Density convergence for some nonlinear Gaussian stationary sequences

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Joint work with

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Based on a joint work with

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Density convergence in the Breuer-Major theorem for nonlinear Gaussian stationary sequences.

To appear in Bernoulli.

Outline

1. Motivation
2. Main results
3. Idea of the proof

1. Motivation

Central limit theorem:
Let $X_1, \cdots, X_n$ be independent, identically distributed random variables with mean m and variance σ^2.

$$F_n := \sqrt{n}\left(\frac{X_1 + \cdots + X_n}{n} - \mathbb{E}(X_1)\right) \longrightarrow N(0, \sigma^2)$$

$$\frac{X_1 + \cdots + X_n}{n} - \mathbb{E}(X_1) \approx \frac{\xi}{\sqrt{n}}, \quad \text{where } \xi \sim N(0, \sigma^2).$$

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The above convergence is in the sense of distribution

$$F_n \to N(0, \sigma^2) \quad \text{in distribution}$$

$$P(F_n \le a) \to \int_{-\infty}^{a} \phi_\sigma(x) dx \quad \forall \ \ a \in \mathbb{R}$$

where $\phi_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$.

Other examples of multiple Itô integral F_n

$$F_n = \int_{[0,T]^q} f_n(t_1, \cdots, t_q) dB_{t_1} \cdots dB_{t_q} \,,$$

where q is an fixed positive integer, $(B_t, t \geq 0)$ is a standard Brownian motion, f_n is a sequence of deterministic functions such that

$$\int_{[0,T]^q} f_n^2(t_1, \cdots, t_q) dt_1 \cdots dt_q$$

Theorem

If $F_n = I_q(f_n)$, then the following are equivalent:

(i) $\lim_{n\to\infty} \mathbb{E}[F_n^4] = 3\sigma^4$,

(ii) *For all* $1 \leq r \leq q-1$, $\lim_{n\to\infty} \|f_n \otimes_r f_n\|_{H^{\otimes 2(q-r)}} = 0$,

(iii) $\|DF_n\|_H^2 \to q\sigma^2$ *in* $L^2(\Omega)$ *as* $n \to \infty$.

(iv) F_n *converges in distribution to the normal law* $N(0, \sigma^2)$ *as* $n \to \infty$.

Nualart, D.; Peccati, G.
Central limit theorems for sequences of multiple stochastic integrals.
Ann. Probab. 33 (2005), 177-93.

Nualart, D.; Ortiz-Latorre, S.
Central limit theorems for multiple stochastic integrals and Malliavin calculus.
Stochastic Process. Appl. 118 (2008), 614-628.

Let $X = \{X_k;\ k = 0, 1, 2, \cdots\}$ be a centered Gaussian stationary sequence with unit variance. For all v, we set

$$\rho(v) = \mathbb{E}[X_0 X_{|v|}]$$

Let γ be the standard Gaussian probability measure and $f \in L^2(\gamma)$ be a fixed deterministic function such that $\mathbb{E}[f(X_1)] = 0$.

We expand f in the orthonormal basis of Hermite polynomials

$$f(x) = \sum_{j=d}^{\infty} a_j H_j(x),$$

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The Breuer-Major Theorem

Suppose that $\sum_{v=-\infty}^{\infty} |\rho(v)|^d < \infty$ and suppose

$$\sigma^2 = \sum_{j=d}^{\infty} j! a_j^2 \sum_{v=-\infty}^{\infty} \rho(v)^j$$

is in $(0, \infty)$. Then we have

$$V_n \xrightarrow{\text{Law}} N(0, \sigma^2)$$

P. Breuer and P. Major
Central limit theorems for non-linear functionals of Gaussian fields.
J. Mult. Anal. **13** (1983), 425–441.

Corollary

Consider $2 \le d \le q < \infty$ *and a family of real numbers* $\{a_j;\, j = d, \ldots, q\}$. *Let* H_j *be the* j*th order Hermite polynomial, and assume that* $\sigma^2 \in (0, \infty)$, *where* $\sigma^2 \equiv \sum_{j=d}^{q} j!a_j^2 \sum_{v \in \mathbb{Z}} \rho(v)^j$. *Set*

$$V_n^{d,q} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \sum_{j=d}^{q} a_j\, H_j(X_k).$$

Then $V_n^{d,q} \xrightarrow{Law} \mathcal{N}(0, \sigma^2)$ *as* n *tends to infinity. In particular, we have:*

$$\lim_{n \to 0} \mathbf{E}\left[\left(V_n^{d,q} \right)^4 \right] = 3\sigma^4.$$

There are some development along this direction.

