Perturbation analysis for continuous-time Markov chains

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Outline

Perturbed Markov chains

Let Φ_t be a time-homogeneous continuous-time Markov chain (CTMC) on a countable state space $\mathbb{E} = \mathbb{Z}_+$, with the irreducible and regular intensity matrix $Q = (q_{ij})$ (Q is regular means that Q is conservative and Q-process is unique). The unique Q-function (transition function) is denoted by $P^t = (P^t_{ij})$. Suppose that Φ_t is positive recurrent with the unique invariant probability measure π. We suppose that Q is perturbed to be another irreducible intensity matrix $\tilde{Q} = (\tilde{q}_{ij})$. Let $\Delta = \tilde{Q} - Q$.

Note:

▷ A typical perturbation form: $\tilde{Q} = Q + \varepsilon G$.

▷ Throughout we assume the unperturbed component Q is irreducible. This type of perturbation is called regular perturbation, otherwise is called singular perturbation. For singular perturbation, please refer to the book G.G. Yin and Q. Zhang (2013).

We are interested in two things:

▷(i) Condition sufficient for the regularity of $\tilde{Q}$ and then the stability of the perturbed $\tilde{Q}$-process;

▷(ii) Computable bounds on $\nu - \pi$ when $\tilde{Q}$-process is ergodic with the invariant distribution ν.

The perturbation Δ is supposed to be small. However small perturbation may result in a big change of the stability of a process.

Illustrative Example:

Consider a birth-death process whose intensity matrix $Q = (q_{ij})$ is conservative and its birth (death) coefficients b_i (a_i) are given by:

$$b_0 = 1, \quad b_i = a_i = i^\gamma, i \geq 1,$$

where $\gamma \geq 0$.

From M.F.Chen (2003), we know that the Q-process is

- ergodic if and only if $\gamma > 1$,
- exponentially ergodic if and only if $\gamma \geq 2$,
- and strongly ergodic if and only if $\gamma > 2$.

Cont'd

Now we consider another intensity matrix $\tilde{Q} = (\tilde{q}_{ij})$ given by

$$\tilde{q}_{ij} = \begin{cases} q_{i0} + \varepsilon, & \text{if } \ i \geq 1, j = 0, \\ q_{ii} - \varepsilon, & \text{if } \ i \geq 1, j = i, \\ q_{ij}, & \text{else.} \end{cases}$$

or in matrix form $\tilde{Q} = Q + \varepsilon G$, where

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Cont'd

It can be shown that $\tilde{Q}$ is also regular (see also Zhang Y.H. 1998). Define a drift function V by $V_0 = 0$ and $V_i = \frac{1}{\varepsilon}$, $i \geq 1$. Then it is easy to show

$$\sum_{j\in\mathbb{E}} q_{ij} V_j \leq -1 + \frac{1}{\varepsilon} I_{\{0\}}(i),$$

where $I_A(\cdot)$ denotes the indicator function about the set A. From M.F.Chen (2003), we know that the $\tilde{Q}$-process is always strongly ergodic (independently of γ) whenever $\varepsilon > 0$.

Observe that both Q and $\tilde{Q}$ can be understood to be a perturbed matrix from each other. The value of ε reflects the size of the perturbation. By the above analysis, a very small perturbation may change a non-ergodic Markov chain into a strongly ergodic one and vice versa. Hence the small perturbation may cause drastic influence on the stability and the invariant measure of a process.

Types of the bounds

Three types of bounds: component-wise, $|\nu_k - \pi_k|$; norm-wise, $\|\nu - \pi\|_1$; V-norm-wise bounds $\|\nu - \pi\|_V$,

For a finite measure μ, define $\|\mu\|_V = \sum_{i\in\mathbb{E}} |\mu_i| V_i$. The V-norm for any matrix $L = (L_{ij})$ on $\mathbb{E} \times \mathbb{E}$ is given by

$$\|L\|_V = \sup_{i\in\mathbb{E}} \frac{1}{V_i} \sum_{j\in\mathbb{E}} |L_{ij}| V_j .$$

The V-normwise perturbation bounds enable us to measure the perturbation of the moments of the invariant distribution, which causes an essential difference from the component-wise or normwise bounds.

