Quasi-invariance of the stochastic flow associated to Itô's SDE with singular time-dependent drift

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Itô's SDE

- $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^m \otimes \mathbb{R}^d$ matrix-valued function,
- $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ vector field,
- $(W_t)_{0 \le t \le T}$: m-dimensional standard Brownian motion.

Consider Itô's SDE

$$\mathrm{d}X_t = \sigma(t, X_t)\,\mathrm{d}W_t + b(t, X_t)\,\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d. \tag{1}$$

Two types of conditions insuring existence of strong solution:

- modulus of continuity on the coefficients σ, b,
- non-degenerate diffusion σ and irregular drift b.

Conditions on modulus of continuity of coefficients

- σ and b are globally Lipschitz continuous in x, uniformly in t $\Longrightarrow$ SDE (1) has a unique strong solution X_t which is Hölder continuous w.r.t. x.
- In the 1-dim case: σ is $(1/2)$-Hölder continuous and b is Lipschitz.
- For $d \geq 1$, S. Fang and T. Zhang (PTRF, 2005):

$$\|\sigma(x) - \sigma(y)\|^2 \leq C|x-y|^2 \log \frac{1}{|x-y|},$$

$$|b(x) - b(y)| \leq C|x-y| \log \frac{1}{|x-y|}.$$

- ......

The non-degenerate case

Non-degeneracy: $\exists\,\lambda > 0$ s.t. $\sigma(t,x)\sigma(t,x)^* \geq \lambda$, $\forall\,(t,x)$.

- Veretennikov (1979): $\sigma(t,\cdot)$ is non-degenerate and bounded Lipschitz continuous, and b is bounded measurable.
- Gyöngy–Martinez (2001): $\sigma(t,\cdot)$ is locally Lipschitz continuous, and $|b(t,x)| \leq C + F(t,x)$, where $F \in L^{d+1}([0,T]\times\mathbb{R}^d)$.
- X. Zhang (2005): replacing the locally Lipschitz continuity of $\sigma(t,\cdot)$ with some integrability of $\nabla_x\sigma(t,\cdot)$.
- Krylov–Röckner (2005): $\sigma \equiv Id$ and $b \in L^q\big([0,T], L^p(\mathbb{R}^d)\big)$, i.e.

$$\int_0^T \left(\int_{\mathbb{R}^d} |b(t,x)|^p \,\mathrm{d}x\right)^{\frac{q}{p}} \mathrm{d}t < +\infty \qquad (2)$$

where $\frac{d}{p} + \frac{2}{q} < 1$. (LPS-type condition).

Regularity of Krylov–Röckner's flow

From now on, we focus on the case studied by Krylov–Röckner. That is, we consider

$$\mathrm{d}X_t = \mathrm{d}W_t + b(t, X_t)\,\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d, \tag{3}$$

where $b \in L^q\big([0,T], L^p(\mathbb{R}^d)\big)$ with $\frac{d}{p} + \frac{2}{q} < 1$.

- Fedrizzi–Flandoli (Stoch Anal Appl, 2013): X_t is indeed a stochastic flow of homeomorphisms on $\mathbb{R}^d$ which are Hölder continuous, by Kolmogorov's theorem.
- Fedrizzi–Flandoli (JFA, 2013): if $v_0 \in \cap_{r\geq 1} W^{1,r}(\mathbb{R}^d)$, then $v(t,x) := v_0\big(X_t^{-1}(x)\big)$ is the unique weakly differentiable solution to the stochastic transport eq.

$$\mathrm{d}v + \langle b, \nabla v\rangle\,\mathrm{d}t + \langle \nabla v, \circ\,\mathrm{d}W_t\rangle = 0, \quad v|_{t=0} = v_0.$$

Main result

$\mathcal{L}^d$: Lebesgue measure on $\mathbb{R}^d$.

Since $X_t : \mathbb{R}^d \to \mathbb{R}^d$ is a.s. a homeomorphisms, the push-forward $\mathcal{L}^d \circ X_t^{-1}$ is well defined.

What is the relation between $\mathcal{L}^d \circ X_t^{-1}$ and $\mathcal{L}^d$?

Theorem 1

Assume $b \in L^q\big([0,T], L^p(\mathbb{R}^d)\big)$ with $\frac{d}{p} + \frac{2}{q} < 1$. Then $\forall\, t \in [0,T]$, $\mathcal{L}^d \circ X_t^{-1}$ is equivalent to $\mathcal{L}^d$ almost surely.

In other words, the Lebesgue measure $\mathcal{L}^d$ is quasi-invariant under the flow X_t of homeomorphisms generated by (3).

Outline

Main idea

Recall that if $b \in L^q\big([0,T], L^p(\mathbb{R}^d)\big)$ with $\frac{d}{p} + \frac{2}{q} < 1$, then

$$\mathrm{d}X_t = \mathrm{d}W_t + b(t, X_t)\,\mathrm{d}t$$

generates a flow X_t of homeomorphisms on $\mathbb{R}^d$. Our purpose is to show $\mathcal{L}^d \circ X_t^{-1} \sim \mathcal{L}^d$. This is well known when b is smooth.

