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# Stochastic De Giorgi Iteration and Regularity of SPDE

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## Stochastic Partial Differential Equation

$$\partial_t u = \nabla \cdot (A \nabla u) + f(t,x,u) + \sum_{i=1}^{\infty} g_i(t,x,u) \dot{w}_t^i$$

Assumptions:

(1) $\{w_i\}$ are independent Brownian motions;

(2) $A, f, g$ progressively measurable;

(3) $A$ uniformly elliptic;

(4) $f$ and $g$ at most linear growth in $u$;

Our emphasis is on the assumption that diffusion coefficients $A$ are progressively measrable, hence random.

## What we know from PDE Theory

(1) Homogeneous equation

$$\partial_t u = \nabla \cdot (A \nabla u).$$

We know that the fundamental solution $p(t, x, y)$ is Hölder continuous in $(t, x)$.

This is a classic result of John Nash. The method of proof: De Gorgi-Nash-Moser iteration.

(2) Inhomogeneous equation

$$\partial_t u = \nabla \cdot (A \nabla u) + h(t, x).$$

We have Duhamel's principle:

$$u(t, x) = \int_0^t ds \int_{\mathbb{R}} p(t - s, x, y) h(s, y) \, dy.$$

## Goal of the work

For the SPDE

$$\partial_t u = \nabla \cdot (A\nabla u) + f(t,x,u) + \sum_{i=1}^{\infty} g_i(t,x,u)\dot{w}_t^i,$$

we want to show

(1) the solution is almost surely Hölder in $(t,x)$;

(2) moment estimates for the solution.

**Theorem.** There is an $\alpha \in (0,1)$ such that for any $T > 0$, the solution $u \in C^{\alpha}([T,2T]\times\mathbb{R}^n)$ almost surely. Furthermore, for every $p > 0$, there is a constant $C$ such that

$$\mathbb{E}\|u\|^p_{C^{\alpha}([T,2T]\times\mathbb{R}^n)} \le C\left(\|u_0\|_{L^2(\mathbb{R}^n)} + 1\right)^p.$$

## Problem with an Obvious Approach

The fundamental solution $p(t, x, y)$ of the homogeneous equation

$$\partial_t u = \nabla \cdot (A \nabla u)$$

is progressively measurable and Hölder continuous in $(t, x)$. Applying Duhamel's principle to the (simplified) inhomogeneous equation

$$\partial_t u = \nabla \cdot (A \nabla u) + g(t, x, u) \dot{w}_t,$$

we have

$$u(t, x) = \int_0^t \left[ \int_{\mathbb{R}} p(t - s, x, y) g(s, y, u(y, s))\, dy \right] dw_s.$$

This is NOT an adapted Itô stochastic integral because $p(t - s, x, y)$ is not adapted to the filtration $\mathscr{F}_s$.

A different approach is needed.

## Our Approach

Recall the equation again:

$$\partial_t u = \nabla \cdot (A \nabla u) + f(t,x,u) + \sum_{i=1}^{\infty} g_i(t,x,u) \dot{w}_t^i$$

To prove Hölder continuity for the solution, we use the following steps:

(1) A stochastic De Giorgi iteration to obtain moment estimate for $\|u\|_{\infty,[T,2T]\times\mathbb{R}^n}$;

(2) Krylov's trick to eliminate Brownian motions from the SPDE. This trick reduces Hölder estimates to moment estimates for the random variables $\|u\|_{\infty,[T,2T]\times\mathbb{R}^n}$;

(3) From uniform boundedness to Hölder continuity is a well known result from PDE theory.

Our contribution is (1).

## Krylov's Trick

This is a very nice trip for SPDE. Consider the two equations:

$$\partial_t u = \nabla \cdot (A\nabla u) + f(t,x,u) + \sum_{i=1}^{\infty} g_i(t,x,u)\dot{w}_t^i$$

$$\partial_t v = \Delta v + \sum_{i=1}^{\infty} g_i(t,x,u)\dot{w}_t^i.$$

(1) $v$ has constant diffusion coefficients and can be handled by Duhamel's principle if we have good moment bounds for $\|u\|_{\infty,[T,2T]\times\mathbb{R}^n}$. Thus we reduce the problem to moment estimates for $u$.

(2) Let $\phi = u - v$. We find that it satisfies the equation.

$$\partial_t \phi = \nabla \cdot (A\nabla \phi) + f(\phi + v) + \nabla \cdot (A\nabla v) - \Delta v.$$

It has no Brownian motion terms. This can be handled path by path.

Thus everything reduces to moment estimates for $\|u\|_{\infty,[T,2T]\times\mathbb{R}^n}$.

## What is the De Giorgi Iteration?

For divergence form elliptic (or parabolic) equations, it is easy to obtain $L^p$ estimates by functional analysis, but not easy to get $L^\infty$ estimates, i.e., proving the solution is uniformly bounded.

