Itô formula for generalized white noise functionals, revisited

Yuh-Jia Lee

Department of Applied Mathematics
National University of Kaohsiung
Kaohsiung, TAIWAN 811

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Motivation

What are the relations of the following integral?

- Itô's question:

$$\int_0^1 B(1)\, dB(t) = ?$$

- Hitsuda-Skorokhod integral:

$$\int_0^1 \partial^* B(1)\, dt$$

- Wick Itô integral:

$$\int_0^1 B(1) \diamond dB(t)$$

Abstract

Without the definition of Itô integral, we are able to derive "Itô's formula", in the proof we show that the Hitsuda-Skorokhod integral arises naturally. In this talk we shall show that the Hitsuda-Skorokhod integral may be defined for any Gaussian and Non-Gaussian Lévy functionals. The main idea was initiated from the following papers:

- L. : Generalized Functions on Infinite Dimensional Spaces and its Application to White Noise Calculus, *J. Funct. Anal.* 82 (1989) 429-464.
- L. : Analytic Version of Test Functionals, Fourier Transform and a Characterization of Measures in White Noise Calculus, *J. Funct. Anal.* 100 (1991) 359-380.

Basic Notations

- $\mathcal{S}$: the Schwartz space
- $\mathcal{S}'$: the space of tempered distribution
- $(\cdot, \cdot)$: the $\mathcal{S}'$-$\mathcal{S}$ pairing
- $\mathcal{S}_0 = L^2(\mathbb{R})$
- A : $\quad Au = -u'' + 1 + u^2, \quad A$ is densely defined in $\mathcal{S}_0$
- $\{e_j : j = 0, 1, 2. \dots\}$: CONS of $\mathcal{S}_0$, consisting of eigenfunctions of A with corresponding eigenvalues $\{2j + 2 : j = 0, 1, 2, \dots\}$
  $\mathcal{S}_p = \{f \in \mathcal{S}' : \|f\|_p < \infty\}$ where

$$\|f\|_p^2 = \sum_{j=0}^{\infty} (2j + 2)^p (f, e_j)^2.$$

Basic Notations, cont.

- $\mathcal{S} = \cap_{p\geq 0}\mathcal{S}_p; \quad \mathcal{S}' = \cup_{p\geq 0}\mathcal{S}_{-p}$
- $\mathcal{S} \subset H \subset \mathcal{S}'$ forms a Gel'fand triple.
- μ: a standard Gaussian measure defined on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$ with the characteristic functional $\mathcal{C}$ on $\mathcal{S}$ given by

$$\mathcal{C}(\eta) = \int_{\mathcal{S}'} e^{(x,\eta)}\mu(dx) = e^{-\frac{1}{2}\|\eta\|_0^2}$$

where $\|\eta\|_0 = \left\{\int_{-\infty}^{+\infty} \eta(t)^2\,dt\right\}^{1/2} \quad (\eta \in \mathcal{S})$.

- $(L^2) := L^2(\mathcal{S}', \mu)$

Wiener-Itô decomposition theorem

For $f \in (L^2)$, f enjoys the following orthogonal decomposition

$$f(x) = \sum_{n=0}^{\infty} \oplus \left\{ \frac{1}{n!} \int_{\mathcal{S}'} D^n \mu f(0)(x+iy)^n \mu(dy) \right\}$$

where $\mu f = \mu * f$ and we have

$$\|f\|^2_{L^2(\mathcal{S},\mu)} = \sum_{\mathbb{N}=0}^{\infty} \frac{1}{n!} \|D^n \mu f(0)\|^2_{HS^n(\mathcal{S}_0)}.$$

where $\|T\|_{HS^n(V)}$ denotes the Hilbert-Schmidt norm of the n-linear operator T on the Hilbert space V.

Wiener-Itô decomposition theorem, cont.

Let f_n denote the kernel of $D^n\mu f(0)/n!$ and $B(t)$ denote the Brownian motion, then

$$\begin{aligned} I_n(f_n) &:= \int \dots \int_{\mathbb{R}^n} f_n(t_1, \dots, t_n) dB(t_1, x) \dots dB(t_n, x) \\ &= \frac{1}{n!} \int_{\mathcal{S}'} D^n \mu f(0)(x + iy)^n \mu(dy) \text{ a.e.}(\mu) \end{aligned}$$

In notation, we write $f \sim (f_n)$.

The Segal-Bargmann transform of Gaussian WNF

For $\xi \in \mathcal{S}$ and for $f \in (L^2) = L^2(\mathcal{S}', \mu)$ with $f \sim (f_n)$, define the transform S on (L^2) by

$$S(f)(\xi) = \sum_{n=0}^{\infty} \int \dots \int_{\mathbb{R}^n} f_n(t_1, \dots, t_n)\xi(t_1)\dots\xi(t_n)dt_1\dots dt_n$$

or,

$$S(f)(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n \mu f(0)\xi^n.$$

Clearly,

$$S(f)(\xi) = \mu * f(\xi) = \int_{\mathcal{S}'} f(x+\xi)\mu(dx) = e^{-\frac{1}{2}\|\xi\|_0^2} \int_{\mathcal{S}'} f(x)e^{(x,\xi)}\mu(dx).$$

The test functionals

Let $(\mathcal{S})_p$ denote the collection of functions f such that

$$\|f\|_{2,p} = \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \|D^n \mu f(0)\|^2_{HS^n(\mathcal{S}_{-p})} \right\}^{1/2} < \infty.$$

$(\mathcal{S})_{-p}$ is identified as the dual of $(\mathcal{S})_p$. For $p > q$

$$(\mathcal{S})_p \subset (\mathcal{S})_q \subset (L^2) \subset (\mathcal{S})_{-q} \subset (\mathcal{S})_p.$$

Set $(\mathcal{S}) = \cap_{p \geq 0} (\mathcal{S})_p$ (with projective limit topology). The we have

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$$

forms a Gel'fand triple. Members of $(\mathcal{S})$ are called test (Gaussian) white noise functionals and members of its dual space $(\mathcal{S})^*$ are called generalized white noise functionals.