Indeed, if $\sigma(t,\cdot), b(t,\cdot) \in C_b^2(\mathbb{R}^d)$, uniformly in t, then SDE

$$\mathrm{d}X_t = \sigma(t, X_t)\,\mathrm{d}W_t + b(t, X_t)\,\mathrm{d}t$$

generates a flow X_t of diffeomorphisms on $\mathbb{R}^d$. Let

$$\rho_t = \frac{\mathrm{d}(\mathcal{L}^d \circ X_t^{-1})}{\mathrm{d}\mathcal{L}^d} = \big|\det(\nabla X_t^{-1})\big|, \ \bar\rho_t = \frac{\mathrm{d}(\mathcal{L}^d \circ X_t)}{\mathrm{d}\mathcal{L}^d} = \big|\det(\nabla X_t)\big|.$$

Then we have $\rho_t(x) = \big[\bar\rho_t\big(X_t^{-1}(x)\big)\big]^{-1}$ and

$$\bar{\rho}_t(x) = \exp\bigg\{ \int_0^t \big\langle \mathrm{div}(\sigma)(s, X_s(x)), \mathrm{d}W_s \big\rangle + \int_0^t \Big[\mathrm{div}(b) - \frac{1}{2}\langle \nabla\sigma, (\nabla\sigma)^* \rangle \Big](s, X_s(x))\, \mathrm{d}s \bigg\}. \quad (4)$$

If $\sigma \equiv Id$, the above identity reduces to

$$\bar{\rho}_t(x) = \exp\bigg\{ \int_0^t \mathrm{div}(b)(s, X_s(x))\, \mathrm{d}s \bigg\}.$$

When $b \in L^q\big([0,T], L^p(\mathbb{R}^d)\big)$, it is natural to

- approximate b with smooth vector fields b^n, and
- prove $\bar{\rho}^n_t(x) \to$? by limit theorem.

However, the divergence $\mathrm{div}(b)(s,\cdot)$ does not exist. Therefore, we cannot directly consider the approximation equations of

$$\mathrm{d}X_t = \mathrm{d}W_t + b(t, X_t)\,\mathrm{d}t.$$

Preliminary results

Some functional spaces:

- $L_p^q(T) := L^q\big([0,T], L^p(\mathbb{R}^d)\big)$,
- $\mathbb{H}_{2,p}^q(T) = L^q\big([0,T], W^{2,p}(\mathbb{R}^d)\big)$.

Theorem 2

Fix $\lambda > 0$. Assume $b \in L_p^q(T)$ with $\frac{d}{p} + \frac{2}{q} < 1$. Then $\exists$ a unique solution in $\mathbb{H}_{2,p}^q(T)$ to the backward parabolic system

$$\partial_t u + \frac{1}{2}\Delta u + b \cdot \nabla u = \lambda u - b, \quad u(T,x) = 0. \tag{5}$$

Moreover, $\exists\, C > 0$ depending on d, p, q, T, λ and $\|b\|_{L_p^q(T)}$ s.t.

$$\|\partial_t u\|_{L_p^q(T)} + \|u\|_{\mathbb{H}_{2,p}^q(T)} \le C\|b\|_{L_p^q(T)}.$$

Some properties of the solution

Lemma 3

Let u_λ be the solution of (5). Then

$$\sup_{t\leq T} \|\nabla u_\lambda\|_\infty \to 0 \quad \text{as } \lambda \to \infty,$$

where $\|\cdot\|_\infty$ is the supremum norm in $C(\mathbb{R}^d)$.

In view of the above lemma, we fix $\lambda > 0$ such that

$$\sup_{t\leq T} \|\nabla u_\lambda\|_\infty \leq \frac{1}{2}.$$

Define $\qquad \phi_\lambda(t,x) = x + u_\lambda(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^d.$

Zvonkin type transform

Proposition 4

The following statements hold:

(i) *uniformly in* $t \in [0, T]$, $\phi_\lambda(t, \cdot)$ *has bounded first derivatives which are Hölder continuous;*

(ii) *for every* $t \in [0, T]$, $\phi_\lambda(t, \cdot)$ *is a* C^1*-diffeomorphism on* $\mathbb{R}^d$*;*

(iii) $\phi_\lambda^{-1}(t, \cdot) := (\phi_\lambda(t, \cdot))^{-1}$ *has bounded first spatial derivatives, uniformly in* t*;*

(iv) ϕ_λ *and* ϕ_λ^{-1} *are jointly continuous in* (t, x)*.*

In the following, we omit λ and write $\phi_t(x) = x + u(t, x)$. Define

$$Y_t = \phi_t(X_t) = X_t + u(t, X_t).$$

As ϕ_t is smooth, it is enough to show $\mathcal{L}^d \circ Y_t^{-1} \sim \mathcal{L}^d$.

