorems for supercritical branching nonsymmetric Markov processes. arXiv:1404.0116

[5]. Central limit theorems for supercritical superprocesses. arXiv:1310.5410

[6]. Functional central limit theorems for supercritical superprocesses. Preprint, 2014.

Outline

For discrete time critical branching processes $\{Z(n), n \geq 0\}$, it is known that $P(Z(n) > 0) \to 0$ as $n \to \infty$.

Kesten, Ney and Spitzer (1966) proved that if Z has finite second moment, then

$$\lim_{n\to\infty} nP(Z(n) > 0) = \frac{1}{\sigma^2} \tag{1}$$

and

$$\lim_{n\to\infty} P\left(\frac{1}{n}Z(n) > \frac{\sigma^2}{2}x | Z(n) > 0\right) = e^{-x}, \quad x \geq 0, \tag{2}$$

where σ^2 is the variance of the offspring distribution.

For probabilistic proofs of these results, see Lyons, Pemantle and Peres (1995)

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For continuous time critical branching processes $\{Z(t), t \geq 0\}$, Athreya and Ney proved in their book (*Branching Processes*, 1972) the following limit theorem: Under the finite second moment condition,

$$\lim_{t\to\infty} P\left(\frac{1}{t}Z(t) > \frac{\sigma^2}{2}x|Z(t) > 0\right) = e^{-x}, \quad x \geq 0, \tag{3}$$

where σ^2 is a positive constant determined by the branching rate and the variance of the offspring distribution.

For discrete time multi-type critical branching processes $\{\mathbf{Z}(n), n \geq 0\}$, Athreya and Ney (1972) gave three limit theorems under the finite second moment condition.
Here $\mathbf{Z}(n) = (Z_1(n), Z_2(n), \cdots Z_d(n))$ (d-type BP).

Let $\mathbf{u}$ and $\mathbf{v}$ be a positive right and left eigenvectors of the mean matrix associated with the eigenvalue 1, respectively.

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Let $\mathbf{u}$ and $\mathbf{v}$ be a positive right and left eigenvectors of the mean matrix associated with the eigenvalue 1, respectively.

(1) $\lim_{n\to\infty} nP(\mathbf{Z}(n) \neq \mathbf{0} | \mathbf{Z}(0) = \mathbf{i}) = c^{-1}(\mathbf{i} \cdot \mathbf{u})$, where c is a positive constant.

(2) If $\mathbf{w} \cdot \mathbf{v} > 0$, then

$$\lim_{n\to\infty} P\left(\frac{\mathbf{Z}(n) \cdot \mathbf{w}}{n} > x | \mathbf{Z}(n) > 0\right) = \int_x^\infty f(y)dy, \quad x \geq 0, \qquad (4)$$

where $f(y) = \frac{1}{\gamma_1} e^{-y/\gamma_1}, \quad y \geq 0$, and γ_1 is a positive constant.

(3) If $\mathbf{w} \cdot \mathbf{v} = 0$, then

$$\lim_{n\to\infty} P\left(\frac{\mathbf{Z}(n) \cdot \mathbf{w}}{\sqrt{n}} > x | \mathbf{Z}(n) > 0\right) = \int_x^\infty f_2(y)dy, \quad x \in \mathbb{R}, \qquad (5)$$

where $f_2(y) = \frac{1}{2\gamma_2} e^{-|y|/\gamma_2}, \quad y \in \mathbb{R}$, and γ_2 is a positive constant.

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For continuous time multi-type critical branching processes, Athreya and Ney(1974) proved two limit theorems, similar to results (4) and (5) respectively, under the finite second moment condition.

Asmussen and Hering(*Branching Processes*,1983) discussed similar questions for critical branching Markov processes $\{Y_t, t \geq 0\}$.
(i)Under some conditions, it was shown that

$$\lim_{t\to\infty} tP_\nu(\|Y_t\| \neq 0) = c^{-1}\int_E \phi_0(x)\nu(dx).$$

uniformly in ν with ν satisfying supp$(\nu) = n$ for any integer n, where c is a positive constant and ϕ_0 is the first eigenfunction of the mean semigroup of $\{Y_t, t \geq 0\}$.
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Outline

Superprocesses

E: a locally compact separable metric space.
m: a σ-finite Borel measure on E with full support.
∂: a separate point not contained in E. ∂ will be interpreted as the cemetery point.