Analyticity of test functionals

Theorem

$[L, 1991]$ *For any $f \in (\mathcal{S})$, there exist an analytic function $\widetilde{f}$ defined on the complexification $\mathcal{CS}'$ such that $f = \widetilde{f}$ a.e. (μ). Moreover, for each $p \geq 0$, there exist a constant C_f, depending only on f, such that*

$$|\widetilde{f}(z)| \leq C_f e^{\frac{1}{2}\|z\|_{-p}^2}.$$

In what follow we identify φ with $\widetilde{\varphi}$ for any $\varphi \in (\mathcal{S})$.

Ananlytic version of $(\mathcal{S})$

For $p \in \mathbb{R}^1$, denote by $\mathcal{A}_p$ the class of entire functions f defined on $\mathcal{S}_{-p}$ which has an entire extension $\widetilde{f}$ to $\mathcal{C}\mathcal{S}_{-p}$ such that

$$\|f\|_{\mathcal{A}_p} := \sup_{z \in \mathcal{C}\mathcal{S}_{-p}} \left\{ |\widetilde{f}(z)| e^{-\frac{1}{2}\|z\|_{-p}^2} \right\} < \infty.$$

In the sequel we shall identify f with $\widetilde{f}$ for $f \in \mathcal{A}_p$.

The space $\mathcal{A}_\infty$

Let $\mathcal{A}_\infty = \cap_{p>0}\mathcal{A}_p$. Endow $\mathcal{A}_\infty$ with the projective topology. Then $\mathcal{A}_\infty$ becomes a topological space.

Basic properties of the test functionals

- If $f \in \mathcal{A}_\infty$, then, for $h_1, \ldots, h_n \in \mathcal{S}$ and for $p \in \mathbb{N}$,

$$|D^n f(z) h_1 \cdots h_n| \leq \|f\|_{\mathcal{A}_p} \exp\Big[\|z\|_{-p}^2 \Big(\sum_{j=1}^{\infty} \|h_j\|_{-p}\Big)^2\Big].$$

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- $\sum_{n=0}^{\infty} \frac{1}{n!} D^n f(0) z^n$ converges to f in $\mathcal{A}_\infty$ for any $f \in \mathcal{A}_\infty$.

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- $\mathcal{A}_\infty$ is an algebra.
- The Wiener–Ito decomposition of $f \in \mathcal{A}_\infty$ converges to f in $\mathcal{A}_\infty$.
- For $f \in \mathcal{A}_\infty$, define $\mathcal{F}_{\alpha,\beta} f(y) = \int_{\mathcal{S}^*} f(\alpha x + \beta y) \mu(dx)$ for $\alpha, \beta \in \mathbb{C}$. Then $\mathcal{F}_{\alpha,\beta}(\mathcal{A}_\infty) \subset \mathcal{A}_\infty$ and $\mathcal{F}_{\alpha,\beta}$ is continuous on $\mathcal{A}_\infty$.

Topological Equivalence of $(\mathcal{S})$ and $\mathcal{A}_\infty$[12]

Let $f \in (\mathcal{S})$ and $\widetilde{f}$ be its analytic version of f. Let $p > \frac{1}{2}$ and $r > \frac{1}{2}$. There exist some constants α_p and β_p such that

$$\alpha_p \|\widetilde{\varphi}\|_{\mathcal{A}_{p-1}} \leq \|\varphi\|_{2,p} \leq \beta_p \|\widetilde{\varphi}\|_{\mathcal{A}_{p+r}}. \tag{3.1}$$

S-transform for GWF

Given $F \in (\mathcal{S})^*$, recall that the S–transform of F is defined as follows:

$$SF(\xi) = \begin{cases} \mu * F(\xi), & \text{if } F \in L^2[\mathcal{S}', \mu]; \\ e^{-\frac{1}{2}|\xi|^2} \langle\langle F, e^{(\cdot,\xi)} \rangle\rangle, & \text{if } F \in (\mathcal{S})^*, \end{cases}$$

where $\xi \in \mathcal{S}$.

SF is also denoted by U_F, U_F is called the U–functional F.

Locality

In [12], it has been shown that, for any real number p,

$$\begin{aligned}\|F\|_{2,p}^2 &= \sum_{n=0}^{\infty} \frac{1}{n!} \|D^n U_F(0)\|^2_{\mathcal{HS}^n[\mathcal{S}_{-p}]} \\ &= \lim_{n\to\infty} \int_{\mathcal{S}'} \Big| \int_{\mathcal{S}'} U_F(A^p P_n x + iA^p P_n y)\mu(dy)\Big|^2 \mu(dx),\end{aligned}$$

where P_n's are orthogonal projections of H which tend to the identity I_H.

Proposition

Let $p \in \mathbb{R}^1$ and $r > \frac{1}{2}$. Then we have

$$\|F\|_{2,p} \le C_r \|U_F\|_{\mathcal{A}_{p+r}}.$$

Browanian motion as a functional in $L^2(\mathcal{S}')$

Let (H, B) be an abstract Wiener space with abstract Wiener measure $\mu = p_1$. Let B^* be the dual space which is regarded as the subspace of H.

- Let $\xi \in B^*$. Define

$$\widetilde{\xi}(x) = (x, \xi).$$

  Then $\widetilde{\xi} \in \mathcal{A}_\infty$ and $\widetilde{\xi}$ is normal distributed with mean zero and variance $\|\xi\|_H^2$.

