Asymptotic properties of supercritical branching processes in random environments

Quansheng LIU 刘全升

Univ. Bretagne-Sud (France) & Changsha Univ. of Science and Technology (China)

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Based on joint works with
Zhiqiang GAO, Chunmao HUANG, Yingqiu LI, Xingang LIANG, Hesong WANG

LMBA
Laboratoire de Mathématiques de Bretagne Atlantique
UMR6205 - CNRS/UBO/UBS

UBS
UNIVERSITÉ DE BRETAGNE-SUD

Outline

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Introduction

We consider a supercritical branching process (Z_n) in an independent and identically distributed random environment ξ, and present some recent results on the asymptotic properties of the branching process.

In particular, we show:

1. a criterion for the existence of weighted moments of the limit variable W of the normalized population size $W_n = Z_n/\mathbb{E}[Z_n|\xi]$;
2. limit theorems (such as moderate and large deviations principles) on $(\log Z_n)$;
3. the convergence rates of $W_n - W$ (a.s., in law, or in L^p).

The talk is mainly based on the short survey:
Y. Li, Q. Liu, Z. Gao, H. Wang. Front. Math. China, 9(4) 2014: 737 – 751.

Outline

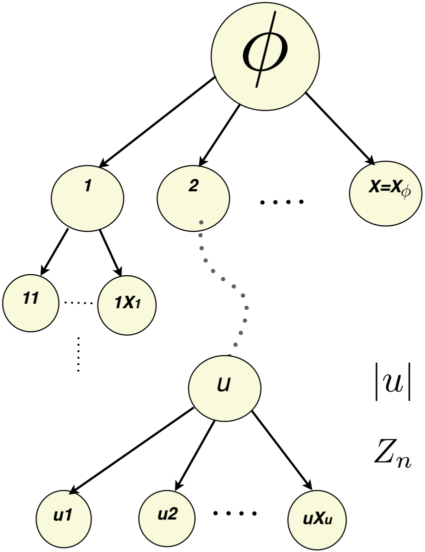
Why Random Environment

In random environment models, the controlling distributions are realizations of a stochastic process, rather then a fixed (deterministic) distribution.

The random environment hypothesis is very natural, because in practice the distributions that we observe are just realizations of a (measure-valued) stochastic process, rather then being constant.

This explains partially why random environment models attract much attention of many mathematicians and physicians.

Branching Process in a Random Environment



$$\xi = (\xi_n)_{(n \geq 0)} \quad i.i.d.$$

$$|u| = n, P_\xi(X_u = k) = p_k(\xi_n)$$

$Z_n = {}^\sharp\{u : |u| = n\}$ — the population size of nth generation

Description of a BPRE (Z_n)

By definition,

$$Z_0 = 1, \quad Z_{n+1} = \sum_{|u|=n} X_u, \quad (n \geq 0).$$

where given ξ , $\{X_u : |u| = n\}$ are conditionally independent of each other and have a common distribution

$$p(\xi_n) = \{p_k(\xi_n) : k \in \mathbb{N}\}$$

on $\mathbb{N} = \{0, 1, ...\}$, Z_n represents the population size of nth generation, and X_u the number of offspring of u. First introduced by:

- Smith (1968), Smith-Wilkinson (1969): *iid environment*, i.e. the offspring distributions $p(\xi_n), n \geq 0$, are iid;
- Athreya-Karlin (1971): *stationary and ergodic environment*, i.e. the offspring distributions $p(\xi_n), n \geq 0$, constitute a stationary and ergodic sequence.

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Galton-Watson Process and its Classification

A Galton-Watson process is a branching process (Z_n) with constant environment:

$$\xi_n = const.$$

This is the case where all the offspring distributions are the same deterministic distribution $\{p_k : k \in \mathbb{N}\}$. Let

$$m = \mathbb{E}Z_1 = \sum kp_k.$$

Classification of Galton-Watson processes:

- Supercritical: $\log m > 0$. Then $Z_n \to \infty$ with positive prob.
- Critical: $\log m = 0$. Then $Z_n \to 0$ a.s.
- Subcritical: $\log m < 0$. Then $Z_n \to 0$ a.s.