$\xi = \{\xi_t, \Pi_x\}$: a Hunt process on E.
$\zeta := \inf\{t > 0 : \xi_t = \partial\}$ is the lifetime of ξ.
$\{P_t : t \geq 0\}$: the semigroup of ξ.

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The superprocess $X = \{X_t : t \geq 0\}$ we are going to work with is determined by three parameters:
(i) a spatial motion $\xi = \{\xi_t, \Pi_x\}$ on E,
(ii) a branching rate function $\beta(x)$ on E which is a non-negative bounded measurable function.
(iii) a branching mechanism φ of the form

$$\varphi(x,z) = -a(x)z + b(x)z^2 + \int_{(0,+\infty)} (e^{-zy} - 1 + zy)n(x,dy), x \in E, z > 0, \tag{6}$$

where $a \in \mathcal{B}_b(E)$, $b \in \mathcal{B}_b^+(E)$ and n is a kernel from E to $(0,\infty)$ satisfying

$$\sup_{x\in E} \int_{(0,+\infty)} y^2 n(x,dy) < \infty. \tag{7}$$

$\mathcal{M}_F(E)$ denote the space of finite measures on E.
$\langle f, \mu \rangle := \int_E f(x)\mu(dx)$ and $\|\mu\| := \langle 1, \mu \rangle = \mu(E)$.

The superprocess X is a Markov process taking values in $\mathcal{M}_F(E)$. For any $\mu \in \mathcal{M}_F(E)$, we denote the law of X with initial configuration μ by $\mathbb{P}_\mu$. Then for every $f \in \mathcal{B}_b^+(E)$ and $\mu \in \mathcal{M}_F(E)$,

$$-\log \mathbb{P}_\mu \left(e^{-\langle f, X_t \rangle} \right) = \langle u_f(t, \cdot), \mu \rangle, \qquad (8)$$

where $u_f(t,x)$ is the unique positive solution to the equation

$$u_f(t,x) + \Pi_x \int_0^{t\wedge\zeta} \varphi(\xi_s, u_f(t-s, \xi_s))\beta(\xi_s)ds = \Pi_x f(\xi_t), \qquad (9)$$

Define

$$\alpha(x) := \beta(x)a(x) \quad \text{and} \quad A(x) := \beta(x)\left(2b(x) + \int_0^\infty y^2 n(x, dy)\right).$$

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For any $f \in \mathcal{B}_b(E)$ and $(t, x) \in (0, \infty) \times E$, define

$$T_t f(x) := \Pi_x \left[e^{\int_0^t \alpha(\xi_s)\, ds} f(\xi_t) \right].$$

First moment: For any $f \in \mathcal{B}_b(E)$,

$$\mathbb{P}_\mu \langle f, X_t \rangle = \langle T_t f, \mu \rangle.$$

Second moment: For any $f \in \mathcal{B}_b(E)$,

$$\mathbb{V}\mathrm{ar}_\mu \langle f, X_t \rangle = \langle \mathbb{V}\mathrm{ar}_{\delta_\cdot} \langle f, X_t \rangle, \mu \rangle = \int_E \int_0^t T_s [A (T_{t-s} f)^2](x)\, ds \mu(dx), \tag{10}$$

where $\mathbb{V}\mathrm{ar}_\mu$ stands for the variance under $\mathbb{P}_\mu$.

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Outline

Assumptions on the spatial process

We assume that that there exists a family of continuous strictly positive functions $\{p_t(x,y) : t > 0\}$ on $E \times E$ such that

$$P_t f(x) = \int_E p(t,x,y) f(y) m(dy).$$

Define

$$a_t(x) := \int_E p(t,x,y)^2 m(dy), \qquad \hat{a}_t(x) := \int_E p(t,y,x)^2 m(dy).$$

Assumption 1

(i) For any $t > 0$, $\int_E p(t,x,y) m(dx) \leq 1$.

(ii) For any $t > 0$, we have $a_t(x), \hat{a}_t(x) \in L^1(E, m(dx))$. Moreover, the functions $x \to a_t(x)$ and $x \to \hat{a}_t(x)$ are continuous on E .

