Law of large numbers for some Markov chains along non-homogeneous genealogies

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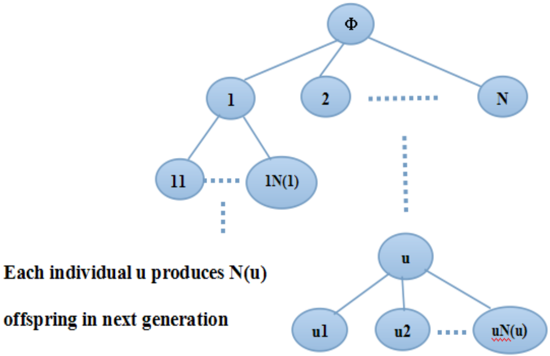
Model — Markov Chain indexed by the genealogical tree

- Genealogical tree $\mathbb{T}$



We consider a population with non-overlapping generations. The genealogical tree $\mathbb{T}$ describes the genealogy of the population in discrete time, and the nodes of the tree are the individuals.

- Markov Chain indexed by the genealogical tree

  We consider a trait in the population. Let $(\mathcal{X}, B_{\mathcal{X}})$ be the state space of this trait. For each $u \in \mathbb{T}$, denote its trait by $X(u) \in \mathcal{X}$. The process $X = (X(u))_{u\in\mathbb{T}}$ satisfies: for u of generation n,

$$\mathbb{P}(X(u1) \in dx_1, \cdots, X(uk) \in dx_k | N(u) = k, X(u) = x) = p_n^{(k)}(x, dx_1, \cdots, dx_k),$$

  where for each k, n and x, $p_n^{(k)}(x, \cdot)$ is a probability on $\mathcal{X}^k$.

  We call $X = (X(u))_{u\in\mathbb{T}}$ a Markov Chain indexed by the genealogical tree $\mathbb{T}$.

- Objective

  Let $\mathbb{T}_n := \{u \in \mathbb{T} : |u| = n\}$ be the set of all individuals in generation n and

$$Z_n := \sum_{u \in \mathbb{T}_n} \delta_{X(u)}$$

be the counting measure of individuals of generation n. In fact, for $A \in B_{\mathcal{X}}$,

$$Z_n(A) = \#\{u \in \mathbb{T}_n : X(u) \in A\}$$

denotes the number of individuals of generation n whose traits belong to A. In particular, we write

$$N_n := Z_n(\mathcal{X}) = \#\mathbb{T}_n.$$

Objective:

$$\frac{Z_n(A)}{N_n} \to ?$$

- Case I: fixed genealogical tree

  We first consider the case where $\mathbb{T}$ is fixed (no random).

Theorem 1.1

Let $A \in B_{\mathcal{X}}$. We assume that

(i) $N_n \to \infty$ as $n \to \infty$;

(ii) $\limsup_{n\to\infty} \mathbb{P}(|U_n \wedge V_n| \geq K) \to 0$ as $K \to \infty$, where U_n, V_n are two individuals uniformly and independently chosen in $\mathbb{T}_n$;

(iii) there exists $\mu(A) \in \mathbb{R}$ such that for all $u \in \mathbb{T}$ satisfying $N_n(u) > 0$ for all $n \geq 1$, and for all $x \in \mathcal{X}$, where $U_n^{(u)}$ denotes an individual uniformly chosen in $\mathbb{T}_{|u|+n}$,

$$\lim_{n\to\infty} \mathbb{P}\left(X(U_n^{(u)}) \in A \middle| X(u) = x \right) = \mu(A).$$

Then

$$\frac{Z_n(A)}{N_n} \to \mu(A) \qquad \text{in } \mathbb{P}\text{-probability}.$$

Assumptions (i) and (ii) hold for many genealogies, such as supercritical branching genealogies. The assumption (iii) is difficult to obtain in general. Here we provide a simple example where it holds.

Example 1: Homogeneous kernels

Assume that

$$\frac{1}{k}\sum_{i=1}^{k} p_n^{(k)}(x, \mathcal{X}^{i-1} dy \mathcal{X}^{k-i}) =: p(x, dy)$$

depends neither of k nor of n. Let $Y = (Y_n)$ be the Markov chain with transition kernel p. The assumption (iii) in fact is

$$\mathbb{P}_x(Y_n \in A) \to \mu(A)$$

for every $x \in \mathcal{X}$. This convergence is related to the ergodicity of the Markov chain Y, for which sufficient conditions are known, see e.g.[4].