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$$\widetilde{\xi}(x) = (x, \xi).$$
Then $\widetilde{\xi} \in \mathcal{A}_\infty$ and $\widetilde{\xi}$ is normal distributed with mean zero and variance $\|\xi\|_H^2$.
- For any $h \in H$, there exists a sequence $(\xi_n) \subset B^*$ such that $|\xi_n - h|_H \to 0$. It follows that $\int_B |\widetilde{\xi}_n - \widetilde{\xi}_m|^2 \mu(dx) = \|\xi_n - \xi_m\|_H^2 \to 0$ as $n, m \to \infty$. Thus $\{\widetilde{\xi}_n\}$ forms a Cauchy sequence in $L^2(B)$ so that the $L^2(B)$-limit of $\{\widetilde{\xi^n}\}$ exists. Define
$$\widetilde{h} = L^2(B) - \lim_{n\to\infty} \widetilde{\xi}.$$
Then $\widetilde{h} \sim N(0, \|h\|_H^2)$. In notation, we also write
$$\widetilde{h}(x) = \langle x, h \rangle.$$

The Brownian motion as a functional in $\mathcal{S}'$, cont.

When $H = L^2(\mathbb{R})$, we consider $(L^2(\mathbb{R}), \mathcal{S}')$ as the union of the abstract Wiener spaces $(L^2(\mathbb{R}), \mathcal{S}_p)$. Then $\widetilde{h}$ is well-defined as a normal distributed random variable with mean 0 and variance $|h|_0^2$.

- The Brownian motion on the probability space $(\mathcal{S}', \mathcal{B}(\mathcal{S}), \mu)$ may be represented by $B(t)$ defined by

$$B(t,x) = \begin{cases} \langle x, 1_{(0,t]} \rangle, & t \geq 0 \\ -\langle x, 1_{(t,0]} \rangle, & t < 0, \end{cases}$$

  for almost all $x \in \mathcal{S}'$.

The Brownian motion as a functional in $\mathcal{S}'$, cont.

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  for almost all $x \in \mathcal{S}'$.

- Let

$$h_t = \begin{cases} 1_{(0,t]}, & t \geq 0 \\ -1_{(t,0]}, & t < 0, \end{cases}$$

  then

$$B(t,x) = \langle x, h_t \rangle.$$

White noise as a GWF

For any test functional φ, we have

$$\begin{aligned}\langle\langle \dot{B}(t), \varphi \rangle\rangle &= \frac{d}{dt}\langle\langle B(t), \varphi \rangle\rangle \\ &= \lim_{\epsilon \to 0} \int_{\mathcal{S}'} \frac{1}{\epsilon}\langle x, h_{t+\epsilon} - h_t \rangle \varphi(x)\mu(dx) \\ &= \lim_{\epsilon \to 0} D\mu\varphi(0)\left\{\frac{1}{\epsilon}(h_{t+\epsilon} - h_t)\right\} \\ &= D\mu\varphi(0)\delta_t.\end{aligned}$$

It is easy to see that the mapping $\varphi \to D\mu\varphi(0)\delta_t$ is continuous on $\mathcal{A}_\infty$. This leads to the definition of white nose given as follows

$$\langle\langle \dot{B}(t), \varphi \rangle\rangle = D\mu\varphi(0)\delta_t.$$

Composition of generalized function with random vectors

For any $f \in \mathcal{S}'(\mathbb{R}^n)$, and $h_i \in L^2$, $i = 1, 2, 3 \ldots,$, we may define $f(\widetilde{h}_1, \widetilde{h}_2, \ldots, \widetilde{h}_n)$ formally by

$$f(\widetilde{h}_1, \widetilde{h}_2, \ldots, \widetilde{h}_n) = \frac{1}{2\pi^n} \int_{\mathbb{R}^n} \hat{f}(u_1, \ldots, u_n) e^{i \sum_{j=1}^n u_i \widetilde{h}_i} \, du_1 \ldots du_n.$$

Then for $\varphi \in \mathcal{A}_\infty$, we have

$$\langle\!\langle f(\widetilde{h}_1, \widetilde{h}_2, \ldots, \widetilde{h}_n), \varphi \rangle\!\rangle = (f, \hat{G}_{\mathbf{h},\varphi}),$$

$$G_{\mathbf{h},\varphi}(\mathbf{u}) = (1/\sqrt{2\pi})^n \mathcal{F}_{1,i}\varphi([\mathbf{u}, \mathbf{h}]) \exp(-\frac{1}{2} \|[\mathbf{u}, \mathbf{h}]\|_0^2),$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ and $[\mathbf{u}, \mathbf{h}] = \sum_{j=1}^n u_i h_i$.

Donsker delta function for Brownian motion

The Donsker delta function $\delta_x(B(t))(t > 0)$ may be defined by

$$\begin{aligned}\langle\langle \delta_x(\widetilde{h}_t), \varphi \rangle\rangle &:= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ias - \frac{1}{2}s^2 t} \mathcal{F}_{1,i}\varphi(sh_t) ds \\ &:= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ias - \frac{1}{2}s^2 t} \left\{ \int_{\mathcal{S}'} \varphi(y + i\, sh_t) \mu(dy) \right\} ds.\end{aligned}$$

Itô's formula

For $f \in \mathcal{S}'$, define $f(B(t)) = f(\widetilde{h_t})$ for $t > 0$ by

$$\langle\langle f(B(t)), \varphi \rangle\rangle := (f, \hat{G}_{t,\varphi})$$

where $\hat{G}_{t,\varphi}(u) = (1/\sqrt{2\pi})\mathcal{F}_{1,i}\varphi(u1_{(0,t]})\exp(-\frac{1}{2}u^2 t)$.
If we differentiate $f(B(t))$ with respect to t we immediately obtain:

$$\begin{aligned}
&\frac{d}{dt}\langle\langle f(B(t)), \varphi \rangle\rangle \\
&= (f_{[u]}, iu\left\{1/\sqrt{2\pi}\mathcal{F}_{1,i}\partial_t\varphi(u1_{(0,t]})e^{-\frac{1}{2}u^2 t}\right\} \\
&\qquad + (f_{[u]}, -\frac{1}{2}u^2\left\{(1/\sqrt{2\pi})\mathcal{F}_{1,i}\partial_t\varphi(u1_{(0,t]})e^{-\frac{1}{2}u^2 t})\right\} \\
&= \langle\langle f(B(t)), \partial_t\varphi \rangle\rangle + \langle\langle \frac{1}{2}f''(B(t)), \varphi \rangle\rangle.
\end{aligned}$$

The generalized Itô's formula follows:

$$\frac{d}{dt}f(B(t)) = \partial_t^* f'(B(t)) + \frac{1}{2}f''(B(t)).$$

It can be shown that

$$\int_a^b \partial_t^* f'(B(t))dt = \int_a^b f'(B(t))dB(t).$$

If one replace the Brownian motion by any other normal processes

$$X_t(x) = \langle x, \beta_t \rangle,$$

one may derive a new Itô formula by differentiating $f(X_t)$ with respect to t.