Equation for Y_t

Since $u(t,x)$ solves the equation (5):

$$\partial_t u + \frac{1}{2}\Delta u + b \cdot \nabla u = \lambda u - b,$$

Itô's formula leads to

$$\begin{aligned} \mathrm{d}Y_t &= (Id + \nabla u(t, X_t))\,\mathrm{d}W_t + \lambda u(t, X_t)\,\mathrm{d}t \\ &= \tilde{\sigma}(t, Y_t)\,\mathrm{d}W_t + \tilde{b}(t, Y_t)\,\mathrm{d}t, \end{aligned} \tag{6}$$

where $\quad \tilde{\sigma}(t,y) = Id + \nabla u\big(t, \phi_t^{-1}(y)\big),\ \ \tilde{b}(t,y) = \lambda u\big(t, \phi_t^{-1}(y)\big).$

Proposition 5

We have $\nabla \tilde{b} \in C\big([0,T], C_b(\mathbb{R}^d)\big)$ *and*

$$\tilde{\sigma} \in C\big([0,T], C_b(\mathbb{R}^d)\big) \cap L^q\big(0,T, W^{1,p}(\mathbb{R}^d)\big).$$

We still cannot directly apply density formula (4) to SDE (6) for Y_t. Nevertheless, $\mathrm{div}(\tilde{\sigma})$ and $\mathrm{div}(\tilde{b})$ make sense.

We need an approximation argument.

Let b^n be a sequence of smooth compactly supported vector fields converging to b in $L_p^q(T)$. Consider

$$\mathrm{d}X_t^n = \mathrm{d}W_t + b^n(t, X_t^n)\,\mathrm{d}t.$$

An approximation result

Lemma 6

Let b^n be a sequence of smooth compactly supported vector fields converging to b in $L_p^q(T)$, and u^n the solution to

$$\partial_t u^n + \frac{1}{2}\Delta u^n + b^n \cdot \nabla u^n = \lambda u^n - b^n.$$

Then we have

(i) *as $n \to \infty$, $u^n(t,x) \to u(t,x)$ and $\nabla u^n(t,x) \to \nabla u(t,x)$ locally uniformly;*

(ii) $\lim_{n\to\infty} \|u^n - u\|_{\mathbb{H}^q_{2,p}(T)} = 0$;

(iii) $\sup_{n\geq 1} \sup_{t,x} |\nabla u^n(t,x)| \leq \frac{1}{2}$ *for λ big enough;*

We fix $\lambda > 0$ s.t. (iii) holds.

Approximating SDEs

Let $\phi_t^n(x) = x + u^n(t,x)$ and $Y_t^n = \phi_t^n(X_t^n)$. Then we have

$$\mathrm{d}Y_t^n = \tilde{\sigma}^n(t, Y_t^n)\,\mathrm{d}W_t + \tilde{b}^n(t, Y_t^n)\,\mathrm{d}t$$

with

$$\tilde{\sigma}^n(t,y) = Id + \nabla u^n\big(t, \phi_t^{n,-1}(y)\big),\ \tilde{b}^n(t,y) = \lambda u^n\big(t, \phi_t^{n,-1}(y)\big).$$

Let

$$\rho_t^n := \frac{\mathrm{d}(\mathcal{L}^d \circ Y_t^{n,-1})}{\mathrm{d}\mathcal{L}^d} = \big|\det\big(\nabla Y_t^{n,-1}\big)\big|,$$

$$\bar{\rho}_t^n := \frac{\mathrm{d}(\mathcal{L}^d \circ Y_t^{n})}{\mathrm{d}\mathcal{L}^d} = \big|\det\big(\nabla Y_t^{n}\big)\big|.$$

Convergence of densities

We have

$$\bar{\rho}_t^n(x) = \exp\left\{ \int_0^t \left\langle \mathrm{div}(\tilde{\sigma}^n)(s, Y_s^n(x)), \mathrm{d}W_s \right\rangle + \int_0^t \left[\mathrm{div}(\tilde{b}^n) - \frac{1}{2}\langle \nabla\tilde{\sigma}^n, (\nabla\tilde{\sigma}^n)^* \rangle \right](s, Y_s^n(x))\, \mathrm{d}s \right\}.$$

Using Lemma 6, we can show that

$$\bar{\rho}_t^n(x) \to \exp\left\{ \int_0^t \left\langle \mathrm{div}(\tilde{\sigma})(s, Y_s(x)), \mathrm{d}W_s \right\rangle + \int_0^t \left[\mathrm{div}(\tilde{b}) - \frac{1}{2}\langle \nabla\tilde{\sigma}, (\nabla\tilde{\sigma})^* \rangle \right](s, Y_s(x))\, \mathrm{d}s \right\}.$$

which implies $\mathcal{L}^d \circ Y_t^{-1} \sim \mathcal{L}^d$

Thanks for your attention!