Convergence in density of multiple integrals

Convergence in density of multiple integrals
Are there $f_n(x)$ such that

$$P(F_n \leq a) = \int_{-\infty}^{a} f_n(x)dx$$

and

$$f_n(x) \longrightarrow \phi_\sigma(x)\,?$$

2. Main result

Let $X = \{X_k\,, k \geq 0\}$ be a Gaussian stationary sequence whose spectral density $f_\rho(\lambda) = \dfrac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} \rho(\nu) e^{i\nu\lambda}$ satisfies $f_\rho \in L^{1/2}([-\pi, \pi])$ and $\log(f_\rho) \in L^1([-\pi, \pi])$, where for all v, we set

$$\rho(v) = \mathbb{E}[X_0 X_{|v|}]$$

Let

$$V_n^{d,q} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \sum_{j=d}^{q} a_j \, H_j(X_k) \, ,$$

where $d \geq 2$.

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where $d \geq 2$. Assume

$$\sigma^2 \equiv \sum_{j=d}^{q} j! a_j^2 \sum_{v \in \mathbb{Z}} \rho(v)^j \in (0, \infty) \, .$$

Then for any $p \geq 1$, there exists n_0 such that

$$\sup_{n \geq n_0} \mathbf{E} \left[\| D V_n^{d,q} \|^{-p} \right] < \infty.$$

In the case of a fixed Wiener chaos we can obtain the following consequence.

Corollary

Under the conditions of above theorem, if $q = d$*, and we define* $F_n = V_n^{d,d}/\sigma_n$*, where* $\sigma_n^2 = \mathbf{E}[(V_n^{d,d})^2]$*, then, for all* $m \geq 0$ *there exists an* n_0 *(depending on* m*) such that*

$$\sup_{n \geq n_0} \sup_{x \in \mathbb{R}} |p_{F_n}^{(m)}(x) - \phi^{(m)}(x)| \leq c_m \sqrt{\mathbf{E}[F_n^4] - 3}.$$

In the case $q \neq d$,

Corollary

Under the conditions of the above theorem, if we define $F_n = V_n^{d,q}/\sigma_n$, where $\sigma_n^2 = \mathbf{E}[(V_n^{d,q})^2]$, then, for all $m \geq 0$ we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |p_{F_n}^{(m)}(x) - \phi^{(m)}x)| = 0.$$

Discussion of the hypothesis

$$f_\rho(\lambda) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \rho(k)\, e^{\imath k \lambda}, \qquad \lambda \in [-\pi, \pi].$$

We assume that

$$f_\rho \in L^{1/2}([-\pi,\pi]) \quad \text{and} \quad \log(f_\rho) \in L^1([-\pi,\pi]).$$

This condition $\log(f_\rho) \in L^1([-\pi,\pi])$ is referred to as *purely nondeterministic* property in the literature.

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(i) If $\rho \in \ell^1$, then the spectral density f_ρ exists and is a nonnegative L^2 function defined on $[-\pi, \pi]$. Then the condition is thus fulfilled.

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Proposition

Let ρ be the covariance function of X. We have the following statements.

(i) If $\rho \in \ell^1$, then the spectral density f_ρ exists and is a nonnegative L^2 function defined on $[-\pi, \pi]$. Then the condition is thus fulfilled.

(ii) If $\lim_{k\to\infty} |k|^\alpha \rho(k) = c_\rho$ for some $\alpha \in (0,1)$ and some positive constant c_ρ, then the spectral density exists, is strictly positive almost everywhere and satisfies $\lim_{\lambda\to 0} |\lambda|^{1-\alpha} f_\rho(\lambda) = c_f$. In particular, the condition is satisfied.