Hitsuda Formula (cf. Kuo [10])

Define

$$\langle\langle f(B(t), B(1), \varphi\rangle\rangle) = (f, \hat{H}_{t,\varphi}), \quad \text{with}$$

$$\hat{H}_{t,\varphi} = \frac{1}{\sqrt{2\pi}^2} \mathcal{F}_{1,i}\varphi(u\,h_t + v\,h_1) e^{-\frac{1}{2}u^2\|u\,h_t + v\,h_1\|_0^2}.$$

Differentiating with respect to t and then integrating from $a > 0$ to $b > a (1 > b)$ we obtain

$$f(B(b), B(1)) - f(B(a), B(1)) = \int_a^b \partial_t^* f_x(B(t), B(1))\, dt + \int_a^b f_{xy}(B(t), B(1)) dt + \frac{1}{2}\int_a^b f_{xx}(B(t), B(1))\, dt.$$

An application of Hitsuda formula

Apply Hitsuda formula with $f(xy) = xy$, we immediately have

$$B(b)B(1) - B(a)B(1) = \int_a^b \partial_t^* B(1)\,dt + (b-a),$$

or,

$$\int_a^b \partial_t^* B(1)\,dt = (B(b) - B(a))B(1) - (b-a).$$

Itô formula for non-adapted Processes

For the more general case $f(X_t)$ with $X_t(x) = \langle x, h_t \rangle$ with $\{X_t\}$ being a normal processes (which is non-adapted generally), one may also apply the same argument above to derive the following "Itô" formula:

$$f(X(b)) = f(X(a)) + \int_a^b D_{\dot{h}_t}^* f'(B(t))\, dt + \int_a^b \{\frac{d}{dt}\|h_t\|_0\} f''(X(t))\, dt,$$

where $\dot{h}_t = \frac{d}{dt} h_t$.
Again a new integral such as $\int_a^b D_{\dot{h}_t}^* f(t)\, dt$ arises.

Itô formula for Brownian Bridge

The Brownian Bridge $X(t)$ may represented by

$$X(t) = B(t) - tB(1) = \widetilde{\beta}_t = \widetilde{h}_t - t\widetilde{h}_1, \ \ (\beta_t = h_t - th_1).$$

Clearly $\|\beta_t\|_0^2 = t - t^2$. Let $k_t = \frac{d}{dt}\beta_t$. Then, for $f \in \mathcal{S}'$, we have

$$f(X(b)) - f(X(a)) = \int_a^b D_{k_t}^* f'(X(t))\, dt + \int_a^b \frac{1}{2}(1-2t) f''(X(t))\, dt$$

which exist in the generalized sense, where $0 < a < b < 1$.

Let $\{Y_t : a \le t \le b\}$, $0 < a < b < 1$ be a continuous $(\mathcal{S})^*$-valued process, we define

$$\int_a^b Y_t \, dX(t+) := \lim_{|\Gamma|\to 0} \sum_{j=1}^{n} (\widetilde{\beta}_{t_j} - \widetilde{\beta}_{t_{j-1}}) Y_{t_{j-1}}$$

provided that the limit exist in $(\mathcal{S})^*$, where
$\Gamma = \{a = t_0 < t_1 < \cdots < t_n = b\}$ Then one can show that

$$\int_a^b D_{k_t}^* f'(X(t)) \, dt = \int_a^b f'(X(t)) dX(t+) + \int_a^b t \, f''(X(t)) \, dt.$$

The above identity also give the probabilistic meaning of the stochastic integral

$$\int_a^b D_{k_t}^* f'(X(t)) \, dt$$

Kuo's stochastic integral

In [W. Ayed and H.H. Kuo: An extension of the Itô formula, v.2, COSA(2008),323-333], the authors define the following stochastic integral: Let $f(t), \quad a \leq t \leq b$, be adapted and $\varphi(t), a \leq t \leq b$ be instantly independent (i.e. $\varphi(t)$ is independent of $\sigma\{B(s), s \leq t\}$). Define the stochastic integral of $f(t)\varphi(t)$ by

$$I(f\varphi) = \int_a^b f(t)\varphi(t)\, dB(t) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(t_{i-1})\varphi(t_i)(B(t_i) - B(t_{i-1}))$$

provided the limit exist in probability.
The main results showed that

$$\int_a^b f(t)\varphi(t)\, dB(t) = \int_a^b \partial^*[f(t)\varphi(t)]\, dB(t).$$

Composition of tempered distribution with Lévy process

Assumption: $\sigma^2 = \beta(0) - \beta(0-) > 0$.
Let Λ be the Lévy probability measure. Applying the inversion formula of Fourier transform, we have for $f \in \mathcal{S}$,

$$f(X(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\mathcal{F}f)(r)\, e^{irX(t)}\, dr,$$

where $\mathcal{F}f$ is the Fourier transform of f. Then, for any test functionals φ, we have

$$\langle\langle\, f(X(t)),\, \varphi \,\rangle\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\mathcal{F}f)(r) \left\{ \int_{\mathcal{S}'} \varphi(x)\, e^{irX(t;\,x)}\, \Lambda(dx) \right\} dr. \tag{5.1}$$