For example, let Φ_t be the queue length of the M/M/1 queue and $V_i = i$, then

$$|\nu(V) - \pi(V)| \le \|\nu - \pi\|_V$$

(the perturbation of the expected stationary queue length).

Review the literature: discrete-time case

For discrete-time Markov chains:

(1) finite state space:

▷ C.D. Meyer (1980), G.E.Cho and Meyer (2000), [group inverse, norm-wise bounds]

▷ E. Seneta (1988, 1991) [ergodicity coefficient, group inverse, norm-wise bounds]

▷ J.J. Hunter (2003, 2006) [first hitting times component-wise bounds]

(2) Infinitely many state space

▷ N.V. Kartashov (1980, 1986, 1996), Aissani and Kartashov (1983) [Lyapunov-like method, V-norm bounds]

▷ Aissani D. and his students (2008, 2010...)

▷ Y.Y. Liu (2012) [Proposing two new norm-wise bounds and one V-normwise bound in sprit of Lyapunov-like method.]

CTMCs

(1) finite state space:

▷ A.Y. Mitrophanov (2004, 2006)[convergence rates]

(2) countable state space with bounded intensity matrices:

▷ E. Altman, C.E. Avrachenkov, R. Núñez-Queija (2004) [regular and singular perturbation]

▷ * Heidergott B, Hordijk A, Leder N. (2009,2010) [series expansion]

▷ A.I. Zeifman (1995, 2009, 2014,...) [non-homogeneous CTMCs]

▷ All these bounds depend on the accuracy of the convergence rate

▷ Y.Y. Liu (2012) [extend the Lyapunov-like method to continuous-time case. The bounds are better.]

(3) infinitely countable state space with general intensity matrices:

▷ R.L. Tweedie (1980) [the component-wise bounds, specific perturbation]

Outline

Deviation matrix

Our focus: V-norm bounds for general CTMCs

To perform the perturbation analysis, one key tool is the deviation matrix $D = (D_{ij})$, defined by

$$D_{ij} = \int_0^\infty (P_{ij}^t - \pi_j)dt.$$

The formulas

$$\nu\tilde{Q} = 0, \quad QD = \Pi - I$$

leads to a key fact:

$$\nu - \pi = \nu(\tilde{Q} - Q)D = \nu\Delta D.$$

This is the starting point for perturbation analysis.

Drift condition for exponential ergodicity

D(V, λ, b, C):There exist a finite function V satisfying $V \geq 1$, some finite set C, and positive constants $\lambda, b < \infty$ such that

$$\sum_{j \in \mathbb{E}} q_{ij} V_j \leq -\lambda V_i + b I_C(i), \quad i \in \mathbb{E}.$$

Remark: (i) The drift condition is equivalent to exponential ergodicity. The explicit drift function V has been found for many continuous-time Markov models, see, e.g. Y.H. Mao and Y.H. Zhang (2004) for single birth process, multidimensional Q-processes, and branching processes.

(ii) From P. Coolen-Schrijner and E A.Van Doorn (2003), we know that the deviation matrix exists (i.e. $D < \infty$) if and only if the process is ℓ-ergodic of order 2 (see, M.F. Chen and Y.Z. Wang (2013), Y.H. Mao (2004) or Y.Y. Liu et al (2010)). Obviously, the drift condition is sufficient for the existence of the deviation matrix.

Theorem 1: bound on $\|\Delta D\|_V$.

The first result gives the V-norm-wise bound on $\|\Delta D\|_V$ in terms of the drift condition.