Example 1

Gaussian autoregressive fractionally integrated moving-average (Gaussian ARFIMA) processes. Denote by B the one lag backward operator ($BX_k = X_{k-1}$). Let $\phi(z)$ and $\theta(z)$ be two polynomials which have no common zeros and such that the zeros of ϕ lie outside the closed unit disk $\{z\,,\ |z| \leq 1\}$. Suppose that X_k is given by

$$\phi(B)X_k = (\mathrm{Id} - B)^{-d}\theta(B)w_k\,,$$

where $-1 < d < 1/2$, and where the operator $(\mathrm{Id} - B)^{-d}$ is defined by:

$$(\mathrm{Id} - B)^{-d} = \sum_{j=1}^{\infty} \eta_j B^j \quad \text{with} \quad \eta_j = \frac{\Gamma(d+j)}{\Gamma(j+1)\Gamma(d)}\,.$$

The sequence $(w_k)_{k\in\mathbb{Z}}$ is a discrete Gaussian white noise. It is well-known that under the above conditions, $\{X_k, k \in \mathbb{N}\}$ admits a spectral density whose exact expression is:

$$f(\lambda) = \frac{1}{2\pi}\left[2\sin\frac{\lambda}{2}\right]^{-2d}\frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}.$$

It is thus readily checked that the conditions are satisfied, and hence X_k has a causal representation.

Example 2

Our second example is the fractional Gaussian noise. Let $\{B_t, t \geq 0\}$ be a fractional Brownian motion of Hurst parameter $H \in (0,1)$. Then $\{X_k = B_{k+1} - B_k\,, k \in \mathbb{N} \cup \{0\}\}$ is a stationary Gaussian process with correlation

$$\rho(k) = \frac{1}{2}\left[(k+1)^{2H} - 2k^{2H} - (k-1)^{2H}\right] .$$

Its spectral density is:

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \rho(|k|)e^{i\lambda} = 2c_f(1-\cos(\lambda)) \sum_{j=-\infty}^{\infty} |2\pi j + \lambda|^{-2H-1} ,$$

where $c_f = (2\pi)^{-1} \sin(\pi H)\Gamma(2H+1)$.

If $H \leq 1/2$, it is clear that $\sum_{k=-\infty}^{\infty} |\rho(|k|)| < \infty$. This implies

$$\sup_{\lambda \in [-\pi,\pi]} |f(\lambda)| < \infty \,.$$

Thus $f \in L^{1/2}$. If $1/2 < H < 1$, then

$$0 \leq f(\lambda) \;\; \leq \;\; 2c_f(1 - \cos(\lambda))|\lambda|^{-2H-1} + 2c_f \sum_{j \neq 0} |2\pi j + \lambda|^{-2H-1} \,.$$

The first term is in L^1 since $H < 1$. When $j \neq 0$, $\int_{-\pi}^{\pi} |2\pi j + \lambda|^{-2H-1} d\lambda \leq Cj^{-2H}$ for some positive constant C. Thus $\int_{-\pi}^{\pi} \sum_{j\neq 0} |2\pi j + \lambda|^{-2H-1} \, d\lambda < \infty$, owing to the fact that $H > 1/2$. Therefore, we have $f \in L^1$ and hence $f \in L^{1/2}$. Summarizing we have $f \in L^{1/2}$ for all $H \in (0,1)$. This also implies $\log^+ f(\lambda) \in L^1$. To see $\log^- f(\lambda) \in L^1$, we notice that

$$f(\lambda) \geq 2c_f(1 - \cos(\lambda))|\lambda|^{-2H-1} \, .$$

So $\log^- f(\lambda) \leq C + \left| \log \left[(1 - \cos(\lambda))|\lambda|^{-2H-1} \right] \right|$ which is in L^1. In conclusion, the sequence X satisfies Hypothesis.

3. Idea of the proof

Causal representation

Proposition

Let X be a Gaussian stationary sequence satisfying the hypothesis. Then for each $k \in \mathbb{N} \cup \{0\}$ the random variable X_k can be decomposed as

$$X_k = \sum_{j \geq 0} \psi_j \, w_{k-j},$$

where $(w_k)_{k \in \mathbb{Z}}$ is a discrete Gaussian white noise and the coefficients ψ_j are deterministic. With a slight abuse of notation, extend the sequence ψ to $(\psi_j)_{j \in \mathbb{Z}}$ by setting $\psi_{-j} = 0$ for $j \geq 0$.

Then one can choose ψ such that it enjoys the following properties:
(i) The sequence ψ admits a spectral density f_ψ such that
$f_\psi = \frac{f_\rho^{1/2}}{2\pi}$.
(ii) In particular, $\psi_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_\rho^{1/2}(\lambda)\, d\lambda$ and $\psi_0 > 0$.
(iii) For all $k_1, k_2 \in \mathbb{N}$ we have $\rho(k_1 - k_2) = \sum_{l=-\infty}^{k_1 \wedge k_2} \psi_{k_1 - l}\, \psi_{k_2 - l}$.

