[10th Workshop on Markov Processes and Related Topics, August 14, 2014]

On the hitting times of continuous-state branching processes with immigration

Chunhua Ma

Nankai University

(A joint work with Xan Duhalde, Clément Foucart)

1. Continuous state branching processes

- Galton-watson branching processes:

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)},$$

where $\xi_i^{(n)} =$ the number of children of i at generation n. $\{\xi_i^{(n)}\}$ i.i.d.

- Let $m = \mathbf{E}[\xi_1^{(1)}]$:

$$X_n = X_0 + \sum_{k=1}^{n} \sum_{i=1}^{X_{k-1}} (\xi_i^{(k)} - m) + \sum_{k=1}^{n} (m-1) X_{n-1}.$$

A scaling limit leads to a continuous state branching process (CB) as the unique solution of stochastic equation ($m - 1 \rightsquigarrow \gamma$):

$$X_t = X_0 + \int_0^t \int_0^{X(s-)} \int_0^{\infty} \zeta \tilde{N}(ds, du, d\zeta) + \gamma \int_0^t X_s ds.$$

See Dawson-Li (2006).

Discrete Lamperti transformation

• Order the particles in in breadth-first order. ξ_i: the number of children of the i-th particle. A certain random walk

$$S_n = \sum_{i=1}^{n} (\xi_i - 1), \quad S_0 = 0$$

with jumps in $\{-1, 0, 1, 2, \cdots\}$.

• Galton-watson process

$$X_{n+1} = X_0 + S_{\sum_{k=0}^{n} X_k}$$

A scaling limit leads to a CB process taking the form:

$$X_t = X_0 + Z_{\int_0^t X_s ds},$$

where $\{Z_t\}$ is a spectrally positive Lévy process with Lévy-Khinchin formula Ψ.

• X_t is called a CBI process if its transition semigroup $(P_t)_{t\geq 0}$ is given by

$$\int_0^\infty e^{-\lambda y} P_t(x, dy) = \exp\{-x v_t(\lambda)\},$$

where $v_t(\lambda)$ is the unique solution of the ODE:

$$\frac{\partial}{\partial t} v_t(\lambda) = -\Psi(v_t(\lambda)), \quad v_0(\lambda) = \lambda.$$

• Grey (1974) : the asymptotic behavior of a CB (Ψ)

$$p = \mathbb{P}_x(\lim_{t\to\infty} X_t = 0) = e^{-xv}$$

where v is the largest root of $\Psi(q) = 0$.

Supercirtical: $\Psi'(0) < 0$, subcritical: $\Psi'(0) > 0$, critical: $\Psi'(0) = 0$.

In the (sub) critical case, p=1 and Grey's condition

$$\int_\theta^\infty \frac{dq}{\Psi(q)} < \infty \iff \mathbb{P}_x(X_t = 0 \text{ for some } t > 0) = 1.$$

2. Immigration processes

Kawazu and Watanabe (1971) : GW processes with immigration

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)} + \eta_n,$$

where $\eta_n =$ the number of immigrants at generation n. $\{\eta_n : n \geq 1\}$ i.i.d.

$$X_{n+1} = X_0 + S_{\sum_{k=0}^{n} X_k} + \sum_{k=1}^{n} \eta_k.$$

• A scaling limit leads to CB processes with immigration (CBI)

$$X_t = X_0 + Z_{\int_0^t X_s ds} + Y_t,$$

where (Z_t) is a spectrally positive Levy process with Levy-Khinchin formula

$$\Psi(q) \quad = \gamma q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu1_{\{u\in(0,1)\}})\pi(du)$$

and (Y_t) a Levy subordinator with Levy-Khinchin formula

$$\Phi(q) = bq + \int_0^\infty (1 - e^{-qu})\nu(du).$$

(1) Positive OU-type processes: (when $Z_t = -\gamma t$ and Y_t subordinator)

(2) Feller diffusion (when Z_t: a B.M. with $\Psi(q) = \frac{\sigma^2}{2} q^2$ and $Y_t = bt$)

$$X_t = X_0 + \sigma \int_0^t \sqrt{X_s} dB_t + bt$$

(3) Positive self similar Markov process (when Z_t is a spectrally positive α-stable process, Y_t is a $(\alpha - 1)$-stable subordinator)

$$X_t = X_0 + \int_0^t \sqrt[\alpha]{X_{s-}}\, dZ_t + Y_t.$$

(4) A critical CB process conditioned to be non extinct (a special CBI process): a time-changed Levy process conditioned to stay positive.

