On the Eigenfunctions of Complex Ornstein-Uhlenbeck Operators and Applications

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第十届马氏过程及其相关课题研讨会

[1] Chen Y., Liu Y., On the eigenfunctions of the complex Ornstein-Uhlenbeck operators, *Kyoto J. Math.,* Vol. 54(3), 577-596, (2014)

[2] Chen Y., Liu Y., On the fourth moment theorem for the complex multiple Wiener-Itô integrals, *Preprint* (2014)

Outline

1. Background
   - Chaos decomposition
   - 4th moment theorem
2. Eigenfunction of Complex OU Process
3. Application: 4th moment theorem

Outline

- Wiener (1938): *The Homogeneous Chaos*
  - Generalized harmonic analysis
  - Singular signal processes: power spectral analysis
  - Generalized space-time Birkhoff ergodic theorem
  - More than normal distribution
  - Stratonowich multiple integrals

☐ Wiener (1938): *The Homogeneous Chaos*

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☐ Cameron-Martin (1947): *in the series of Fourier-Hermite...*

- $(C_0[0,1], \mathbf{P}^W)$, $L^2(C_0[0,1], \mathbf{P}^W)$
- $\{\alpha_p\}$ orthonormal basis in $L^2(0,1)$, H_m Hermite polynomial of degree m

$$\Phi_{m,p} = H_m\Big(\int_0^1 \alpha_p(s)\mathrm{d}\omega(s)\Big)$$

- Not connect with Itô multiple integrals

- Itô (1951): *Multiple Wiener Integrals*
  - System of normal random measures $\mathbf{B}$

$$\int \cdots \int \varphi(t_1) \cdots \varphi(t_n) \mathrm{d}\beta(t_1) \cdots \mathrm{d}\beta(t_n)$$
$$= \frac{1}{\sqrt{2}^n} H_n\Big(\frac{1}{\sqrt{2}} \int \varphi(s) \mathrm{d}\beta(s)\Big)$$

  - Iterated stochastic integrals
  - Orthogonalizing Wiener's chaos polynomials

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□ Itô (1953): *Complex Multiple Wiener Integrals*

- System of complex normal random measures $\mathbf{M}$
- $H_{p,q}(z,\overline{z})$: complex Hermite polynomial of degree (p,q), $\|f\|_2 = 1$

$$\int \cdots \int f(t_1)\cdots f(t_p)\overline{f(t_{s_1})}\cdots\overline{f(t_{s_q})} \, \mathrm{d}M(t_1)\cdots \mathrm{d}M(t_p)\mathrm{d}\overline{M(s_1)}\cdots \mathrm{d}\overline{M(s_q)}$$

$$= H_{p,q}\Big(\int f(s)\mathrm{d}M(s), \overline{\int f(s)\mathrm{d}M(s)}\Big)$$

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- A theory of integration over Hilbert spaces, QFT
- Harmonic analysis, Fourier-Plancherel transform
- Algebra of symmetric tensors $\cong_U$ square integrable functions
- Finitely additive (cylindrical) measure

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☐ Hu and Meyer (1988): *Hu-Meyer formula*

- Relation between Stratonowich multiple integrals and Wiener-Itô multiple integrals

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THEOREM (WIENER-ITÔ CHAOS DECOMPOSITION)

$$L^2(C_0[0,1], \mathbf{P}^W) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

$$\mathcal{H}_n = \overline{\text{span}}\Big\{H_n\Big(\int_0^1 h(s)\mathrm{d}W(s)\Big), h \in H, \|h\|_{L^2(0,1)} = 1\Big\},$$

$$H_n\Big(\int_0^1 h(s)\mathrm{d}W(s)\Big)$$
$$= n! \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} h(t_1), \ldots, h(t_n)\mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_n}.$$

Theorem (Chaos decomposition: Gaussian Hilbert space)

$\mathfrak{H}$: *seperable Hilbert space*

$\mathcal{H}$: *closed subspace of zero-mean Gaussian r.v.s*

$\mathcal{H} \cong \mathfrak{H}, \ \mathbf{E}(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_{\mathfrak{H}}$

$$L^2(\Omega, \mathcal{G}(W(h)), \mathbf{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

$$\begin{aligned} \mathcal{H}_n &= \overline{\text{span}}\{H_n(W(h)), h \in H, \|h\|_{\mathfrak{H}} = 1\} \\ &= \overline{\text{span}}\{I_n(f), \ f \in \mathfrak{H}^{\odot n}\} \end{aligned}$$

Theorem (Nualart-Peccati Criterion, 2005)