- Case II: in random environment
  - In random environment
    Let $\xi = (\xi_0, \xi_1, \cdots)$ be a stationary and ergodic process. Each ξ_n corresponds to a distribution $p(\xi_n) = (p_k(\xi_n))_{k=0}^{\infty}$ on $\mathbb{N}_0 = \{0, 1, \cdots\}$ and a class of probabilities $p_{\xi_n}^{(k)}(x, dx_1, \cdots, dx_k)$ on $\mathcal{X}^k$ for each k, n, x. Such ξ is called a random environment.

    We consider the case where the population evolves following a branching process in a random environment (so that $\mathbb{T}$ is random). Given ξ, the offspring number $N(u)$ of individual u of generation n is distributed as $p(\xi_n)$ and its offspring traits $\{X(ui)\}$ are determined by

$$\begin{aligned}\mathbb{P}_\xi(X(u1) \in dx_1, \cdots, X(uk) \in dx_k | N(u) = k, X(u) = x) \\ = p_{\xi_n}^{(k)}(x, dx_1, \cdots, dx_k),\end{aligned}$$

    where $\mathbb{P}_\xi$ represents the conditional probability given ξ and is usually called quenched law.

- Branching process in a random environment (BPRE)
  The population of generation n, N_n, is a BPRE. Put

$$m_n = \sum_k k p_k(\xi_n) \ (n \geq 0) \quad \text{and} \quad P_n = m_0 \cdots m_{n-1} \ (n \geq 1).$$

It is well known that the normalized population

$$W_n = \frac{N_n}{P_n}$$

is a non-negative martingale and its limit $W = \lim_{n\to\infty} W_n$ exists a.s.
Consider the supercritical non-degenerated case

$$0 < \mathbb{E}(\log m_0) < \infty \quad \text{and} \quad \mathbb{E}\left(\log \mathbb{E}_\xi W_1^2\right) < \infty. \qquad \text{(A)}$$

This assumption ensures that $W_n \to W$ in L^2 under $\mathbb{P}_\xi$ and so the limit $W > 0$ on the non-extinction event $\{N_n \to \infty\}$, see [2].

- The auxiliary Markov Chain $Y = (Y_n)$
  Let
  $$P_{\xi_n}^{(k,i)}(x, \cdot) = p_{\xi_n}^{(k)}(x, \mathcal{X}^{i-1} \times \cdot \times \mathcal{X}^{k-i})$$
  be the ith marginal distribution of $p_{\xi_n}^{(k)}(x, \cdot)$ and we introduce the random transition probability
  $$Q_n(x, \cdot) := \frac{1}{m_n} \sum_{k=0}^{\infty} p_k(\xi_n) \sum_{i=1}^{k} P_{\xi_n}^{(k,i)}(x, \cdot).$$
  Given the environment ξ, we define an auxiliary Markov chain in varying environment $Y = (Y_n)$, whose transition probability in time j is Q_j :
  $$\mathbb{P}_\xi(Y_{j+1} \in dy | Y_j = x) = Q_j(x, dy).$$
  We'll see that the convergence of the measure $Z_n(\cdot)/N_n$ comes from the ergodic behavior of Y_n.

- Law of large numbers in generation n
  Similarly to the result of Delmas and Marsalle [1] for deterministic environment case, we have

Theorem 2.1 Law of large numbers in generation n

Let $A \in B_{\mathcal{X}}$. We assume that there exists a sequence $(\mu_{\xi,n}(A))_n \subset \mathbb{R}$ such that for almost all ξ and for each $r \in \mathbb{N}$,

$$\lim_{n\to\infty} (\mathbb{P}_{T^r\xi,x}(Y_{n-r} \in A) - \mu_{\xi,n}(A)) = 0 \qquad \text{for every } x \in \mathcal{X}. \tag{1}$$

Then we have for almost all ξ, conditionally on the non-extinction event,

$$\frac{Z_n(A)}{N_n} - \mu_{\xi,n}(A) \to 0 \qquad \text{in } \mathbb{P}_\xi\text{-probability.}$$

$\mathbb{P}_{\xi,x}$ denotes the quenched law when the process Y starts from the initial value x and $T\xi = (\xi_1, \xi_2, \cdots)$ if $\xi = (\xi_0, \xi_1, \cdots)$.
The condition (1) holds if Y is weakly ergodic. For sufficient conditions of weak ergodicity in the non-homogeneous case, see Mukhamedov [4].