• A well-know result in Itô and Mckean (1996; book) for Feller's diffusion:

If $2b > \sigma^2$, the process is transient, or the process is recurrent.

If $2b \geq \sigma^2$, the point 0 is polar, i.e. for any $x > 0$,

$$\mathbb{P}_x(\sigma_0 < \infty) = 0,$$

where σ_0 the hitting time; otherwise (X_t) hits 0 with positive probability.

In particular, if $2b = \sigma^2$, then 0 is polar and the process is recurrent.

3. First entrance times for CBI processes

Recall that (Z_t) is a spectrally positive Levy process with Levy-Khinchin formula

$$\Psi(q) \quad = \gamma q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu\mathbf{1}_{\{u\in(0,1)\}})\pi(du)$$

Usually assume that the effective drift **d** defined by

$$\mathbf{d} := \begin{cases} \gamma + \int_0^1 z\pi(dz) \text{ if the process } Z_t \text{ has bounded variation paths} \\ \\ \infty \text{ if the process } Z_t \text{ has unbounded variation paths,} \end{cases}$$

belongs to $(0, \infty]$.

Recall that (Y_t) a Levy subordinator with Levy-Khinchin formula

$$\Phi(q) = bq + \int_0^\infty (1 - e^{-qu})\nu(du).$$

- $\liminf_{t\to\infty} X_t \geq \ell := b/\mathbf{d}$.

Denote the first entrance time in $[0, a]$ by σ_a :

$$\sigma_a := \inf\{t > 0; X_t \leq a\}.$$

We will consider the law of σ_a when (X_t) starts from $x \geq a$.

Remark that the process has no downward jumps, therefore $X_{\sigma_a} = a$ almost surely.

Theorem 1 *Let $x > a \geq \ell$. For every $\lambda > 0$ and $\mu \geq 0$, we have*

$$\mathbb{E}_x\Big[\exp\Big\{-\lambda\sigma_a - \mu\int_0^{\sigma_a} X_t \mathrm{d}t\Big\}\Big] = \frac{\int_{q(\mu)}^{\infty} \frac{\mathrm{d}z}{\Psi(z)-\mu} \exp\left(-xz + \int_\theta^z \frac{\Phi(u)+\lambda}{\Psi(u)-\mu}\mathrm{d}u\right)}{\int_{q(\mu)}^{\infty} \frac{\mathrm{d}z}{\Psi(z)-\mu} \exp\left(-az + \int_\theta^z \frac{\Phi(u)+\lambda}{\Psi(u)-\mu}\mathrm{d}u\right)},$$

where $q(\mu) := \sup\{q \geq 0 : \Psi(q) = \mu\}$, and θ is an arbitrary constant larger than $q(\mu)$.

Corollary 1 *For all* $\lambda \in (0, \infty)$*, and* $x > a \geq \ell$

$$\mathbb{E}_x \left[e^{-\lambda \sigma_a} \right] = \frac{f_\lambda(x)}{f_\lambda(a)},$$

where

$$f_\lambda(x) = \int_{q(0)}^{\infty} \frac{e^{-xz}}{\Psi(z)} \exp\left[\int_\theta^z \frac{\Phi(u) + \lambda}{\Psi(u)} \mathrm{d}u \right].$$

Corollary 2 *In the critical, or sub-critical case, for all* $x > a \geq \ell$*,*

$$\mathbb{E}_x \left[\sigma_a \right] = \int_0^{\infty} \frac{\mathrm{d}z}{\Psi(z)} \left(e^{-az} - e^{-xz} \right) \exp\left(\int_0^z \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u \right).$$