Cf. e.g. Harris (1963), Athreya-Ney (1972).

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Classification of BPRE

Let

$$m_0 = \mathbb{E}_\xi Z_1 = \sum k p_k(\xi_0).$$

- Supercritical: $\mathbb{E} \log m_0 > 0$. Then $Z_n \to \infty$ with positive prob.
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- Subcritical: $\mathbb{E} \log m_0 < 0$. Then $Z_n \to 0$ a.s.

Cf. Athreya-Karlin (1971), Tanny (1977)

The martingale in BPRE

Denote

$$m_n = \sum_k k p_k(\xi_n)$$

$$P_0 = 1, \qquad P_n = m_0 \cdots m_{n-1} \text{ for } n \geq 1.$$

Then the normalized population size

$$W_n = \frac{Z_n}{P_n}$$

is a nonnegative martingale, so that for some real r.v. W,

$$W_n \to W \quad \text{a.s.}$$

Non-degeneration of W (Kesten -Stigum type theorem): for iid environment,

$$\mathbb{P}(W = 0) < 1 \Leftrightarrow \mathbb{E}W = 1 \Leftrightarrow \mathbb{E}\frac{Z_1}{m_0}\log^+ Z_1 < \infty$$

Cf. Athreya-Karlin (1971) for "$\Leftarrow$", Tanny (1988) for "$\Rightarrow$".

Supercritical BPRE

We consider the *supercritical* case where

$$\mathbb{E} \log m_0 \in (0, \infty) \quad \text{and} \quad \mathbb{E} \frac{Z_1}{m_0} \log^+ Z_1 < \infty.$$

The first condition implies that the process is supercritical ($Z_n \to \infty$ with positive probability) ; the second implies that W is non-degenerate ($\mathbb{P}(W = 0) < 1$, which implies $\mathbb{E}W = 1$). Moreover, $\mathbb{E}_\xi W = 0$ or 1, and

$$\mathbb{P}_\xi(W > 0) = \mathbb{P}_\xi(Z_n \to \infty) = \lim_{n \to \infty} \mathbb{P}_\xi(Z_n > 0) \quad a.s..$$

For simplicity, we also assume that the environment sequence (ξ_n) is i.i.d., although some results that we will present also hold for a stationary and ergodic environment.

We are interested in the asymptotic properties of W, the limit theorems on $\log Z_n$, and the convergence rate of $W_n - W$.

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Outline

Weighted moments for a Galton-Watson process

For a supercritical branching process (Z_n) , many limit theorems depend on the existence of moments or weighted moments of W.

The existence of moments has been studied by many authors: see e.g.

Harris (1963),
Athreya and Ney (1972),
Bingham and Doney (1974),
Alsmeyer and Rösler (2004).

Weighted Moments for a Galton-Watson process

Of particular interest is the following comparison theorem about weighted moments of W and Z_1, for a Galton-Watson process Z_n with $\mathbb{E}Z_1 \in (1, \infty)$:

1. Bingham and Doney (1974) (via Tauberian theorems): when $p > 1$ is not an integer and ℓ is a positive function slowly varying at ∞ (that is, $\lim_{x\to\infty} \ell(\lambda x)/\ell(x) = 1 \ \forall \lambda > 0$),

$$\mathbb{E}W^p \ell(W) < \infty \Leftrightarrow \mathbb{E}Z_1^p \ell(Z_1) < \infty. \tag{3.1}$$

2. Alsmeyer and Rösler (2004) showed that the equivalence remains true when p is not of the form 2^n for some integer $n \geq 1$, by a nice martingale argument.
3. Liang and Liu (2013) showed that the equivalence is always true whenever $p > 1$.

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Moments for a BPRE

For a branching process in an iid random environment, a necessary and sufficient condition for the existence of the moments of W was first announced by Guivarc'h and Liu (2001): for $p > 1$, writing $m_0 = \sum_k kp_k(\xi_0) = \mathbb{E}_\xi Z_1$, we have

$$\mathbb{E}W^p < \infty \Leftrightarrow \mathbb{E}W_1^p < \infty \text{ and } \mathbb{E}m_0^{-(p-1)} < 1. \tag{3.2}$$

The result suggests that under a moment condition on m_0, W_1 and W have similar tail behavior. This is confirmed by the following comparison theorem between weighted moments of W_1 and W.