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One can check that there exists a family $\{q_t(x,y) : t > 0\}$ of continuous strictly positive symmetric functions on $E \times E$ such that

$$T_t f(x) = \Pi_x \left[e^{\int_0^t \alpha(\xi_s)\, ds} f(\xi_t) \right] = \int_E q_t(x,y) f(y) m(dy).$$

Let $\{\widehat{T}_t, t > 0\}$ be the adjoint operators on $L^2(E,m)$ of $\{T_t, t > 0\}$, that is, for $f, g \in L^2(E,m)$,

$$\int_E f(x) T_t g(x)\, m(dx) = \int_E g(x) \widehat{T}_t f(x)\, m(dx)$$

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It follows from (i) above that, for any $t > 0$, T_t is a Hilbert-Schmidt operator and thus a compact operator. Let L and $\widehat{L}$ be the infinitesimal generators of the semigroups $\{T_t\}$ and $\{\widehat{T}_t\}$ in $L^2(E,m)$ respectively.

Define $\lambda_0 := \sup \Re(\sigma(L)) = \sup \Re(\sigma(\widehat{L}))$.
By Jentzsch's theorem, λ_0 is an eigenvalue of multiplicity 1 for both L and $\widehat{L}$.

Assume that ϕ_0 and ψ_0 are the eigenfunctions of L and $\widehat{L}$ respectively associated with λ_0. ψ_0 and ϕ_0 can be chosen to be continuous and strictly positive satisfying $\|\phi_0\|_2 = 1$ and $\langle \phi_0, \psi_0 \rangle_m = 1$.

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More Assumptions

Assumption 2

(i) ϕ_0 is bounded.

(ii) The semigroup $\{T_t, t > 0\}$ is intrinsically ultracontractive, that is, there exists $c_t > 0$ such that

$$q(t, x, y) \leq c_t \phi_0(x) \psi_0(y). \tag{11}$$

Assumption 3 The superprocess is critical: $\lambda_0 = 0$.

Assumption 4 Define $q_t(x) := \mathbb{P}_{\delta_x}(\|X_t\| = 0)$. We also assume that There exists $t_0 > 0$ such that,

$$\inf_{x \in E} q_{t_0}(x) > 0. \tag{12}$$

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Remarks on our assumptions

In Liu, Ren and Song (2011), quite a few examples of Hunt processes satisfying Assumptions 1 and 3 were given.

If E consists of finitely many points, and $\xi = \{\xi_t : t \geq 0\}$ is a conservative irreducible Markov process on E, then ξ satisfies the Assumptions 1 and 3 for some finite measure m on E with full support. So, as special cases, our results give the analogs of the results of Athreya and Ney (1974) for critical super-Markov chains.

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Outline

Main Result

Theorem 1 For any non-zero $\mu \in \mathcal{M}_F(E)$,

$$\lim_{t\to\infty} t\mathbb{P}_\mu(\|X_t\| \neq 0) = \nu^{-1}\langle \phi_0, \mu\rangle. \quad (13)$$

Main Result

Define $\mathbb{P}_{t,\mu}(\cdot) := \mathbb{P}_{\mu}\left(\cdot \mid \|X_t\| \neq 0\right)$.

Assume that $Y_t, t > 0$, and Y are random variables on $(\Omega, \mathcal{G})$. We write

$$Y_t|_{\mathbb{P}_{t,\mu}} \to Y \quad \text{in probability,}$$

if, for any $\epsilon > 0$,

$$\lim_{t\to\infty} \mathbb{P}_{t,\mu}(|Y_t - Y| \geq \epsilon) = 0.$$

Suppose that Z is a random variable on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{G}}, P)$, we write

$$Y_t|_{\mathbb{P}_{t,\mu}} \xrightarrow{d} Z,$$

if, for all $a \in \mathbb{R}$ with $P(Z = a) = 0$,

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Main Result

Define $\mathbb{P}_{t,\mu}(\cdot) := \mathbb{P}_\mu\left(\cdot \mid \|X_t\| \neq 0\right)$.

Assume that $Y_t, t > 0$, and Y are random variables on $(\Omega, \mathcal{G})$. We write

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if, for any $\epsilon > 0$,

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Suppose that Z is a random variable on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{G}}, P)$, we write

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Define

$$\nu := \frac{1}{2}\langle A(\phi_0)^2, \psi_0\rangle_m. \tag{14}$$

It is easy to see that $0 < \nu < \infty$. Define

$$\mathcal{C}_p := \{f \in \mathcal{B}(E) : \langle |f|^p, \psi_0\rangle_m < \infty\}.$$

Theorem 2 If $f \in \mathcal{C}_2$ then, for any non-zero $\mu \in \mathcal{M}_F(E)$, we have

$$t^{-1}\langle f, X_t\rangle|_{\mathbb{P}_{t,\mu}} \xrightarrow{d} \langle f, \psi_0\rangle_m W, \tag{15}$$

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Remark Our assumptions imply that $1 \in \mathcal{C}_2$. Thus the limit result above implies that

$$t^{-1}\langle 1, X_t\rangle|_{\mathbb{P}_{t,\mu}} \xrightarrow{d} \langle 1, \psi_0\rangle_m W,$$

which says that, conditioned on no-extinction at time t, the growth rate of the total mass $\langle 1, X_t\rangle$ is t as $t \to \infty$.