Let $G_{t,\varphi}$ be the map from $\mathbb{R}$ to $\mathbb{C}$ defined by

$$G_{t,\varphi}(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{S}'} \varphi(x)\, e^{i\, rX(t\,;\,x)}\, \Lambda(dx).$$

Recall the relation between $\mathcal{T}$-transform and S-transform

$$\mathcal{T}\varphi(\eta) = \mathbb{E}[e^{i\langle\cdot,\,\eta\rangle}]\, S\varphi(\phi_{i\eta}). \tag{5.2}$$

for $\eta \in L^1 \cap L^2(\mathbb{R}, dt)$, where $\phi_\xi(t,u) = (e^{\xi^*(t,u)} - 1)/u$ if $u \neq 0$; otherwise, $\phi_\xi(t,u) = \xi(t,u)$.
It follows from the identity (5.2) that we obtain

$$G_{t,\varphi}(r) = \frac{\mathbb{E}[e^{irX(t)}]}{\sqrt{2\pi}} \times S\varphi(\phi(t,r)),$$

where $\phi : (0,+\infty) \times \mathbb{R} \to L^2_c(\mathbb{R}^2, \lambda)$ is given by
$\phi(t,r)(s,u) = (e^{iru1_{[0,t]}(s)} - 1)/u$, if $u \neq 0$; otherwise,
$\phi(t,r)(s,u) = i\, r\, 1_{[0,t]}(s)$.

Lemma[16]

For a fixed $\varphi \in \mathcal{L}$, $G_{t,\varphi}$ is a function in $\mathcal{S}_c$. In fact, for any $q \geq 0$ and $0 < a < b < +\infty$, there exists a positive real number p, depending only on q, a, b, such that

$$|G_{t,\varphi}|_q \leq \|\varphi\|_p$$

uniformly in t on the compact interval $[a, b]$.

The above lemma implies that the mapping $\varphi \in \mathcal{L} \mapsto G_{t,\varphi}$ is a continuous $\mathcal{S}_c$-valued map. It follows that the composition $F(X(t))$ is well-defined for $F \in \mathcal{S}'$ in the following

Definition

For $F \in \mathcal{S}'$ and $t > 0$, we define $\langle\langle F(X(t)), \varphi \rangle\rangle = (\mathcal{F}F, G_{t,\varphi})$ for $\varphi \in \mathcal{L}$, where $\mathcal{F}F$ is the Fourier transform of F and $(\cdot, \cdot)$ is the $\mathcal{S}_c'$-$\mathcal{S}_c$ pairing. In particular, when $F = \delta_a$, the Dirac delta function concentrated on the point a, $\delta_a(X(t))$ $(= \delta(X(t) - a))$ is called the Donsker's delta function of the Lévy process X.

Note that

$$\delta(X(t)-a) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ir(X(t)-a)}\,dr \quad \text{in } \mathcal{L}_{-p,c},$$

for any sufficiently large $p > 0$ so that $\sigma^2 - (\tau_2/\lambda_0^{2p}) > 0$, where the integral exists in the sense of Bochner (see [16]). Moreover,

$$\langle\!\langle\, F(X(t)-a),\, \varphi \,\rangle\!\rangle = \left(F_{[s]},\, \langle\!\langle \delta(X(t)-a-s),\, \varphi \rangle\!\rangle\right),$$

where $(\cdot,\cdot)$ is the $\mathcal{S}'$-$\mathcal{S}$ pairing, and $F_{[s]}$ means that F acts on the test functions in the variable s.

Itô formula for $F(X(t))$[17]

We are ready to show the Itô formula for the $\mathcal{L}'$-valued process $F(X(t))$ with $F \in \mathcal{S}'$, $t > 0$.
By differentiating $F(X(t)$ with respect to t, we obtain Let $F \in \mathcal{S}'$. Then, for $b > a > 0$,

$$
\begin{aligned}
F(X(b)) - F(X(a)) = \; & \tau_1 \int_a^b F'(X(t))\, dt \\
& + \int_a^b \int_{-\infty}^{+\infty} \frac{\kappa_u F(X(t)) - F(X(t)) - u\, F'(X(t))}{u^2}\, d\lambda(t,u) \\
& + \int_a^b \int_{-\infty}^{+\infty} \partial^*_{(t,u)} \frac{\kappa_u F(X(t)) - F(X(t))}{u}\, d\lambda(t,u),
\end{aligned}
$$

in $\mathcal{L}'$, where F' is the first distribution derivative of F, $\kappa_u F = F(\cdot + u)$ is the translate of F; and the integrals exist in the sense of Bochner.

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• Let $(\mathcal{S}_c', \mathcal{B}(\mathcal{S}_c'), \nu(dz))$ be the underlying probability space, where $\mathcal{S}$ is the Schwartz space with dual space $\mathcal{S}'$, $\mathcal{S}_c'$ is the complexification of $\mathcal{S}_c'$ which is identified as the product space $\mathcal{S}' \times \mathcal{S}'$, $\mathcal{B}(\mathcal{S}_c')$ the Borel field of $\mathcal{S}' \times \mathcal{S}'$ and $\nu(dz)$ denotes the product measure $\mu_{1/2}(dx)\mu_{1/2}(dy)$,

where μ_t denotes the Gaussian measure defined on $\mathcal{S}'$ with characteristic function given by

$$C(\xi) = \int_{\mathcal{S}'} e^{i(x,\xi)} \mu_t(dx) = e^{-t|\xi|^2/2}.$$

- The complex Brownian motion on $(\mathcal{S}'_c, \mathcal{B}(\mathcal{S}'_c), \nu(dz))$ may be represented by where μ_t denotes the Gaussian measure defined on $\mathcal{S}'$ with characteristic function given by

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- The complex Brownian motion on $(\mathcal{S}'_c, \mathcal{B}(\mathcal{S}'_c), \nu(dz))$ may be represented by

$$Z_t(x,y) = \langle x, h_t \rangle + i\langle y, h_t \rangle,$$

where

$$h_t = \begin{cases} 1_{(0,t]} & , t > 0, \\ -1_{[t,0]} & , t < 0. \end{cases}$$

The calculus of complex Brownian functional is then performed with respect to the measure $\mu(dz)$.