Theorem 1. (Y.Y. Liu 2014) Let Q be irreducible and regular, and let i_0 be any fixed state in $\mathbb{E}$. Suppose that Q satisfies the drift condition $\mathbf{D}(V, \lambda, b, C)$ for $C = \{i_0\}$ and $V_{i_0} = 1$. Let $c = 1 + \pi(V)$. Then we have

$$\begin{aligned} \|\Delta D\|_V &\leq \frac{c}{\lambda}\|\Delta\|_V \\ &\leq \frac{\lambda + b}{\lambda^2}\|\Delta\|_V. \end{aligned}$$

Theorem 2: bound on $\|\nu - \pi\|_V$.

Theorem 2. (Y.Y. Liu 2014) Let Q be irreducible and conservative, and let i_0 be any fixed state in $\mathbb{E}$. Suppose that Q satisfies the drift condition $\mathbf{D}(V, \lambda, b, C)$ for $C = \{i_0\}$, $V_{i_0} = 1$ and $\lim_{i \to \infty} V_i = \infty$. Let $c = 1 + \pi(V)$.

(i) If $\|\Delta\|_V < \frac{\lambda}{c}$, then $\tilde{Q}$ is regular, the $\tilde{Q}$-process $\tilde{\Phi}(t)$ is exponentially ergodic, $\nu = \pi \sum_{n=0}^{\infty} (\Delta D)^n$, and

$$\|\nu - \pi\|_V \leq \frac{c\pi(V)\|\Delta\|_V}{\lambda - c\|\Delta\|_V}.$$

(ii) If $\|\Delta\|_V < \frac{\lambda^2}{b+\lambda}$, then $\tilde{Q}$ is regular, the $\tilde{Q}$-process $\tilde{\Phi}(t)$ is exponentially ergodic, $\nu = \pi \sum_{n=0}^{\infty} (\Delta D)^n$, and

$$\|\nu - \pi\|_V \leq \frac{b[(b+\lambda)\|\Delta\|_V]}{\lambda^3 - \lambda[(b+\lambda)\|\Delta\|_V]}.$$

Remark on Theorem 2

Remark:

▷(i) In this theorem, the drift function V is required to satisfy $\lim_{i\to\infty} V_i = \infty$, which however is not needed for the other two theorems. This condition is imposed for guaranteeing the regularity of Q and $\tilde{Q}$.

▷(ii) Theorem 1 reveals an interesting phenomenon that the exponentially ergodic process Φ_t displays good stability:

◇ (i) Φ_t is always strongly stable (i.e. every intensity matrix $\tilde{Q}$ in the set $\{\tilde{Q} : \|\Delta\|_V < \varepsilon\}$ is regular and has a unique invariant probability measure ν such that $\|\nu - \pi\|_V \to 0$ as $\|\Delta\|_V \to 0$);

◇ (ii) the perturbed process $\tilde{\Phi}_t$ is also exponentially ergodic with respect to small perturbation in the sense of V-norm.

Cont'd

▷(iii) We could have obtained bounds on $\|D\|_V$ more directly through the estimate of the convergence rates. If Q satisfies $\mathbf{D}(V, \lambda, b, \{i_0\})$, then it follows from S.P Myen and R.L. Tweedie (1993) that there exist positive constants α, β such that

$$\|P^t - \Pi\|_V \leq \alpha e^{-\beta t},$$

which implies that $\|D\|_V \leq \frac{\alpha}{\beta}$. However, the constants α, β are difficult to determine except for some special cases, for example, Q is monotone.

Suppose that Q is monotone and that Q satisfies $\mathbf{D}(V, \lambda, b, \{0\})$ for a non-decreasing function V, then it follows from R.B. Lund etal (1996) that (17) holds for $\alpha = 2\left(1 + \frac{b}{\lambda}\right)$ and $\beta = \lambda$, which implies $\|D\|_V \leq \frac{2(\lambda+b)}{\lambda^2}$, $\|\Delta D\|_V \leq \frac{2(\lambda+b)}{\lambda^2}\|\Delta\|_V$. Hence if $\|\Delta\|_V < \frac{\lambda^2}{2(\lambda+b)}$, then

$$\|\nu - \pi\|_V \leq \frac{b[2(b+\lambda)\|\Delta\|_V]}{\lambda^3 - \lambda[2(b+\lambda)\|\Delta\|_V]}.$$

Theorem 3: bound on $\|D\|_V$.