For $q \geq 2$, $F_n = I_q(f_n)$, $f_n \in \mathfrak{H}^{\odot q}, n \geq 1$. $\mathbf{E}(F_n^2) \to 1$. *The following 4 conditions are equivalent, as* $n \to \infty$,

$$\begin{aligned}
&\text{(i)} && F_n \xrightarrow{d} \mathcal{N}(0,1); \\
&\text{(ii)} && \mathbf{E}(F_n^4) \to 3; \\
&\text{(iii)} && \|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} \to 0, \;\; r = 0, 1 \cdots, q-1; \\
&\text{(iv)} && \|D(F_n)\|_{\mathfrak{H}}^2 \to 0 \text{ in } L^2.
\end{aligned}$$

Approaches to prove

- Nualart, Peccati (05, *Ann. Probab.*)
  Dambis-Dubins-Schwarz Theorem
- Nourdin, Peccati (09, *P.T.R.F.*)
  Stein's method and Malliavin calculus
- Nourdin (11, *Electron.Comm.Probab.*)
  k-moment, combinatorial techniques
- Ledoux (12, *Ann. Probab.*)
  Markov operators, spectral theory
- Nourdin, Peccati and Swan (13, *arXiv*)
  Relative entropy

Does 4th Moment Theorem hold for complex Wiener-Itô integrals?

Outline

Generating function of real Hermite polynomial

$$\exp\left(tx - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} H_n(x) t^n.$$

$$H_n'(x) = H_{n-1}(x),\ (n+1)H_{n+1}(x) = nH_n(x) - H_{n-1}(x)$$

Generating function of Complex Hermite polynomial

$$\exp\left(-t\bar{t} + t\bar{z} + \bar{t}z\right) = \sum_{p,q=0}^{\infty} \frac{1}{p!q!} H_{p,q}(z,\bar{z}) \bar{t}^p t^q.$$

$$\frac{\partial}{\partial z} H_{p,q} = pH_{p-1,q},\ \frac{\partial}{\partial \bar{z}} H_{p,q} = H_{p,q-1}, \ldots$$

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$$\int \cdots \int f(t_1) \cdots f(t_p) \overline{f(t_{s_1})} \cdots \overline{f(t_{s_q})}$$

$$\mathrm{d}M(t_1) \cdots \mathrm{d}M(t_p) \mathrm{d}\overline{M(s_1)} \cdots \mathrm{d}\overline{M(s_q)}$$

$$= H_{p,q}\Big(\int f(s)\mathrm{d}M(s), \overline{\int f(s)\mathrm{d}M(s)} \Big)$$

1-d complex OU process

$$dZ_t = -\alpha Z_t dt + \sqrt{2\sigma^2} d\zeta_t.$$

$$\alpha = ae^{i\theta} = r + i\Omega,\ \zeta_t = B_1(t) + iB_2(t)$$

$$\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} -a\cos\theta & a\sin\theta \\ -a\sin\theta & -a\cos\theta \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} dt + \sqrt{2\sigma^2} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}$$
$$= \begin{bmatrix} -r & \Omega \\ -\Omega & -r \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} dt + \sqrt{2\sigma^2} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}.$$

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Invariant measure, Generator of 1-d complex OU process

Invariant measure

$$\mathrm{d}\mu = \frac{r}{2\pi\sigma^2} \exp\left\{-\frac{r(x^2+y^2)}{2\sigma^2}\right\} \mathrm{d}x\mathrm{d}y.$$

Generator

$$\begin{aligned} A &= (-rx+\Omega y)\frac{\partial}{\partial x} + (-\Omega x - ry)\frac{\partial}{\partial y} + \sigma^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \\ &= -r\left[\left(1+\mathrm{i}\frac{\Omega}{r}\right)z\frac{\partial}{\partial z} + \left(1-\mathrm{i}\frac{\Omega}{r}\right)\overline{z}\frac{\partial}{\partial \overline{z}} - \frac{4\sigma^2}{r}\frac{\partial^2}{\partial z \partial \overline{z}}\right] \end{aligned}$$

PROPOSITION (NORMAL OPERATOR (正常,正规,规范算子))

- $$AA^* = A^*A$$

- $\mathcal{A}_s = \sigma^2\Delta - rx\frac{\partial}{\partial x} - ry\frac{\partial}{\partial y}, \quad \mathcal{J} = -\mathrm{i}\Omega(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})$

$$\mathcal{A}_s\mathcal{J} = \mathcal{J}\mathcal{A}_s$$

PROPOSITION (Metafune, Pallara, Priola 02, Chen & L. 14a)

$$\sigma(A) = \{-(m+n)r + \mathrm{i}(m-n)\Omega, \ m, n = 0, 1, 2\dots\}$$

THEOREM (EIGENFUNCTION OF COMPLEX OU OPERATOR)