Theorem 2 *(Pinsky (1972))*

i) If $\int_0^1 \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u < \infty$, *then the CBI*$(\Psi, \Phi)$ *process,* $(X_t, t \geq 0)$, *has an invariant probability distribution. In the subcritical case* ($\Psi'(0+) > 0$)*, this integral condition is equivalent to*

$$\int_1^\infty \log(u)\nu(\mathrm{d}u) < \infty.$$

ii) If $\int_0^1 \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u = \infty$, *then for all* $x, b \in \mathbb{R}_+$,

$$\lim_{t\to\infty} \mathbb{P}_x(X_t \leq b) = 0.$$

Case (i): $\mathbb{E}_x[\sigma_a] < \infty$; Case (ii): $\mathbb{E}_x[\sigma_a] = \infty$

We adopt the following definition of recurrence and transience.

Definition 1 We say that the process $(X_t, t \geq 0)$ is recurrent if there exists $x \in \mathbb{R}_+$ such that

$$\mathbb{P}_x(\liminf_{t\to\infty} |X_t - x| = 0) = 1.$$

On the other hand, we say that the process is transient if

$$\mathbb{P}_x(\lim_{t\to\infty} X_t = \infty) = 1 \text{ for every } x \in \mathbb{R}_+.$$

4. A recurrence criterion for CBI processes

Theorem 3 *(a) In the critical or subcritical case, the CBI*(Ψ, Φ) *process is recurrent or transient accordingly as*

$$\int_0^1 \frac{\mathrm{d}z}{\Psi(z)} \exp\left[-\int_z^1 \frac{\Phi(x)}{\Psi(x)}\mathrm{d}x\right] = \infty \text{ or } < \infty.$$

(b) In the supercritical case, the CBI(Ψ, Φ) *process is transient.*

For a critical CBI process with finite variance, we have a simpler criterion for its recurrence and transience, which somehow is reminiscent of the Feller diffusion.

Corollary 3 *Assume* $\int_1^\infty u^2 \log u \, \pi(\mathrm{d}u) < \infty$ *and* $\int_1^\infty u^2 \nu(\mathrm{d}u) < \infty$ *and consider*

$$\Psi(q) = \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\pi(\mathrm{d}u),$$
$$\Phi(q) = bq + \int_0^\infty (1 - e^{-qu})\nu(\mathrm{d}u).$$

Let $\tilde{\sigma}^2 := \Psi''(0)$ *and* $\tilde{b} := \Phi'(0)$. *The process is transient if and only if*

$$2\tilde{b} > \tilde{\sigma}^2.$$

5. $\mathbf{0}$ is polar or not

Theorem 4 *The only point that may be polar is* ℓ*. If* $\mathrm{d} < \infty$ *then* ℓ *is polar. In the unbounded variation case,* $\ell = \mathbf{0}$ *and* $\mathbf{0}$ *is polar or hit with positive probability accordingly as*

$$\int_{\theta}^{\infty} \frac{\mathrm{d}z}{\Psi(z)} \exp\left[\int_{\theta}^{z} \frac{\Phi(x)}{\Psi(x)} \mathrm{d}x\right] = \infty \text{ or } < \infty.$$

• Foucart and Uribe Bravo (2013) proved the result for $\ell = \mathbf{0}$ based on a connection between the zero set of a CBI and the random cutout sets defined by Mandelbrot (1972)

• We recover and complete the results of Foucart and Uribe Bravo (2013) through more classic techniques.

Example 1 Consider $\Psi(q) = dq^{\alpha}$, $\Phi(q) = d'q^{\beta}$ with $\alpha \in (1, 2]$ and $\beta \in (0, 1]$.

- If $\beta > \alpha - 1$, the process is recurrent and 0 is polar.
- If $\beta < \alpha - 1$, the process is transient and 0 is not polar.
- If $\beta = \alpha - 1$ and $\alpha \in (1, 2]$, the process is recurrent if $d'/d \leq \alpha - 1$ and transient if $d'/d > \alpha - 1$.

  The point 0 is polar if and only if $d'/d \geq \alpha - 1$.

  In particular if $d'/d = \alpha - 1$, then 0 is polar but $\liminf_{t \to \infty} X_t = 0$.

Patie (2009) obtained the condition for 0 to be polar via other arguments.

Thanks!