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$$E[|f(Z(t))|^2] = \int_{\mathcal{S}'} \int_{\mathcal{S}'} |f(\langle x + iy, h_t \rangle)|^2 \, \mu_{1/2}(dx)\mu_{1/2}(dy)$$

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The above identity gives a connection between the function of complex Brownian motion and the Segal-Bargmann entire functionals.

Itô formula for entire Brownian Functionals

We shall show that, for any Segal-Bargmann entire function F, the Itô formula is given by

$$F(Z(b)) - F(Z(a)) = \int_a^b F'(Z(t))dZ(t).$$

Definition of Segal-Bargmann space L[12]

A single-valued function f defined on H_c is called a Segal-Bargmann entire function if it satisfies the following conditions:

(i) f is analytic in H_c.

(ii) The number

$$M_f := \sup_P \int_H \int_H |f(Px + iPy)|^2 n_t(dx) n_t(dy)$$

is finite, where n_t denoted as the Gaussian cylinder measure on H with variance parameter $t > 0$ and P's run through all orthogonal projections on H.

Denote the class of Segal-Bargmann entire function on H by $\mathcal{SB}_t[H]$ and define $\|f\|_{\mathcal{SB}_t[H]} = \sqrt{M_f}$. Then $(\mathcal{SB}_t[H], \|\cdot\|_{\mathcal{SB}_t[H]})$ is a Hilbert space.

It follows immediately from L[12] that we have

$$\begin{aligned} \|f\|^2_{\mathcal{SB}_t[H]} &= \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \left(\sum_{i_1,\ldots,i_k=1}^{N} \left| D^k f(0) e_{i_1} \cdots e_{i_k} \right|^2 \right) \\ &= \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \|D^n f(0)\|^2_{HS^2[H]}, \end{aligned} \tag{6.1}$$

When $t = 1/2$, we simply denote $\mathcal{SB}_t[H]$ by $\mathcal{SB}[H]$.

where $\|S\|_{HS^n[H]}$ denotes the Hilbert-Schmidt norm of a n-linear operator $S \in L^n(H)$ defined by

$$\|S\|_{HS^n(H)} := \left(\sum_{i_1,\ldots,i_k=1}^{\infty} |Se_{i_1} \cdots e_{i_k}|^2 \right)^{1/2}$$

which is independent of the choice of CONS $\{e_i\}$ of H.

Definition of Infinite-dimensional Segal-Bargman entire functionals

Definition

For each $p \in \mathbb{R}$, define

$$\|\phi\|_p = \left(\sum_{n=0}^{\infty} \frac{\|D^n \phi(0)\|^2_{HS^n[\mathcal{S}_{-p}]}}{n!} \right)^{1/2}$$

and set

$$\mathcal{SB}_p = \{\phi \in \mathcal{SB}[\mathcal{S}_{-p}] : \|\phi\|_p < \infty\}$$

Let $\mathcal{SB}_\infty$ be the projective limit of $\mathcal{SB}_p$ for $p \geq 0$ and let $\mathcal{SB}'_\infty$ be the dual space of $\mathcal{SB}_\infty$. We note that

$$\mathcal{SB}_\infty = \mathcal{A}_\infty.$$

$\mathcal{SB}_\infty$ is a nuclear space and we have the following continuous inclusions:

$$\mathcal{SB}_\infty \subset \mathcal{SB}_p \subset \mathcal{SB}[L^2] = \mathcal{SB} \subset \mathcal{SB}'_p \subset \mathcal{SB}'_\infty.$$

The space $\mathcal{SB}_\infty$ will serve as test functionals and $\mathcal{SB}'_\infty$ is referred as the generalized complex Brownian functionals.

The space $\mathcal{SB}'_p$ may be identified as the space of entire functions defined on $\mathcal{S}_{p,c}$ such that : $|\!|\!|\phi|\!|\!|_{-p} < \infty$ and the pairing of $\mathcal{SB}'_\infty$ and $\mathcal{SB}_\infty$ is defined by

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle\!\langle D^n \overline{\Phi}(0), D^n \varphi(0) \rangle\!\rangle_{HS^n},$$

where

$$\langle\!\langle D^n \overline{\Phi}(0), D^n \varphi(0) \rangle\!\rangle_{HS^n} := \sum_{i_1,\ldots,i_n=1}^{n} \left[\overline{D^n \Phi(0) e_{i_1} \cdots e_{i_n}} D^n \varphi(0) e_{i_1} \cdots e_{i_n} \right].$$

One-dimensional Segal-Bargman entire functions

If $\phi(z)$ can be represented by a formal power series $\sum_{n=0}^{\infty} a_n z^n$, we define

$$|\phi|_p = \left(\sum_{n=0}^{\infty} (2n+2)^{2p} n! |a_n|^2 \right)^{1/2}$$

and let

$$\mathcal{SB}_p(\mathbb{R}) = \{\phi : |\phi|_p < \infty\}$$

One-dimensional Segal-Bargman entire functions, cont.

If $\phi(z)$ is a formal power series represented by $\sum_{n=0}^{\infty} b_n z^n$, we define

$$|\phi|_{-p} = \left(\sum_{n=0}^{\infty} n! |b_n|^2 (2n+2)^{-2p} \right)^{1/2} .$$

Then the dual space $\mathcal{SB}_p'$ of $\mathcal{SB}_p$ is characterized by

$$\mathcal{SB}_{-p}(\mathbb{R}) = \{\phi : |\phi|_{-p} < \infty\}$$

The space $\mathcal{SB}_\infty[\mathbb{R}]$ is defined as the projective limit of $\mathcal{SB}_p[\mathbb{R}]$ with dual space $\mathcal{SB}_\infty'[\mathbb{R}] = \bigcup_{p>0} \mathcal{SB}_p'[\mathbb{R}]$.