The eigenfunction associated with the eigenvalue $-r(m+n) - \mathrm{i}(m-n)\Omega$ *of* A *is*

$$J_{m,n}(z,\,\rho) = \begin{cases} z^{m-n} \sum_{r=0}^{n} (-1)^r r! \binom{m}{r} \binom{n}{r} |z|^{2(n-r)} \rho^r, & m \geq n, \\ \bar{z}^{n-m} \sum_{r=0}^{m} (-1)^r r! \binom{m}{r} \binom{n}{r} |z|^{2(m-r)} \rho^r & m < n, \end{cases}$$

$$= \begin{cases} z^{m-n}(-1)^n n! L_n^{m-n}(|z|^2\,,\,\rho), & m \geq n, \\ \bar{z}^{n-m}(-1)^m m! L_m^{n-m}(|z|^2\,,\,\rho), & m < n. \end{cases}$$

$$J_{m,n}(x,y) = \begin{cases} (-1)^n n!(x+\mathrm{i}y)^{m-n} L_n^{m-n}(x^2+y^2,\rho), & m \geq n, \\ (-1)^m m!(x-\mathrm{i}y)^{n-m} L_m^{n-m}(x^2+y^2,\rho), & m < n, \end{cases}$$

$\rho = \frac{2\sigma^2}{r}$, $L_n^{\alpha}(z,\rho)$ *is Laguerre Polynomial.*

Remark

If $\rho = 1$, $J_{m,n}(z,1)$ is called the Hermite polynomials of complex variables by K. Itô. We name $J_{m,n}(z,\rho)$ the Hermite-Laguerre-Itô Polynomials.

The first few Hermite-Laguerre-Itô polynomials are

$$J_{m,0} = z^m, \quad J_{0,n} = \bar{z}^n,$$
$$J_{1,1} = |z|^2 - \rho, \quad J_{2,1} = z(|z|^2 - 2\rho), \quad J_{3,1} = z^2(|z|^2 - 3\rho), \ldots$$
$$J_{1,2} = \bar{z}(|z|^2 - 2\rho), \; J_{2,2} = |z|^4 - 4\rho\,|z|^2 + 2\rho^2, \; J_{3,2} = z(|z|^4 - 6\rho\,|z|^2$$
$$\ldots$$

Proof: 2 Approaches

☐ Key point: $\mathcal{A}_s\mathcal{J} = \mathcal{J}\mathcal{A}_s$, common eigenfunctions. Solving β_k

$$\begin{cases} J_{m,n}(x,y) &= \sum_{k=0}^{l} \beta_k H_k(x, \frac{\rho}{2}) H_{l-k}(y, \frac{\rho}{2}) \\ J_{m,n}(x,y) &= -\mathrm{i}\lambda\Omega J_{m,n}(x,y) \\ \mathsf{M}\big(-\mathrm{i}(m-n)\big)\vec{\beta} &= 0 \end{cases}$$

☐ Complex creation and annihilation operator

$$(\partial^*\phi)(z) = -\frac{\partial}{\partial\bar{z}}\phi(z) + \frac{z}{\rho}\phi(z), \quad (\bar{\partial}^*\phi)(z) = -\frac{\partial}{\partial z}\phi(z) + \frac{\bar{z}}{\rho}\phi(z).$$

$$\begin{aligned} J_{0,0}(z,\rho) &= 1 \\ J_{m,n}(z,\rho) &= \rho^{m+n}(\partial^*)^m(\bar{\partial}^*)^n 1. \end{aligned}$$

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PROPOSITION (Chen & L. 14a, 14b)

$z = x + \mathrm{i}y$. *Then*

$$J_{m,l-m}(z) = \sum_{k=0}^{l} \mathrm{i}^{l-k} \sum_{r+s=k} \binom{m}{r}\binom{l-m}{s}(-1)^{l-m-s} H_k(x) H_{l-k}(y),$$

$$H_k(x) H_{l-k}(y) = \frac{\mathrm{i}^{l-k}}{2^l} \sum_{m=0}^{l} \sum_{r+s=m} \binom{k}{r}\binom{l-k}{s}(-1)^s J_{m,l-m}(z).$$