Malliavin calculus

Let $\mathfrak{H}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$. The norm of $\mathfrak{H}$ will be denoted by $\| \cdot \| = \| \cdot \|_{\mathfrak{H}}$. Recall that we call *isonormal Gaussian process* over $\mathfrak{H}$ any centered Gaussian family $W = \{W(h) : h \in \mathfrak{H}\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and such that $\mathbf{E}[W(h)W(g)] = \langle h, g \rangle_{\mathfrak{H}}$ for every $h, g \in \mathfrak{H}$.

In our application the underlying Gaussian family will be a discrete Gaussian white noise $(w_k)_{k \in \mathbb{Z}}$. The space $\mathfrak{H}$ is given here by $\mathfrak{H} = \ell^2(\mathbb{Z})$ (the space of square integrable sequences indexed by $\mathbb{Z}$) equipped with its natural inner product. Set $\{\varepsilon^j; \, j \in \mathbb{Z}\}$ for the canonical basis of $\ell^2(\mathbb{Z})$, that is $\varepsilon^j_k = \delta_j(k)$. We thus identify w_j with $W(\varepsilon^j)$. Assume from now on that our underlying σ-algebra $\mathcal{F}$ is generated by W.

For any integer $q \in \mathbb{N} \cup \{0\}$, we denote by $\mathcal{H}_q$ the qth *Wiener chaos* of W. $\mathcal{H}_q$ is the closed linear subspace of $L^2(\Omega)$ generated by the family of random variables $\{H_q(W(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, with H_q the q-th Hermite polynomial given by

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} \left(e^{-\frac{x^2}{2}} \right).$$

Let $\mathcal{S}$ be the set of all cylindrical random variables of the form

$$F = g(W(h_1), \ldots, W(h_n)),$$

where $n \geq 1$, $h_i \in \mathfrak{H}$, and g is infinitely differentiable such that all its partial derivatives have polynomial growth. The Malliavin derivative of F is the element of $L^2(\Omega; \mathfrak{H})$ defined by

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(W(h_1), \ldots, W(h_n))\, h_i.$$

By iteration, for every $m \geq 2$, we define the mth derivative $D^m F$. This is an element of $L^2(\Omega; \mathfrak{H}^{\odot m})$, where $\mathfrak{H}^{\odot m}$ designates the symmetric mth tensor product of $\mathfrak{H}$.

For $m \geq 1$ and $p \geq 1$, $\mathbb{D}^{m,p}$ denote the closure of $\mathcal{S}$ with respect to the norm $\| \cdot \|_{m,p}$ defined by

$$\|F\|_{m,p}^p = \mathbf{E}[|F|^p] + \sum_{j=1}^{m} \mathbf{E}\left[\|D^j F\|_{\mathfrak{H}^{\otimes j}}^p\right].$$

Set $\mathbb{D}^\infty = \cap_{m,p} \mathbb{D}^{m,p}$. One can then extend the definition of D^m to $\mathbb{D}^{m,p}$. When $m = 1$, one simply write D instead of D^1. As a consequence of the hypercontractivity property of the Ornstein-Uhlenbeck semigroup, all the $\| \cdot \|_{m,p}$-norms are equivalent in any *finite* sum of Wiener chaoses.

Finally, let us recall that the Malliavin derivative D satisfies the following *chain rule*: if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is in $\mathcal{C}_b^1$ (that is, belongs to the set of continuously differentiable functions with a bounded derivative) and if $\{F_i\}_{i=1,\dots,n}$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F_1, \dots, F_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n)\, DF_i.$$

Main results we shall use

Let $\{F_n\}$ be a sequence of random variables belonging to a fixed chaos of order greater than or equal to 2.

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(i) Suppose that for some $\epsilon > 0$,

$$\sup_n \mathbb{E}\left[\|DF_n\|^{-4-\epsilon}\right] < \infty.$$

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(i) Suppose that for some $\epsilon > 0$,

$$\sup_n \mathbb{E}\left[\|DF_n\|^{-4-\epsilon}\right] < \infty.$$

Then, there exists a constant c such that for all $n \geq 1$,

$$\sup_{x\in\mathbb{R}} |p_{F_n}(x) - \phi(x)| \leq c\sqrt{\mathbb{E}[F_n^4] - 3}.$$

(ii) Suppose that for all $p \geq 1$,

$$\sup_n \mathbb{E}\left[\|DF_n\|^{-p}\right] < \infty.$$

Then, for any $m \geq 0$, there exists a constant c_m such that for all $n \geq 1$,

$$\sup_{x \in \mathbb{R}} |p_{F_n}^{(m)}(x) - \phi^{(m)}(x)| \leq c_m \sqrt{\mathbb{E}[F_n^4] - 3}.$$

Hu, Y. ; Lu, F. and Nualart, D.