Composition of generalized function with complex Brownian motion

Let $\psi \in \mathcal{SB}_\infty$. Then, for any one dimensional generalized Segal-Bargman entire function $f \in \mathcal{SB}'_\infty(\mathbb{R})$, represented by $\psi(z) = \sum_{n=0}^\infty a_n z^n$, we have

$$\langle\langle f(Z(t)), \psi\rangle\rangle_c = \sum_{n=0}^{\infty} b_n D^n \psi(0) h_t^n. \tag{6.2}$$

(6.2) gives the definition of $f(Z(t))$.

It can be proved that the composition $f(Z(t))$ defined in Example 6 is in fact a generalized Segal-Bargmann functional. This follows straight forward from the identity (6.2). We state it as a theorem without proof.

Theorem

Let $\psi \in \mathcal{SB}_\infty$. If $f \in \mathcal{SB}'_\infty(\mathbb{R})$, then $f(Z(t))$, defined by (6.2), is a member of $\mathcal{SB}'_\infty$. More precisely, for each $t > 0$, $\exists p \ni |h_t|_{-p} \leq 1$, and

$$|\langle\!\langle f(Z(t)), \psi \rangle\!\rangle_c| \leq |f|_{-p} \|\psi\|_p$$

Itô formula for Generalized Complex Brownian functionals

Let $f \in \mathcal{SB}'_\infty(\mathbb{R})$. Then we have

$$\begin{aligned} & \frac{d}{dt}\langle\!\langle f(Z(t)), \phi \rangle\!\rangle_c \\ = & \sum_{n=0}^{\infty} b_n D^n \phi(0) h_t^{n-1} \delta_t = \sum_{n=0}^{\infty} b_n n D^{n-1}(D\phi(0)\delta_t) h_t^{n-1} \\ = & \sum_{n=0}^{\infty} b_n n D^{n-1}(\partial_t \phi)(0) h_t^{n-1} = \langle\!\langle \partial_t^* f'(Z(t)), \phi \rangle\!\rangle_c \end{aligned}$$

where $\partial_t = \partial_{\delta_t}$ and ∂_t^* is the adjoint operator of ∂_t. It follows that

$$\frac{d}{dt} f(Z(t)) = \partial_t^* f'(Z(t)).$$

Itô formula

This proves the Itô formula for complex Brownian motion. As a summary, we state the above result as a theorem.

Theorem

Let $f \in \mathcal{SB}'_\infty(\mathbb{R})$. Then we have

$$\frac{d}{dt} f(Z(t)) = \partial_t^* f'(Z(t)).$$

or in the integral form,

$$f(Z(b)) - f(Z(a)) = \int_a^b \partial_t^* f'(Z(t)) dt.$$

As in the case of real Brownian motion, the term on the right hand side of Itô formula may be interpreted as stochastic integral as shown below.

Definition

Suppose that $f \in \mathcal{SB}'_{\infty}$. Define the stochastic integral $f(Z(t))$ as follows:

$$\langle\langle \int_a^b f(Z(t))dZ(t), \phi \rangle\rangle_c$$

$$:= \lim_{\|\triangle_n\| \to 0} \langle\langle \sum_{i=1}^{n} f(Z(t_{i-1}))(Z(t_i) - Z(t_{i-1})), \phi \rangle\rangle_c$$

where $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ and $\|\triangle_n\| = max_j |t_j - t_{j-1}|$.

Stochastic integral

Theorem

Let $f \in \mathcal{SB}_\alpha(\mathbb{R})$ *and* $\phi \in \mathcal{SB}_\alpha$. *Then*

$$\langle\!\langle \int_a^b f(Z(t))dZ(t), \phi \rangle\!\rangle_c = \langle\!\langle \int_a^b \partial_t^* f(Z(t))dt, \phi \rangle\!\rangle_c.$$

A connection between the Itô formulas for the complex and real Brownian motion

Recall the Itô formula of $f(t, B_t)$,

$$f(b, B_b) - f(a, B_a) = \int_a^b f_t(t, B_t)\, ds + \int_a^b f_x(t, B_t) dB_t + \frac{1}{2} \int_a^b f_{xx}(t, B_t) dt.$$

Take $S-$transform, we obtain

$$\mu_b f(\langle \xi, h_b \rangle) - \mu_a f(\langle \xi, h_a \rangle) = \int_a^b \xi(t)(\mu_t f)'(\langle \xi, h_t \rangle) dt + \frac{1}{2} \mu_t f''(\langle \xi, h_t \rangle) dt,$$

where $\mu_t f(u) = \int_{\mathbb{R}} f(u + \sqrt{t} v) \mu(dv)$.

Replace ξ by $\dot{Z}(t)$ in the above equation, we have

$$\mu_b f(Z_b) - \mu_a(Z_a) = \int_a^b \mu_t f'(Z_t) dZ_t + \frac{1}{2}\int_a^b \mu f''(Z_t) dt.$$

The above formula indeed follows from the Itô formula of complex Brownian motion by applying the Itô formula to $\mu_t f(t, Z_t)$:

$$\mu_b f(Z_b) - \mu_a(Z_a) = \int_a^b \mu_t f'(Z_t) dZ_t + \int_a^b \frac{d}{dt}(\mu_t f)(Z_t) dt,$$

where the last term is verified by the following computation

$$\begin{aligned}
&\int_a^b \frac{d}{dt}(\mu_t f)(Z_t) dt \\
&\quad = \frac{1}{2}\int_a^b \frac{1}{\sqrt{t}} \int_{\mathbb{R}} [f'(Z_t + \sqrt{t}u)] \cdot u\, \mu(du) dt \\
&\quad = \frac{1}{2}\int_a^b \mu_t f''(Z_t) dt.
\end{aligned}$$