Weighted Moments for a BPRE

Theorem 3.1 (Weighted moments, Liang and Liu 2013)

Let $p > 1$ *be such that* $\mathbb{E}m_0^{1-p} < 1$ *and* $\mathbb{E}m_0^{1-(p+\delta)} < \infty$ *for some* $\delta > 0$*. Let* $\ell : [0, \infty) \mapsto [0, \infty)$ *be a function slowly varying at* ∞*. Set* $W^* = \sup_{n \geq 1} W_n$*. Then the following assertions are equivalent:*

(a) $\mathbb{E}W_1^p \ell(W_1) < \infty$;

(b) $\mathbb{E}W^{*p} \ell(W^*) < \infty$;

(c) $0 < \mathbb{E}W^p \ell(W) < \infty$.

The argument in the proof is a refinement of that of Alsmeyer and Rösler (2004), and is based on the Burkholder-Davis-Gundy inequalities for martingales.

The case where $p = 1$ was also considered in Liang and Liu (2013).

The method leads to a new proof for a criterion of non-degeneration of W.

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Quenched Moments for a BPRE

The above results concern the annealed moments. We can also consider the quenched moments $\mathbb{E}_\xi W^p$. Actually, Huang and Liu (2014) have proved the following criterion:

Theorem 3.2 (Quenched moments, Huang and Liu 2014)

Let $p > 1$. Then $\mathbb{E}_\xi W^p < \infty$ a.s. if and only if $\mathbb{E} \log \mathbb{E}_\xi \left(\frac{Z_1}{m_0}\right)^p < \infty$.

The sufficiency of the condition was first proved in Li, Hu and Liu (2011) by a different method.

Tail behavior : a conjecture

The tail behavior of W is not fully known. Inspired by the criterion (3.2) for existence of moments, we can formulate the following conjecture:

Conjecture 3.3

Let $p > 1$ be such that $\mathbb{E}m_0^{-(p-1)} = 1$. Under a finite moment condition on W_1 (e.g. $\mathbb{E}W_1^{(p+\varepsilon)} < \infty$) and a non-lattice condition on $\log m_0$ (i.e. m_0 is not concentrated on a geometric progression), we should have

$$0 < \lim_{x \to \infty} x^p \mathbb{P}(W > x) < \infty.$$

Outline

Limit theorems on $\log Z_n$

For simplicity, in this section we assume always that

$$p_0(\xi_0) = 0 \quad a.s.$$

Therefore $W > 0$ and $Z_n \to \infty$ a.s.. Notice that

$$\log Z_n = \log P_n + \log W_n. \tag{4.1}$$

Since $W_n \to W > 0$ a.s., certain asymptotic properties of $\log Z_n$ would be determined by those of $\log P_n$. We shall show that $\log Z_n$ and $\log P_n$ satisfy the same limit theorems under suitable moment conditions.

Law of large numbers

It is well known (see e.g. Tanny(1977)) that $\log Z_n$ satisfies a law of large numbers:

$$\lim_{n\to\infty} \frac{\log Z_n}{n} = \mathbb{E} \log m_0 \quad a.s. \text{ (on } \{Z_n \to \infty\}).$$

We are interested in the asymptotic properties of the corresponding deviation probabilities.

Central Limit Theorem

It can be easily seen that $\log Z_n$ satisfies the same central limit theorem as $\log P_n = \log m_0 + ... + \log m_{n-1}$:

Lemma 4.1 (Central Limit Theorem, Huang and Liu 2012)

If $\sigma^2 = var(\log m_0) \in (0, \infty)$, *then*

$$\frac{\log Z_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \to N(0,1) \quad \textit{in law.}$$

Large Deviation Principle: the rate function

We find that $\log Z_n$ and $\log P_n$ satisfy the same large deviation principe.