Convergence of densities of some functionals of Gaussian processes.

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A key lemma

Our future computations will heavily rely on an efficient way to compute conditional expectations. Towards this aim, we state here some general results. Let us start with a decomposition for Hermite polynomials:

Lemma

For any $q \geq 1$, let H_q be the Hermite polynomial. Consider $y, z \in \mathbb{R}$ and two real parameters a, b with $a^2 + b^2 = 1$. Then the following relation holds true:

$$H_q(ay + bz) = \sum_{\ell=0}^{q} \binom{q}{\ell} a^{q-\ell} b^{\ell} \, H_{q-\ell}(y) \, H_{\ell}(z).$$

short proof

By the definition of the Hermite polynomials, we have

$$e^{aty-\frac{(at)^2}{2}} = \sum_{i=0}^{\infty}(at)^i H_i(y), \quad \text{and} \quad e^{tbz-\frac{(bt)^2}{2}} = \sum_{j=0}^{\infty}(bt)^j H_j(z);$$

$$e^{t(ay+bz)-t^2/2} = \sum_{q=0}^{\infty} t^q H_q(ay+bz).$$

Since $a^2+b^2=1$, we obviously have
$e^{aty-\frac{(at)^2}{2}} e^{tbz-\frac{(bt)^2}{2}} = e^{t(ay+bz)-t^2/2}$. Thus, we have

$$\sum_{q=0}^{\infty} t^q H_q(ay+bz) = \sum_{i=0}^{\infty}(at)^i H_i(y) \sum_{j=0}^{\infty}(bt)^j H_j(z),$$

which easily yields the desired identity.

This is to be used in the following computation of conditional expectations:

Proposition

Let Y and Z be two centered Gaussian random variables such that Y is measurable with respect to a σ-algebra $\mathcal{G} \subset \mathcal{F}$ and Z is independent of $\mathcal{G}$. Assume that $\mathbf{E}[Y^2] = \mathbf{E}[Z^2] = 1$. Then for any $q \geq 1$, and real parameters a, b such that $a^2 + b^2 = 1$, we have:

$$\mathbf{E}[H_q(aY + bZ)|\mathcal{G}] = a^q H_q(Y).$$

Short proof

Apply the key lemma in order to decompose $H_q(aY + bZ)$. Then identity follows easily from the fact that Y is $\mathcal{G}$-measurable, Z is independent from $\mathcal{G}$ and Hermite polynomials have 0 mean under a centered Gaussian measure except for $H_0 \equiv 1$.

Carbery-Wright inequality

Proposition

Let $X = (X_1, \ldots, X_n)$ be a Gaussian random vector in $\mathbb{R}^n$ and $Q : \mathbb{R}^n \to \mathbb{R}$ a polynomial of degree at most m. Then there is a universal constant $c > 0$ such that:

$$(\mathbf{E}[|Q(X_1, \ldots, X_n)|])^{\frac{1}{m}} \ \mathbf{P}(|Q(X_1, \ldots, X_n)| \leq x) \leq c\, m\, x^{\frac{1}{m}}$$

for all $x > 0$.

Sketch of the proof

Step 1: Computation of the Malliavin norm.

$$
\begin{aligned}
DV_n^{d,q} &= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f'(X_k) \left(\sum_{j \geq 0} \psi_j \, \varepsilon^{k-j} \right) \\
&= \frac{1}{\sqrt{n}} \sum_{l=-\infty}^{n-1} \left(\sum_{k=l^+}^{n-1} \psi_{k-l} \, f'(X_k) \right) \varepsilon^l ,
\end{aligned}
$$

where $l^+ = \max\{l, 0\}$.

$$
\left\langle DV_n^{d,q}, \, DV_n^{d,q} \right\rangle_{\mathfrak{H}} = \frac{1}{n} \sum_{k_1, k_2 = 0}^{n-1} f'(X_{k_1}) \rho(k_1 - k_2) f'(X_{k_2}),
$$

$$
\left\langle DV_n^{d,q}, \, DV_n^{d,q} \right\rangle_{\mathfrak{H}} = \frac{1}{n} \sum_{\ell=-\infty}^{n-1} \left(\sum_{k=\ell^+}^{n-1} \psi_{k-\ell} \, f'(X_k) \right)^2 .
$$