Note that, when $\langle f, \psi_0\rangle_m = 0$, $t^{-1}\langle f, X_t\rangle|_{\mathbb{P}_{t,\mu}} \to 0$ in probability. Therefore it is natural to consider central limit type theorems for $\langle f, X_t\rangle$.

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$$\sigma_f^2 = \int_0^\infty \langle A(T_s f)^2, \psi_0\rangle_m \, ds. \tag{17}$$

Theorem 3

Suppose that $f \in \mathcal{C}_2$ and $\langle f, \psi_0\rangle_m = 0$, then we have, $\sigma_f^2 < \infty$ and for any non-zero $\mu \in \mathcal{M}_F(E)$,

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where $G(f) \sim \mathcal{N}(0, \sigma_f^2)$ is a normal random variable and W is the random variable defined in Theorem 2. Moreover, W and $G(f)$ are independent.

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As a consequence of Theorem 3, we immediately get the following central limit theorem.

Corollary Suppose that $f \in \mathcal{C}_2$ and $\langle f, \psi_0 \rangle_m = 0$, then we have, $\sigma_f^2 < \infty$ and for any non-zero $\mu \in \mathcal{M}_F(E)$,

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Sketch of the proof of Theorem 3

We need to consider the limit of the following $\mathbb{R}^2$-valued random variable:

$$U_1(t) := \left(t^{-1}\langle\phi_0, X_t\rangle, t^{-1/2}\langle f, X_t\rangle\right).$$

Which is equivalent to consider the limit of

$$U_1(s+t) = \left((t+s)^{-1}\langle\phi_0, X_{t+s}\rangle, (t+s)^{-1/2}\langle f, X_{s+t}\rangle\right) \quad \text{as } t\to\infty.$$

First, we consider $U_2(s,t) = \left(t^{-1}\langle\phi_0, X_t\rangle, t^{-1/2}\left(\langle f, X_{s+t}\rangle - \langle T_s f, X_t\rangle\right)\right)$. We prove that

$$U_2(s,t)|_{\mathbb{P}_{t,\mu}} \xrightarrow{d} \left(W, \sqrt{W}G_1(s)\right), \quad \text{as } t\to\infty, \tag{20}$$

where $G_1(s)\sim\mathcal{N}(0,\sigma_f^2(s))$ with $\sigma_f^2(s) = \langle \mathbb{V}\mathrm{ar}_{\delta_\cdot}\langle f, X_s\rangle, \psi_0\rangle_m$ and W is the random variable defined in Theorem 3.

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The characteristic function of $U_2(s,t)$ is

$$
\begin{aligned}
& \mathbb{P}_{t,\mu}(\exp\{i\theta_1 t^{-1}\langle \phi_0, X_t\rangle + i\theta_2 t^{-1/2}\left(\langle f, X_{s+t}\rangle - \langle T_s f, X_t\rangle\right)\}) \\
= \; & \mathbb{P}_{t,\mu}\Big(\exp\{i\theta_1 t^{-1}\langle \phi_0, X_t\rangle - i\theta_2 t^{-1/2}\langle T_s f, X_t\rangle + \\
& \langle \log \mathbb{P}_{\delta_\cdot} \exp\{-i\theta_2 t^{-1/2}\langle f, X_s\rangle\}, X_t\rangle\}\Big) \\
= \; & \mathbb{P}_{t,\mu}\Big(\exp\{i\theta_1 t^{-1}\langle \phi_0, X_t\rangle \\
& + \int_E \int_{\mathbb{D}} \left(e^{i\theta_2 t^{-1/2}\langle f, \omega_s\rangle} - 1 - i\theta_2 t^{-1/2}\langle f, \omega_s\rangle\right) \mathbb{N}_x(d\omega) X_t(dx)\}\Big),
\end{aligned}
$$

where $\mathbb{P}_{\delta_x} \longleftrightarrow \mathbb{N}_x$ for each $x \in E$. For the definition of $\mathbb{N}_x$, see Z. Li's book (Measure-valued Branching Markov Processes, 2011)

Thank you!