Theorem 3. (Y.Y. Liu 2014) Let Q be irreducible and regular, and let i_0 be any fixed state in $\mathbb{E}$. Suppose that Q satisfies the drift condition $\mathbf{D}(V, \lambda, b, C)$ for $C = \{i_0\}$ and $V_{i_0} = 1$. Let $c = 1 + \pi(V)$. Then we have

$$\|D\|_V \leq \frac{c^2}{\lambda} \leq \frac{(\lambda + b)^2}{\lambda^3}.$$

Remark: Let $|g| \leq V$. Then $h = Dg$ is a solution of the Poisson equation $Qh = -[g - \pi(g)]$, and

$$|h_i| = |\sum_{j\in\mathbb{E}} D_{ij} g_j| \leq \sum_{j\in\mathbb{E}} |D_{ij}| V_j| \leq \|D\|_V V_i.$$

While by P. Glynn and S. Meyn (1996), we can only show that $|h_i| \leq d(1 + V_i)$ for an existing but unknown positive constant d.

Outline

Augmented truncation

Suppose that Φ_t is ergodic with intensity matrix Q and the unique invariant probability measure π. Let ${}_{(n)}\mathbb{E} = \{0, 1, 2, \cdots, n\}$ and let K be any fixed state in $\mathbb{E}$. Denote by ${}_{(n)}Q$, $n \geq K$, the $(n+1) \times (n+1)$ northwest corner truncation of Q. The $(K+1)$-th column augmentation, denoted by ${}_{(n,K)}Q = ({}_{(n,K)}q_{ij})$, is given by

$$ {}_{(n,K)}q_{ij} = \begin{cases} q_{ij} + I_{\{K\}}(j) \cdot \sum_{j \notin \mathbb{E}_n} q_{ij}, & \text{if } \ i, j \in \mathbb{E}_n, \\ 0, & \text{otherwise.} \end{cases} $$

From the special construction, we know that ${}_{(n,K)}Q$ has a unique invariant probability measure ${}_{(n,K)}\pi$.

Does ${}_{(n,K)}\pi$ converge to π for an arbitrary K? Not always!

Non-convergence example

Consider a CTMC with the following conservative intensity matrix

$$Q = \begin{pmatrix} q_0 & p_0 - 1 & 0 & 0 & \cdots \\ q_1 & 0 & p_1 - 1 & 0 & \cdots \\ q_2 & 0 & 0 & p_2 - 1 & \cdots \\ \vdots & & & & \ddots \end{pmatrix},$$

where $p_0 = \frac{1}{2}$ and for any $n \geq 1$

$$p_n = \begin{cases} \frac{1}{2}, & n \text{ is an odd}, \\ 1 - \frac{1}{3^{\frac{n}{2}}}, & n \text{ is an even}. \end{cases}$$

Q is irreducible, regular and Q-process is strongly ergodic. If we perform the last column augmentation (i.e. $K = n$) of the truncated intensity matrix, then we know that ${}_{(n,n)}\pi \nrightarrow \pi$. (Reason: Consider $P = Q + I$ and use Example 2.1 in Y.Y. Liu (2010).)

A basic question is: what conditions ensure that ${}_{(n)}\pi \to \pi$.