De Giorgi's iteration is a method to go from $L^p$ to $L^\infty$. It is usually the first step towards smoothness of solutions. Say that we want to show that $u$ is bounded from above by $B$ on $[T, 2T]$. We let $\{I_k\}$ be a sequence of intervals shrinking to $[T, 2T]$ and $B_k$ a sequence increasing to $B$ (say $B_k = (1 - 2^{-(k+1)})B$). Let

$$U_k = \left[\int_{I_k} \|(u(t) - B_k)^+\|_2^4 \, dt\right]^{1/2}.$$

De Giorgi's iteration is (by Sobolev inequalities and uniform ellipticity and proper choices of $I_k$ and $B_k$)

$$U_{k+1} \le C^k U_k^{1+\lambda}$$

for some positive constants $C$ and $\lambda$. In our case $\lambda = 1/(n+1)$.

## De Giorgi iteration (continued)

Recall the iterative inequality $U_{k+1} \le C^k U_k^{1+\lambda}$ for some positive constants $C$ and $\lambda$.

**Lemma.** If $U_0$ is sufficiently small, then $U_k \le K r^k$ for some $K$ and $0 < r < 1$.

If $B$ is siffuciently big, then from

$$U_0 = \left[ \int_{I_k} \|(u(t) - B_0)^+\|_2^4 \, dt \right]^{1/2}$$

and $B_0 = B/2$ we know that $U_0$ is small enough (or $B$ is big enough). Hence $U_k \to 0$, which means

$$\int_T^{2T} \|(u(t) - B)^+\|_2^4 \, dt = 0.$$

This shows that $u \le A$. By symmetry, we can show $\|u\|_{\infty,[T,2T]\times\mathbb{R}^n} \le B$.

## Stochastic De Giorgi Iteration

For the SPDE

$$\partial_t u = \nabla \cdot (A\nabla u) + f(t,x,u) + \sum_{i=1}^{\infty} g_i(t,x,u)\dot{w}_t^i,$$

we need to allow a stochastic error term in the usual De Giorgi iteration

$$U_{k+1} \le C^k U_k^{1+\lambda}.$$

*Stochastic De Giorgi iteration*

$$U_{k+1} \le C^k (U_k + X_k) U_k^{\lambda}.$$

The random variable $X_k$ can be computed explicitly and it turns out that it is the maximum process of a martingale whose quadratic variation is controled by $U_k$. This means that $X_k$ can be controled by $U_k$.

**Lemma.** Let $\{M_t\}$ be a local martingale, then

$$\mathbb{P}\left\{ \max_{0\le t\le T} M_t \ge a, \quad \langle M\rangle_t \le b \right\} \le e^{-a^2/2b}.$$

## Key Estimate

Recall the stochastic De Giorgi estimate

$$U_{k+1} \leq C^k (U_k + X_k) U_k^{\lambda}.$$

We need to show that $X_k$ is comparable with $U_k$. It turns out that all we need to show is that when $X_k$ is big, $U_k$ cannot be small.

**Proposition.** There is a constant $C$ such that for any positive $\alpha$ and $\beta$ we have

$$\mathbb{P}\left\{X_k \geq \alpha\beta, \quad U_k \leq \beta\right\} \leq e^{-\alpha^2/C^k}.$$

By a good choice of $\alpha$ and $\beta$ depending on $k$, we can use a Borel-Cantelli type argument to prove the following comparison result.

**Proposition.** There is a constant $M_0$ such that for all $M \geq M_0$ we have

$$\mathbb{P}\left\{\|u^+\|_{\infty,[T,2T]\times\mathbb{R}^n} > a,\ MU_0 \leq a\right\} \leq e^{-M^{1/(n+1)}}.$$

## Key Estimates (continued)

We see that in a probabilistic sense, we have bounded $\|u\|_\infty$ by $\|u\|_2$. An easy argument shows that

$$\mathbb{E}\|u\|^p_{\infty,[T,2T]\times\mathbb{R}^n} \leq C_p \mathbb{E} U_0^p.$$

Recall that

$$U_0^2 = \int_{I_0} \|u(t)^+\|_2^4 \, dt.$$

Its moments can be estimated by simple $L_p$ theory.

This way we have shown that $\|u\|_{\infty,[T,2T]\times\mathbb{R}^n}$ has finite moments.

## Recapitulation

Stochastic partial differential equation is

$$\partial_t u = \nabla \cdot (A \nabla u) + f(t,x,u) + \sum_{i=1}^{\infty} g_i(t,x,u)\dot{w}_t^i.$$

Compare with the equation

$$\partial_t v = \Delta v + \sum_{i=1}^{\infty} g_i(t,x,u)\dot{w}_t^i.$$

(1) The solution of $v$ is Hölder continuous if we have bounds for $\|u\|_\infty$ because it has constant diffusion coefficients. (2) The difference $\phi = u - v$ satisfies the equation

$$\partial_t \phi = \nabla \cdot (A \nabla \phi) + f(\phi + v) + \nabla \cdot (A \nabla v) - \Delta v.$$

It is Hölder continuous pathwise by PDE theory. (3) We use a stcohastic De Giorgi iteration to show that $\|u\|_\infty$ can be bounded if $\|u\|_2$ can be bounded. (4) $\|u\|_2$ can be bounded by general $L_p$-theory.

**Thank You!**