Rearranging terms (namely, change $k-\ell$ to k and then $n-\ell-1$ to m), we end up with:

$$\begin{aligned}\left\langle DV_n^{d,q},\, DV_n^{d,q}\right\rangle_{\mathfrak{H}} &\geq \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\sum_{k=0}^{n-\ell-1} f'(X_{\ell+k})\,\psi_k\right)^2 \\ &= \frac{1}{n}\sum_{m=0}^{n-1}\left(\sum_{k=0}^{m} f'(X_{n-1-(m-k)})\,\psi_k\right)^2 \equiv A_n.\end{aligned}$$

As a last preliminary step we resort to the fact that $X=\{X_k;\, k\in\mathbb{N}\cup\{0\}\}$ is a Gaussian stationary sequence, which allows to assert that A_n is identical in law to B_n with

$$B_n := \frac{1}{n}\sum_{m=0}^{n-1}\left(\sum_{k=0}^{m} f'(X_{m-k})\,\psi_k\right)^2 = \frac{1}{n}\sum_{m=0}^{n-1}\left(\sum_{k=0}^{m} f'(X_k)\,\psi_{m-k}\right)^2.$$

We will now bound the negative moments of B_n.

Step 2: Block decomposition.

Fix thus an integer $N \geq 1$ and let $M = [n/N]$ be the integer part of n/N. Then $n \geq NM$ and as a consequence,

$$\begin{aligned} B_n &= \frac{1}{n} \sum_{m=0}^{n-1} \left(\sum_{k=0}^{m} f'(X_k)\, \psi_{m-k} \right)^2 \\ &\geq \frac{1}{n} \sum_{i=0}^{N-1} \sum_{m=iM}^{(i+1)M-1} \left(\sum_{k=0}^{m} f'(X_k)\, \psi_{m-k} \right)^2 . \end{aligned}$$

For $i = 0, \ldots, N-1$, define

$$B_n^i = \frac{1}{n} \sum_{m=iM}^{(i+1)M-1} \left(\sum_{k=0}^{m} f'(X_k)\, \psi_{m-k} \right)^2$$

so that $B_n \geq \sum_{i=0}^{N-1} B_n^i$.

Then it is readily checked that:

$$(B_n)^{-\frac{p}{2}} \leq \prod_{i=0}^{N-1} (B_n^i)^{-\frac{p}{2N}}.$$

we obtain:

$$\begin{aligned}
\mathbf{E}\left[(B_n)^{-\frac{p}{2}}\right] &\leq \mathbf{E}\left[\prod_{i=0}^{N-1} (B_n^i)^{-\frac{p}{2N}}\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[(B_n^{N-1})^{-\frac{p}{2N}} | \mathcal{F}_{(N-1)M}\right] \prod_{i=0}^{N-2} (B_n^i)^{-\frac{p}{2N}}\right].
\end{aligned}$$

Step 3: Application of Carbery-Wright. Let us go back to the particular situation of $f = \sum_{j=d}^{q} a_j H_j$, which means in particular that $f' = \sum_{j=d}^{q} j\, a_j\, H_{j-1}$. First, we notice

$$\begin{aligned} & \mathbf{E}\left[(B_n^{N-1})^{-\frac{p}{2N}} | \mathcal{F}_{(N-1)M} \right] \\ \leq \quad & 1 + \frac{p}{2N} \int_0^1 \mathbf{P}\left(B_n^{N-1} \leq x | \, \mathcal{F}_{(N-1)M} \right) x^{-\frac{p}{2N}-1} dx \, . \end{aligned}$$

Since B_n^{N-1} is a polynomial of order $m = 2(q-1)$, Carbery-Wright's inequality yields:

$$\mathbf{P}\left(B_n^{N-1} \leq x | \mathcal{F}_{(N-1)M} \right) \leq \frac{c\, x^{\frac{1}{2(q-1)}}}{\left[\mathbf{E}\left(B_n^{N-1} | \mathcal{F}_{(N-1)M} \right) \right]^{\frac{1}{2(q-1)}}} .$$