Thus, $\{J_{k,l}(z) : k + l = n\}$ *and* $\{H_k(x)H_l(y) : k + l = n\}$ *generate the same linear subspace of* $L^2_{\mathbb{C}}(\mathbb{C}, \nu)$.

$$\overline{J_{m,n}(z,\rho)} = J_{n,m}(z,\rho).$$

$$E_\nu[J_{m,n}(z,\rho)^2] = E_\nu[J_{m,n}(z,\rho)\overline{J_{n,m}(z,\rho)}] = 0, \text{ if } m \neq n.$$

Complex Gaussian Hilbert space, Complex isonormal Gaussian process

Definition

$\mathfrak{H}$: separable Hilbert space

X, Y : i.i.d. real Gaussian isonormal process over $\mathfrak{H}$

$X_{\mathbb{C}}$, $Y_{\mathbb{C}}$: complexicfation of X, Y

$$Z(\mathfrak{h}) = \frac{X_{\mathbb{C}}(\mathfrak{h}) + \mathrm{i}Y_{\mathbb{C}}(\mathfrak{h})}{\sqrt{2}}, \ \mathfrak{h} \in \mathfrak{H}_{\mathbb{C}}$$

Theorem (Chen & L. 14b)

$$\mathscr{H}_{m,n}(Z) = \overline{\text{span}}\{J_{m,n}(Z(\mathfrak{h})),\ \mathfrak{h} \in \mathfrak{H}_{\mathbb{C}}, \|\mathfrak{h}\|_{\mathfrak{H}_{\mathbb{C}}}\}$$

$$L^2_{\mathbb{C}}(\Omega, \sigma(X,Y), P) = \bigoplus_{n=0}^{\infty} \bigoplus_{m=0}^{\infty} \mathscr{H}_{m,n}$$

Remark (Janson 08)

$$L^2_{\mathbb{C}} = \bigoplus_{n=0}^{\infty} H_{\mathbb{C}}^{:n:}.$$

In fact, we can prove

$$H_{\mathbb{C}}^{:n:} = \bigoplus_{m=0}^{\infty} \mathscr{H}_{m,n}.$$

Definition

$$\mathbf{J_{m,n}} := \prod_k \frac{1}{\sqrt{2^{m_k+n_k} m_k! n_k!}} J_{m_k,n_k}(\sqrt{2} Z(\mathfrak{e}_k)).$$

$$\mathscr{I}_{m,n}(\mathrm{symm}(\otimes_{k=1}^{\infty} \mathfrak{e}_k^{\otimes m_k}) \otimes \mathrm{symm}(\otimes_{k=1}^{\infty} \bar{\mathfrak{e}}_k^{\otimes n_k})) = \sqrt{\mathbf{m}!\mathbf{n}!} \mathbf{J_{m,n}}.$$

$$\mathscr{I}_{m,n}(f) : \mathfrak{H}_{\mathbb{C}}^{\odot m} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot n} \mapsto \mathscr{H}_{m,n}.$$

W: Gaussian Hilbert spaces over $\mathfrak{H} \oplus \mathfrak{H}$.
$\mathcal{H}_n(W)$: n-th Chaos decomposition of W.

THEOREM (Chen & L. 14b)

Suppose that $\|f\|_{\mathfrak{H}}^2 + \|g\|_{\mathfrak{H}}^2 = 1$,

$$H_n(X(f)+Y(g)) = \sum_{l=0}^{n} \binom{n}{l} \|f\|^l \|g\|^{n-l} H_l\Big(\frac{X(f)}{\|f\|}\Big) H_{n-l}\Big(\frac{Y(g)}{\|g\|}\Big),$$

$$H_l(X(f))H_{n-l}(Y(g)) = \sum_k \mathsf{M}_{l,k}^{-1} H_n\big(\cos\theta_k X(f) + \sin\theta_k Y(g)\big).$$

$$\mathcal{H}_n(W) = \bigoplus_{k+l=n} \mathcal{H}_k(X)\mathcal{H}_l(Y).$$

THEOREM (Chen & L. 14b)

Suppose that $\|f\|_{\mathfrak{H}}^2 + \|g\|_{\mathfrak{H}}^2 = 1$, $\left\|\tilde{f}\right\|_{\mathfrak{H}}^2 + \|\tilde{g}\|_{\mathfrak{H}}^2 = 1$.