Let

$$\Lambda(t) = \log \mathbb{E} m_0^t,$$

and

$$\Lambda^*(x) = \sup_{t \in \mathbb{R}} \{xt - \Lambda(t)\}$$

be the Fenchel-Legendre transform of Λ.

Large Deviation Principle: Assumption (H)

We will use the following assumption:

Assumption(H)

There exist constants $A > A_1 > 1$ such that

$$A_1 \leq \mathbb{E}_\xi Z_1, \qquad \mathbb{E}_\xi Z_1^2 \leq A^2.$$

Remark. The hypothesis (H) can be relaxed to a more natural moment condition.

Large Deviation Principle

Theorem 4.1 (Large Deviation Principle, Huang and Liu 2012)

Assume (H). If $\mathbb{E}Z_1^s < \infty$ *for all* $s > 1$ *and* $p_1 = 0$ $a.s.$*, then for any measurable subset* B *of* $\mathbb{R}$*,*

$$\begin{aligned} -\inf_{x \in B^o} \Lambda^*(x) &\leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \in B\right) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \in B\right) \\ &\leq -\inf_{x \in \bar{B}} \Lambda^*(x), \end{aligned}$$

where B^o denotes the interior of B, and $\bar{B}$ its closure.

Large Deviation Principle: tail probabilities

From Theorem 4.1, we obtain the following corollary:

Corollary (Bansaye and Berestycki (2009))

Under the conditions of Theorem 4.1, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \leq x\right) = -\Lambda^*(x) \quad \text{for } x \leq \mathbb{E}\log m_0,$$
$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \geq x\right) = -\Lambda^*(x) \quad \text{for } x \geq \mathbb{E}\log m_0.$$

This result was first obtained by Bansaye and Berestycki in 2009. Our approach is different.

Moderate Deviation Principle

- Large deviation principle: $\frac{\log Z_n - n\mathbb{E}\log m_0}{n}$
- Central limit theorem: $\frac{\log Z_n - n\mathbb{E}\log m_0}{\sqrt{n}}$
- Moderate deviation principle: $\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n}$, where $\{a_n\}$ is a sequence of positive numbers satisfying

$$\frac{a_n}{n} \to 0 \quad \text{and} \quad \frac{a_n}{\sqrt{n}} \to \infty, \text{ as } n \to \infty.$$

Moderate Deviation Principle

Theorem 4.2 (Moderate Deviation Principle, Huang and Liu 2012)

Assume (H) and $\sigma^2 = var(\log m_0) \in (0, \infty)$ *. Then for any measurable subset* B *of* $\mathbb{R}$*, we have*

$$\begin{aligned} -\inf_{x \in B^o} \frac{x^2}{2\sigma^2} &\leq \liminf_{n \to \infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \in B \right) \\ &\leq \limsup_{n \to \infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \in B \right) \\ &\leq -\inf_{x \in \bar{B}} \frac{x^2}{2\sigma^2}. \end{aligned}$$

Remark. For the LDP and MDP, the hypothesis (H) can be relaxed to a more natural moment condition: cf. Grama, Liu, Miqueu (2014).

Proof of Theorem LDP (Theorem 4.1)

Notice that the Laplace transform of $\log Z_n$ is $\mathbb{E}Z_n^t = \mathbb{E}e^{t \log Z_n}$. Theorem 4.1 is a consequence of *Gatner-Ellis Theorem* and the following result.

Theorem 4.3 (Moments of Z_n, Huang and Liu 2012)

Under the conditions of Theorem 4.1, we have

$$\lim_{n\to\infty} \frac{\mathbb{E}Z_n^t}{(\mathbb{E}m_0^t)^n} = C(t) \in (0, \infty), \quad \forall t \in \mathbb{R}.$$

Remarks.

1) This is an extension of a result of Ney and Vidyashankar (2003) on the Galton-Watson process.

2) The result suggests more than a LDP; actually we can give a much sharper result like Cramér's large deviation expansion: see Grama, Liu, Miqueu (2014).