Step 4: Estimates for the conditional expectation. We now estimate the conditional expectation $\mathbf{E}[B_n^{N-1}|\mathcal{F}_{(N-1)M}]$. We have:

$$\begin{aligned}
&\mathbf{E}\left[B_n^{N-1}|\mathcal{F}_{(N-1)M}\right] \\
&= \frac{1}{n}\sum_{m=(N-1)M}^{NM-1} \mathbf{E}\left[\left(\sum_{k=0}^{m} f'(X_k)\,\psi_{m-k}\right)^2 \middle| \mathcal{F}_{(N-1)M}\right] \\
&\geq \frac{1}{n}\sum_{m=(N-1)M}^{NM-1} A_m,
\end{aligned}$$

where we have set

$$A_m = \mathbf{Var}\left(\sum_{k=(N-1)M}^{m} f'(X_k)\,\psi_{m-k} \middle| \mathcal{F}_{(N-1)M}\right).$$

Furthermore, notice that

$$f'(X_k) = f'\left(\sum_{\ell=-\infty}^{k} \psi_{k-i}\, w_i\right) = f'(Y_k + Z_k),$$

where $Y_k = \sum_{i=-\infty}^{(N-1)M-1} \psi_{k-i}\, w_i$ is $\mathcal{F}_{(N-1)M}$-measurable and $Z_k = \sum_{i=(N-1)M}^{k} \psi_{k-i}\, w_i$ is independent of $\mathcal{F}_{(N-1)M}$. Recalling that $f' = \sum_{j=d}^{q} j\, a_j\, H_{j-1}$. This gives:

$$\begin{aligned} & H_{q-1}(X_k) - \mathbf{E}[H_{q-1}(X_k)|\mathcal{F}_{(N-1)M}] \\ & = \sum_{j=d}^{q} \sum_{\ell=1}^{j-1} j^2 a_j \binom{j-1}{\ell} \sigma_{Y_k}^{j-1-\ell} H_{j-1-\ell}(\widetilde{Y}_k) \sigma_{Z_k}^{\ell} H_\ell(\widetilde{Z}_k), \end{aligned}$$

where $\sigma_{Y_k} = [\mathbf{Var}(Y_k)]^{1/2}$, $\sigma_{Z_k} = [\mathbf{Var}(Z_k)]^{1/2}$, $\widetilde{Y}_k = Y_k/\sigma_{Y_k}$ and $\widetilde{Z}_k = Z_k/\sigma_{Z_k}$.

Therefore,

$$A_m = \mathbf{E}\left[\left(\sum_{k=(N-1)M}^{m}\sum_{j=d}^{q}\sum_{\ell=1}^{j-1} a_{j,\ell,k}\, H_{j-1-\ell}(\widetilde{Y}_k)\, H_\ell(\widetilde{Z}_k)\,\psi_{m-k}\right)^2 \middle| \mathcal{F}_{(N-1)M}\right]$$

$$= \mathbf{E}\left[\left(\sum_{\ell=1}^{q-1}\sum_{k=(N-1)M}^{m}\sum_{j=(\ell+1)\vee d}^{q} a_{j,\ell,k}\, H_{j-1-\ell}(\widetilde{Y}_k)\, H_\ell(\widetilde{Z}_k)\,\psi_{m-k}\right)^2 \middle| \mathcal{F}_{(N-1)M}\right]$$

where we have set $a_{j,\ell,k} = j^2 a_j \binom{j-1}{\ell}\sigma_{Y_k}^{j-1-\ell}\sigma_{Z_k}^{\ell}$.

Recall that the random variables $\widetilde{Y}_k$ are $\mathcal{F}_{(N-1)M}$-measurable while the random variables $\widetilde{Z}_k$ are independent of $\mathcal{F}_{(N-1)M}$. By decorrelation properties of Hermite polynomials we thus get:

$$A_m = \sum_{\ell=1}^{q-1} \mathbf{E}\left[\left(\sum_{k=(N-1)M}^{m} \sum_{j=(\ell+1)\vee d}^{q} a_{j,\ell,k}\, H_{j-1-\ell}(\widetilde{Y}_k)\, H_\ell(\widetilde{Z}_k)\, \psi_{m-k}\right)^2 \middle| \mathcal{F}_{(N-1)M}\right]$$

and we trivially lower bound this quantity by taking the term corresponding to $\ell = q-1$. In this situation the sum $\sum_{j=(\ell+1)\vee d}^{q}$ is reduced to the term corresponding to $j = q$, and since $a_{q,q-1,k} = q^2 a_q \sigma_{Z_k}^{q-1}$ we obtain:

$$A_m \geq \mathbf{E}\left[\left(\sum_{k=(N-1)M}^{m} q^2\, a_q\, \sigma_{Z_k}^{q-1}\, H_{q-1}(\widetilde{Z}_k)\, \psi_{m-k}\right)^2 \middle| \mathcal{F}_{(N-1)M}\right]$$

$$= q^4\, a_q^2\, \mathbf{E}\left[\left(\sum_{k=(N-1)M}^{m} \sigma_{Z_k}^{q-1}\, H_{q-1}(\widetilde{Z}_k)\, \psi_{m-k}\right)^2\right].$$