Basic properties about augmented truncation

Proposition: Let Q be irreducible and regular. Suppose that Q satisfies the drift condition $\mathbf{D}(V, \lambda, b, \{i_0\})$ with $V_{i_0} = 1$. Then we have

(i) ${}_{(n,i_0)}Q$ satisfy $\mathbf{D}(V, \lambda, b, \{i_0\})$ for any $n \geq i_0$;

(ii) ${}_{(n,i_0)}P_{ij}(t) \to P_{ij}(t), \quad n \to \infty$;

(iii) $\|{}_{(n,i_0)}\pi - \pi\|_V \to 0, \quad n \to \infty$.

▷ (i) follows from

$$\sum_{j \in {}_{(n)}\mathbb{E}} {}_{(n,i_0)}q_{ij} V_j = \sum_{j \in {}_{(n)}\mathbb{E}, j \neq i_0} q_{ij} V_j + q_{ii_0} V_{i_0} + \sum_{k \geq n+1}^{\infty} q_{ik} V_{i_0} \leq \sum_{k \in \mathbb{E}}^{\infty} q_{ik} V_k.$$

▷ Hart and Tweedie (2012) proved (ii) and $\lim_{n\to\infty} \|{}_{(n,i_0)}\pi - \pi\| = 0$. proved. Actually their arguments can be modified easily to obtain the stronger convergence (iii). Key step: For any fixed $i \in {}_{(n)}\mathbb{E}$ and $m \in \mathbb{N}_+$,

$$\begin{aligned} \|{}_{(n,i_0)}\pi - \pi\|_V &\leq \|{}_{(n,i_0)}R^m_{i\cdot} - {}_{(n,i_0)}\pi\|_V + \|R^m_{i\cdot} - \pi\|_V + \|{}_{(n,i_0)}R^m_{i\cdot} - R^m_{i\cdot}\|_V \\ &\leq 2MV(i)\rho^m + \sum_{s=0}^{m-1} \sum_{k \in \mathbb{E}} R^s_{ik} \left(\frac{\beta + b}{\beta + \lambda}\right)^{m-1-s} \|R_{k\cdot} - {}_{(n,i_0)}R_{k\cdot}\|_V \end{aligned}$$

Outline

Proof of Theorem 1: Step 1

$$\Rightarrow \|\Delta D\|_V \leq \frac{c}{\lambda}\|\Delta\|_V \leq \frac{\lambda+b}{\lambda^2}\|\Delta\|_V.$$

We only need to prove the first inequality. The second inequality follows the first one and that $c = 1 + \pi(V) \leq 1 + \frac{b}{\lambda}$ (by the drift condition).

(1) We prove the result for the case that Q is bounded, i.e. there exists a positive constant h such that $\sup_{i\in\mathbb{E}}(-q_{ii}) \leq \frac{1}{h}$. Let $P_h = I + hQ$, $\tilde{P}_h = I + h\tilde{Q}$, and $\Delta_h = \tilde{P}_h - P_h = h\Delta$. Denote by D_h the deviation matrix of P_h. Note that P_h has the same invariant probability measure as Q and $D_h = \frac{D}{h}$. From N.V. Kartashov (1986), we know that

$$D_h = (I - \Pi)\sum_{n=0}^{\infty} T_h^n (I - \Pi).$$

where T_h is the the same as P_h except the first row (all changed into 0).

Cont'd

Since Q satisfies $\mathbf{D}(V, \lambda, b, \{i_0\})$, we have

$$\|T_h\|_V \leq 1 - \lambda h.$$

Since $\Delta_h \Pi = 0$, we have

$$\begin{aligned} \|\Delta D\|_V = \|\Delta_h D_h\|_V &= \left\| \Delta_h \sum_{n=0}^{\infty} T_h^n (I - \Pi) \right\|_V \\ &\leq \|\Delta\|_V \frac{c}{\lambda}. \end{aligned}$$

Step 2

(2) Performing the augmented truncations to both Q and $\tilde{Q}$ obtain ${}_{(n)}Q$ and ${}_{(n)}\tilde{Q}$, respectively. Take ${}_{(n)}\Delta = {}_{(n)}Q - {}_{(n)}\tilde{Q}$. Let ${}_{(n)}\pi$ and ${}_{(n)}\nu$ be the invariant probability measure of ${}_{(n)}Q$ and ${}_{(n)}\tilde{Q}$, respectively. Let ${}_{(n)}D$ be the deviation matrix with respect to ${}_{(n)}Q$.