Example

To evaluate the integral

$$I = \int_a^b \partial_t^* B(1)\, dt.$$

We first take S-transform of I to obtain

$$S(I)(\xi) = e^{-\frac{1}{2}\|\xi\|_0^2} \int_a^b \langle\langle \partial_t^* B(1), e^{\langle \cdot, \xi \rangle} \rangle\rangle\, dt$$
$$= \int_a^b \xi(t) \langle \xi, h_1 \rangle\, dt.$$

Replace ξ by $\dot{Z}$, we obtain

$$S(I)(\dot{Z}) = \int_a^b \dot{Z}(t) \langle \dot{Z}, h_1 \rangle\, dt = (Z(b) - Z(a))Z(1).$$

It follows that

$$I = \int_{\mathcal{S}'} \langle x + iy, h_b - h_a \rangle \langle x + iy, h_1 \rangle\, \mu(dy) = B(1)(B(b) - B(a)) - (b - a).$$

A remark

The above theory remains true that if we replace the Brownian motion and the associated Hilbert space $H = L^2(\mathbb{R}, dx)$ by a Lévy process together with the Hilbert space $L^2(\mathbb{R}^2, d\lambda)$.

Volterra Gaussian processes [1]

Consider the Gaussian Processes of the form

$$Z(t) = \int_0^t K(t, s)\, dB(s), \quad 0 \le t \le 1,$$

where $K(t, s)$ is a kernel function from $[0,\, 1] \times [0,\, 1]$ into $\mathbb{R}$ satisfying

$$\sup_{t \in [0,1]} \int_0^t |K(t, s)|^2\, ds < +\infty.$$

For each $t \in [0,\, 1]$, define

$$K_t(s) = \int_0^{s \wedge t} K(t, u)\, du, \quad 0 \le s \le 1.$$

Then $K_t \in \mathscr{H}$ with $\dot{K}_t = K(t,\, \cdot) \cdot \mathbf{1}_{[0,\, t]}$.

Moreover,

$$Z(t) = \langle \cdot, K_t \rangle, \quad 0 \le t \le 1, \quad \text{on } (\mathscr{C}, \mathscr{B}(\mathscr{C}), \omega).$$

Let $F \in L^1(\mathbb{R})$ and φ be a test Brownian functional. Then

$$\begin{aligned}
\langle\!\langle F(Z(t)), \varphi \rangle\!\rangle &= \int_{\mathscr{C}} F(\langle x, K_t \rangle)\, \varphi(x)\, \omega(dx) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathscr{C}} \left\{ \int_{-\infty}^{\infty} \hat{F}(u)\, e^{\mathrm{i}\,\langle x, K_t \rangle\, u}\, du \right\} \varphi(x)\, \omega(dx) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(u) \left\{ \int_{\mathscr{C}} \varphi(x + \mathrm{i}\, u K_t)\, \omega(dx) \right\} e^{-\frac{1}{2} u^2 \int_0^t |K(t,s)|^2\, ds}\, du \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(u) \cdot S\varphi(\mathrm{i}\, u K_t) \cdot e^{-\frac{1}{2} u^2 \int_0^t |K(t,s)|^2\, ds}\, du,
\end{aligned}$$

where $\hat{F}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(r)\, e^{-\mathrm{i}\, ur}\, dr$.

Define

$$G_{t,\varphi}(u) = \frac{1}{\sqrt{2\pi}} S\varphi(\mathrm{i}\, uK_t) \cdot e^{-\frac{1}{2}u^2 \int_0^t |K(t,s)|^2\, ds}, \quad u \in \mathbb{R}.$$

Then $G_{t,\varphi} \in \mathcal{S}$. Thus, for $F \in \mathcal{S}'$, we can define $F(Z(t))$ as a generalized Brownian functional by

$$\langle\langle F(Z(t)),\, \varphi \rangle\rangle = (\hat{F},\, G_{t,\varphi}),$$

where $(\cdot,\cdot)$ is the $\mathcal{S}'$-$\mathcal{S}$ dual pairing.

Fact

Assume that $K(t,u)$ is differentiable in the variable t in $\{(t,u);\, 0 < u \le t < 1\}$, and both K and $\frac{\partial K}{\partial t}$ are continuous in $\{(t,u);\, 0 < u \le t < 1\}$. For $0 < t < 1$, let

$$h_t(s) = K(t,t) \cdot \mathbf{1}_{[t,1]}(s) + \int_0^{s \wedge t} \frac{\partial K}{\partial t}(t,u)\, du, \quad s \in [0,1].$$

Then

$$\lim_{\epsilon \to 0} \frac{K_{t+\epsilon} - K_t}{\epsilon} = h_t \quad \text{in } L^2([0,1]).$$

There are two typical Volterra Gaussian processes as follows:
(a)(Fractional Brownian motion)
Let

$$K(t, s) := K_H(t, s) =$$

$$\begin{cases} c_H \, \mathbf{1}_{[0,\,t]}(s)\,(t-s)^{H-\frac{1}{2}} \int_0^1 u^{H-\frac{3}{2}} \left(1-\left(1-\frac{t}{s}\right)u\right)^{H-\frac{1}{2}} du, & (H \in (\frac{1}{2},\, 1)), \\ \mathbf{1}_{[0,\,t]}(s), & (H = \frac{1}{2}), \\ b_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \right. \\ \quad \left. - \left(H-\frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} \, du \right], & (H \in (0,\, \frac{1}{2})), \end{cases}$$

where

$$c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H,\, H-\frac{1}{2})}} \text{ and } b_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H,\, H+\frac{1}{2})}}$$

($\beta(z, w) = \int_0^1 x^{z-1}(1-x)^{w-1}\, dx$ with $\Re z, \Re w > 0$). Then $\{Z(t);\, t \in [0,\, T]\}$ is a fractional Brownian motion (fBm for short) of Hurst index $H \in (0, 1)$ (see [2]).

(b) (Brownian bridge) Let $T = 1$ and

$$K(t, s) = \begin{cases} \frac{1-t}{1-s}, & \text{if } 0 \le s \le t < 1, \\ \\ 0, & \text{otherwise.} \end{cases}$$

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Thank You for your attention!