We now invoke the identity $\mathbf{E}[H_p(\tilde{Z}_{k_1})H_p(\tilde{Z}_{k_2})] = \frac{1}{p!}(\mathbf{E}[\tilde{Z}_{k_1}\tilde{Z}_{k_2}])^p$ in order to obtain

$$A_m \geq \frac{q^5\, a_q^2}{q!} \sum_{k_1,k_2=(N-1)M}^{m} \sigma_{Z_{k_1}}^{q-1}\, \sigma_{Z_{k_2}}^{q-1}\, \mathbf{E}\left[\widetilde{Z}_{k_1}\, \widetilde{Z}_{k_2}\right]^{q-1} \psi_{m-k_1}\psi_{m-k_2}.$$

Furthermore, it is readily checked that:

$$\mathbf{E}\left[\widetilde{Z}_{k_1}\,\widetilde{Z}_{k_2}\right] = \frac{1}{\sigma_{Z_{k_1}}\,\sigma_{Z_{k_2}}} \sum_{i=(N-1)M}^{k_1 \wedge k_2} \psi_{k_1-i}\,\psi_{k_2-i},$$

and thus

$$
\begin{aligned}
A_m &\geq \frac{q^5\, a_q^2}{q!} \sum_{k_1,k_2=(N-1)M}^{m} \left(\sum_{i=(N-1)M}^{k_1 \wedge k_2} \psi_{k_1-i}\, \psi_{k_2-i} \right)^{q-1} \psi_{m-k_1} \psi_{m-k_2} \\
&= \frac{q^5\, a_q^2}{q!} \sum_{i_1,\dots,i_{q-1}=(N-1)M}^{m} \sum_{k_1,k_2=\max(i_1,\dots,i_{q-1})}^{m} \psi_{m-k_1} \psi_{m-k_2} \prod_{j=1}^{q-1} \psi_{k_1-i_j}\, \psi_{k_2-i_j} \\
&= \frac{q^5\, a_q^2}{q!} \sum_{i_1,\dots,i_{q-1}=(N-1)M}^{m} \left(\sum_{k=\max(i_1,\dots,i_{q-1})}^{m} \psi_{m-k} \prod_{j=1}^{q-1} \psi_{k-i_j} \right)^2 .
\end{aligned}
$$

Here again, this sum of squares is trivially lower bounded by taking the term corresponding to $i_1 = \cdots = i_{q-1} = m$, which yields:

$$
A_m \geq c_{a,q,\psi} \quad \text{with} \quad c_{a,q,\psi} \equiv \frac{q^5\, a_q^2}{q!}\, \psi_0^{2q} > 0.
$$

Step 5: Conclusion. Recalling that N is a given integer whose exact value will be fixed below, we get:

$$\mathbf{E}\left[B_n^{N-1}|\mathcal{F}_{(N-1)M}\right] \geq \frac{M\, c_{a,q,\psi}}{n} \geq c_{a,q,\psi,N} > 0,$$

as long as N stays bounded. We then get:

$$\begin{aligned} & \mathbf{P}\left(B_n^{N-1} \leq x|\mathcal{F}_{(N-1)M}\right) \\ \leq \quad & 1 + \frac{p\, c_{a,q,\psi,N}}{2N} \int_0^1 x^{\frac{1}{2(q-1)} - \frac{p}{2N} - 1} dx = c_{a,q,\psi,N,p} < \infty, \end{aligned}$$

where we have chosen N such that $\frac{p}{2N} < \frac{1}{2(q-1)}$. Iterating this bound, we have thus obtained:

$$\mathbf{E}\left[(B_n)^{-\frac{p}{2}}\right] \leq c^N_{a,q,\psi,N,p},$$

which is a finite quantity.

Finally recall from Step 1 that $\mathbf{E}[(B_n)^{-\frac{p}{2}}] = \mathbf{E}[\|DV_n^{d,q}\|_{\mathfrak{H}}^{-p}]$, which finishes the proof.

THANKS