Corollary

Let $A \in B_{\mathcal{X}}$. We assume that there exists $\mu(A) \in \mathbb{R}$ such that for almost all ξ,

$$\lim_{n\to\infty} \mathbb{P}_{\xi,x}(Y_n \in A) = \mu(A) \qquad \text{for every } x \in \mathcal{X}. \tag{2}$$

Then we have for almost all ξ, conditionally on the non-extinction event,

$$\frac{Z_n(A)}{N_n} \to \mu(A) \qquad \text{in } \mathbb{P}_\xi\text{-probability.}$$

The condition (2) holds if Y is ergodic. However, the ergodicity in the non-homogeneous case is difficult to get under general assumptions. If Y is homogeneous, the sufficient conditions are known, see e.g.[4].

- Law of large numbers on the whole tree

Theorem 2.2 Law of large numbers on the whole tree

Let $A \in B_{\mathcal{X}}$. We assume that there exists $\mu(A) \in \mathbb{R}$ such that for almost all ξ,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}_{\xi,x}(Y_k \in A) = \mu(A) \qquad \text{for every } x \in \mathcal{X}. \tag{3}$$

Then we have for almost all ξ, conditionally on the non-extinction event,

$$\frac{1}{n} \sum_{k=1}^{n} \frac{Z_k(A)}{N_k} \to \mu(A) \qquad \text{in } \mathbb{P}_\xi\text{-probability.}$$

A sufficient condition for (3) was shown in Seppäläinen [5].

- Central limit theorem
  When the auxiliary Markov chain Y satisfies a central limit theorem, the measure $Z_n(\cdot)/N_n$ maybe also satisfies a central limit theorem.

Theorem 2.3 Central limit theorem

Let $\mathcal{X} \subset \mathbb{R}$. We assume that for almost all ξ, Y_n satisfies a central limit theorem: there exits a sequence of random variables $\{(a_n(\xi), b_n(\xi)\}$ satisfying $b_n(\xi) > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}_{\xi,x}\left(\frac{Y_n - a_n(\xi)}{b_n(\xi)} \le y\right) = \Phi(y) \qquad \text{for every } x \in \mathcal{X}, \tag{4}$$

where Φ is a continuous function on $\mathbb{R}$. If for each $r \in \mathbb{N}$ fixed,

$$\lim_{n\to\infty} \frac{b_n(\xi)}{b_{n-r}(T^r\xi)} = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{a_n(\xi) - a_{n-r}(T^r\xi)}{b_{n-r}(T^r\xi)} = 0 \quad a.s., \tag{5}$$

then we have for almost all ξ, conditionally on the non-extinction event,

$$\frac{Z_n(-\infty, b_n(\xi)y + a_n(\xi)]}{N_n} \to \Phi(y) \qquad \text{in } \mathbb{P}_\xi\text{-probability}.$$

Example 2: Branching random walk with a random environment in time (Huang and Liu [3])

Given the environment $\xi = (\xi_n)$, each particle u of generation n, located at $X(u) \in \mathbb{R}$, is independently replaced by $N(u)$ new particles of generation $n+1$ which scatter on $\mathbb{R}$ with positions determined by

$$X(ui) = X(u) + L_i(u),$$

where the point process $(N(u); L_1(u), L_2(u), \cdots)$ has the normalized intensity measure $q_n = q(\xi_n)$ for $u \in \mathbb{T}_n$. In this model, we can see that

$$Q_n(x, dy) = q_n(dy - x).$$

Let $\mu_n = \int_{\mathbb{R}} t q_n(dt)$ and $\sigma_n^2 = \int_{\mathbb{R}} (t - \mu_n)^2 q_n(dt)$. Huang and Liu [3] obtained (under certain assumptions) the condition (4), with

$$a_n(\xi) = \sum_{i=0}^{n-1} \mu_n, \qquad b_n(\xi) = \left(\sum_{i=0}^{n-1} \sigma_n^2 \right)^{1/2}.$$

References

J-F. Delmas, L. Marsalle.
Detection of cellular aging in a Galton-Watson process. *Stoch. Proc. Appl. 120 (2010), 2495-2519.*

C. Huang, Q. Liu.
Convergence in L^p and its exponential rate for a branching process in a random environment. *Available via http://arxiv.org/abs/1011.0533.*

C. Huang, Q. Liu.
Branching random walk with a random environment in time. *Available via http://arxiv.org/abs/1407.7623.*

F. Mukhamedov.
On L_1-weak ergodicity of nonhomogeneous discrete Markov processes and its applications. *Rev. Mat. Complut. 26 (2013) 799-813.*

T. Seppäläinen.
Large deviations for Markov chains with Random Transitions. *Ann. Prob. 22 (1994), 713-748.*

Thank you !

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