$$H_n\big(X(f)+Y(g)\big) + \mathrm{i}H_n\big(X(\tilde{f})+Y(\tilde{g})\big)$$
$$= \sum_{k=0}^{n} d_k\big(J_{k,n-k}(Z(\mathfrak{h})) + \mathrm{i}J_{k,n-k}(Z(\tilde{\mathfrak{h}}))\big),$$

where $\mathfrak{h} = \sqrt{2}e^{\mathrm{i}\theta}(f-\mathrm{i}g)$, $\tilde{\mathfrak{h}} = \sqrt{2}e^{\mathrm{i}\theta}(\tilde{f}-\mathrm{i}\tilde{g})$,

$$d_k = \frac{1}{2^n}\sum_{r+s=k}(-1)^s\sum_{l=0}^{n}\binom{n}{l}\binom{l}{r}\binom{n-l}{s}(\cos\theta)^l(\mathrm{i}\cdot\sin\theta)^{n-l}.$$

THEOREM (*Continued*)

Suppose that $\mathfrak{H}_{\mathbb{C}} \ni \mathfrak{h}$ *with* $\|\mathfrak{h}\|_{\mathfrak{H}_{\mathbb{C}}} = \sqrt{2}$,

$$J_{k,n-k}(Z(\mathfrak{h})) = \sum_{i=0}^{n} \tilde{c}_i H_n(X(f_i) + Y(g_i)),$$

$f_i + \mathrm{i} g_i = \frac{1}{\sqrt{2}} e^{\mathrm{i}\theta_i} \overline{\mathfrak{h}}$, *and*

$$\tilde{c}_i = \sum_{j=0}^{n} \mathsf{M}_{j,i}^{-1} \mathrm{i}^{n-k} \sum_{r+s=j} \binom{k}{r} \binom{n-k}{s} (-1)^{n-k-s}.$$

$$\mathcal{H}_n^{\mathbb{C}}(W) := \mathcal{H}_n(W) + \mathrm{i}\mathcal{H}_n(W) = \bigoplus_{k+l=n} \mathscr{H}_{k,l}(Z).$$

THEOREM (Chen & L. 14b)

Suppose that $\varphi \in \mathfrak{H}_{\mathbb{C}}^{\odot m} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot n}$ *and* $F = \mathscr{I}_{m,n}(\varphi) = U + \mathrm{i}V$, *There exist real* $u,\, v \in (\mathfrak{H} \oplus \mathfrak{H})^{\odot(m+n)}$ *such that*

$$U = \mathcal{I}_{m+n}(u), \quad V = \mathcal{I}_{m+n}(v),$$

where $\mathcal{I}_p(g)$ *is the* p*-th real Wiener-Itô multiple integral of* g *with respect to* W. *And if* $m \neq n$ *then*

$$E[U^2] = E[V^2], \quad E[UV] = (m+n)!\langle u,\, v\rangle_{(\mathcal{H}\oplus\mathcal{H})^{\otimes(m+n)}} = 0.$$

Outline

Theorem (Chen & L. 14b)

F_k : (m,n)-th complex Wiener-Itô multiple integrals, $m+n \geq 2$, $E[|F_k|^2] \to \sigma^2$ as $k \to \infty$.

(1) If $m \neq n$, as $k \to \infty$, then

(i) $(F_k) \xrightarrow{d} \zeta \sim \mathcal{CN}(0, \sigma^2)$;

$$\Leftrightarrow$$

(ii) $E[|F_k|^4] \to 2\sigma^4$.

Theorem (*Continued*)

(2) *If* $m = n$, $E[F_k^2] \to \sigma^2(a + \mathrm{i}b)$, $a, b \in \mathbb{R}$, $a^2 + b^2 < 1$, *then*

(i) $(\mathrm{Re}F_k, \mathrm{Im}F_k) \xrightarrow{d} \mathcal{N}(0, \frac{\sigma^2}{2}\mathsf{C})$, *where* $\mathsf{C} = \begin{bmatrix} 1 + a & b \\ b & 1 - a \end{bmatrix}$,

$$\Leftrightarrow$$

(ii) $E[|F_k|^4] \to (a^2 + b^2 + 2)\sigma^4$.

(3) *If* $m = n$, $E[F_k^2] \to \sigma^2(a + \mathrm{i}b)$, $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, *then*

(i) $(\mathrm{Re}F_k, \mathrm{Im}F_k) \xrightarrow{d} \mathcal{N}(0, \frac{\sigma^2}{2}\mathsf{C})$

$$\Leftrightarrow$$

(ii) $E[|F_k|^4] \to 3\sigma^4$

$$\Leftrightarrow$$

(iii) $E[{F_k}^4] \to 3(a + \mathrm{i}b)^2\sigma^4$.

Thanks