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Proof of Theorem 4.1 (LDP)

To prove Theorem 4.3, we introduce a new BPRE and need a theorem about the harmonic moments of W:

Theorem 4.4 (Harmonic moments, Huang and Liu 2012)

Assume (H).
(i) (General case). There always exists a constant $a > 0$ such that

$$\mathbb{E}W^{-a} < \infty.$$

(ii) (Special case). If $p_1 \leq \bar{p}_1$ $a.s.$ for some constant $\bar{p}_1 < 1$, then $\forall a > 0$,

$$\mathbb{E}W^{-a} < \infty \quad \text{if and only if} \quad \mathbb{E}p_1 m_0^a < 1.$$

Harmonic moments

Corollary (Critical value)

Assume (H) and $p_1 \leq \bar{p}_1$ $a.s.$ for some constant $\bar{p}_1 < 1$. If $\mathbb{E} p_1 m_0^{a_0} = 1$, then

$$\mathbb{E} W^{-a} < \infty \quad \text{if } 0 < a < a_0,$$
$$\mathbb{E} W^{-a} = \infty \quad \text{if } a \geq a_0.$$

Remark

According to Hambly(1992), under (H), the number $\alpha_0 := -\frac{\mathbb{E} \log p_1}{\mathbb{E} \log m_0}$ is the critical value for the a.s. existence of the quenched moments $\mathbb{E}_\xi W^{-a} (a > 0)$. By Jensen's inequality, it is easy to see that $a_0 \leq \alpha_0$.

Harmonic moments

Corollary (Critical value)

Assume (H) and $p_1 \le \bar{p}_1$ $a.s.$ for some constant $\bar{p}_1 < 1$. If $\mathbb{E} p_1 m_0^{a_0} = 1$, then

$$\begin{aligned} \mathbb{E} W^{-a} &< \infty \quad \text{if } 0 < a < a_0, \\ \mathbb{E} W^{-a} &= \infty \quad \text{if } a \ge a_0. \end{aligned}$$

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Proof of MDP (Theorem 4.2)

Similar to the case of LDP (Theorem 4.1), Theorem 4.2 is a consequence of *Gatner-Ellis Theorem* and the following result.

Theorem 4.5 (Huang and Liu 2012)

Assume (H). We have

$$\lim_{n\to\infty} \frac{\log \mathbb{E} Z_n^{\frac{a_n}{n}t}}{\log \mathbb{E} P_n^{\frac{a_n}{n}t}} = 1 \qquad \forall t \neq 0.$$

Outline

Convergence rates of $W_n - W$

A.s. (In the spirit of LLN of Marcinkiewicz - Zygmund) Under a moment condition of order $p \in (1,2)$, we can find the best $a > 0$ such that $W - W_n = o(e^{-na})$ a.s.; assuming only $\mathbb{E}W_1 \log W_1^{\alpha+1} < \infty$ for some $\alpha > 0$, we can find the best $\alpha > 0$ such that $W - W_n = o(n^{-\alpha})$ a.s. See Huang & Liu (2014)

In law (In the spirit of CLT) Under a second moment condition, there are norming constants $a_n(\xi)$ (that we calculate explicitly) such that $a_n(\xi)(W - W_n)$ converges in law to a non-degenerate distribution: See Wang, Gao & Liu (2011) and Huang & Liu (2014)

In L^p We can find the least $\rho \in (0,1)$ such that $E|W - W_n|^p = O(\rho^n)$: see Huang & Liu (2014).

Outline

BRW with a random environment in time

Many of the preceding results can be extended to branching random walks with a random environment in time: e.g.

1. weighted moments of W: see Liang & Liu (2014),
2. convergence rate in L^p of $W_n - W$: see Huang & Liu (2014).

For limit theorems on the counting measure

$$Z_n(A) = \#\{ \text{ particles of generation situated in } A\}$$

of a BRW in with a random environment in time:

1. CLT, convergence to stable laws, LDP: see Huang & Liu (2014) ;
2. Exact convergence rate in the CLT, local limit theorem: see Gao & Liu (2014).

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Thank you !

quansheng.liu@univ-ubs.fr