$$\lim_{n\to\infty} \sum_{j\in {}_{(n)}\mathbb{E}} |({}_{(n)}\Delta_{(n)}D)_{ij}|V_j = \sum_{j\in\mathbb{E}} |(\Delta D)_{ij}|V_j.$$

To prove the above fact, we need to show $\lim_{n\to\infty} {}_{(n)}D_{ij} = D_{ij}$.
Since ${}_{(n)}Q$ satisfies $\mathbf{D}(V, \lambda, b, \{i_0\})$ uniformly for any $n > i_0$, there exist positive constants d, α such that

$$\sum_{j\in {}_{(n)}\mathbb{E}} |{}_{(n)}P_{ij}^t -_{(n)} \pi_j|V_j \le dV(i)e^{-\alpha t}.$$

Since $\lim_{n\to\infty} {}_{(n)}P_{ij}^t = P_{ij}^t$ and $\lim_{n\to\infty} {}_{(n)}\pi_j = \pi_j$, by the bounded convergence theorem, we obtain

$$\lim_{n\to\infty} {}_{(n)}D_{ij} = \int_0^\infty \lim_{n\to\infty} [{}_{(n)}P_{ij}^t -_{(n)} \pi_j]dt = D_{ij}.$$

Step 3

(3) There exists big enough N such that for any $n \geq N$

$$\|_{(n)}\Delta_{(n)}D\|_V \geq \|\Delta D\|_V - \varepsilon.$$

Since $_{(n)}Q$ satisfies $\mathbf{D}(V, \lambda, b, \{i_0\})$ uniformly for any $n > i_0$, applying the fist part, we

$$\|_{(n)}\Delta_{(n)}D\|_V \leq \|\Delta\|_V \frac{c}{\lambda}.$$

Thus by (2) and (3), we have

$$\|\Delta D\|_V \leq \frac{c}{\lambda}\|\Delta\|_V$$

because ε is arbitrarily small.

Proof of Theorem 2:

(i) Since $\|\Delta\|_V < \frac{\lambda}{c}$, we obtain

$$\sum_{j\in\mathbb{E}} \tilde{q}_{ij} V_j \le -\frac{(c-1)\lambda}{c} V_i + b I_{\{i_0\}}(i).$$

By M.F. Chen (2003), we know that both Q and $\tilde{Q}$ are regular due to the extra assumption that $\lim_{i\to\infty} V_i = \infty$. Moreover, the $\tilde{Q}$-process $\tilde{\Phi}(t)$ is exponentially ergodic. Since $\|\Delta\|_V < \frac{\lambda}{c}$, by Theorem 1, we have $\|\Delta D\|_V < 1$. Since $\nu(I - \Delta D) = \pi$, we have

$$\nu = \pi \sum_{n=0}^{\infty} (\Delta D)^n,$$

which follows

$$\|\nu - \pi\|_V \le \pi(V) \sum_{n=1}^{\infty} \|\Delta D\|_V^n \le \frac{c\pi(V)\|\Delta\|_V}{\lambda - c\|\Delta\|_V}.$$

(ii) Since $\frac{c\pi(V)\|\Delta\|_V}{\lambda - c\|\Delta\|_V}$ is increasing of c, the statements of (ii) hold when c and $\pi(V)$ are changed into $\frac{b+\lambda}{\lambda}$ and $\frac{b}{\lambda}$, respectively.

Outline

Future research

- Developing the technique of augmented truncation for CTMCs:
  - ▷ reducing the exponential ergodicity condition
  - ▷ extending to high-dimensional CTMCs
- Establishing the lower bound for the error. (By far, all the known perturbation bounds are upper bounds)

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