# Limit theory of pruning processes for Galton-Watson trees

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# Aldous and Pitman (1998, AIHP): uniform pruning at edges

Given a tree $\mathcal{G}$,

- Associate each edge an uniformly distributed r.v. $\xi_e$ on $[0,1]$.
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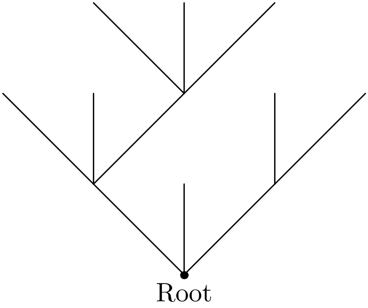

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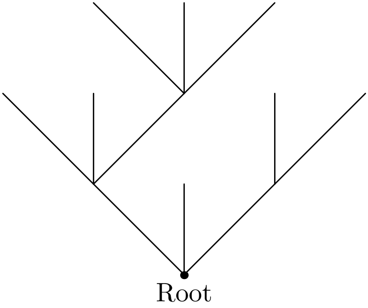

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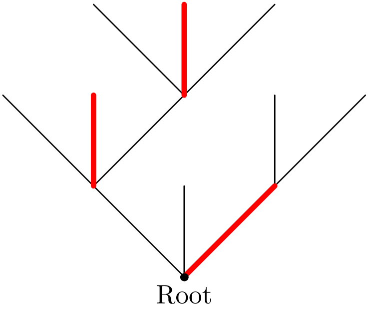

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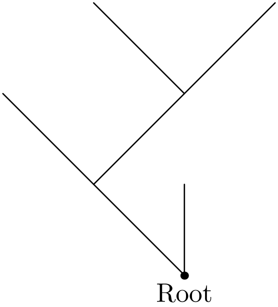

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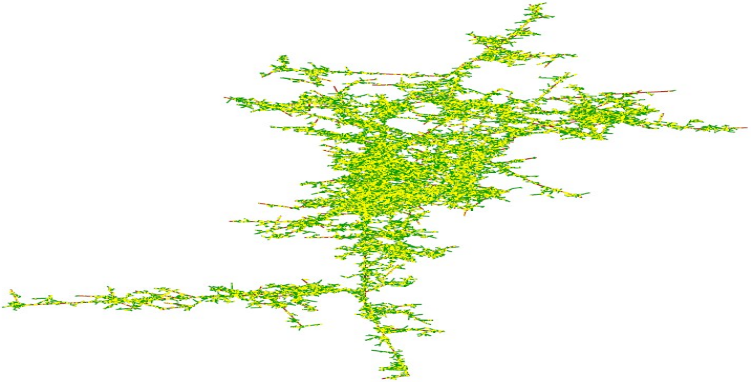
# Lévy trees

Roughly, a Lévy tree is a random locally compact metric space without loops, locally isometric to the real line.



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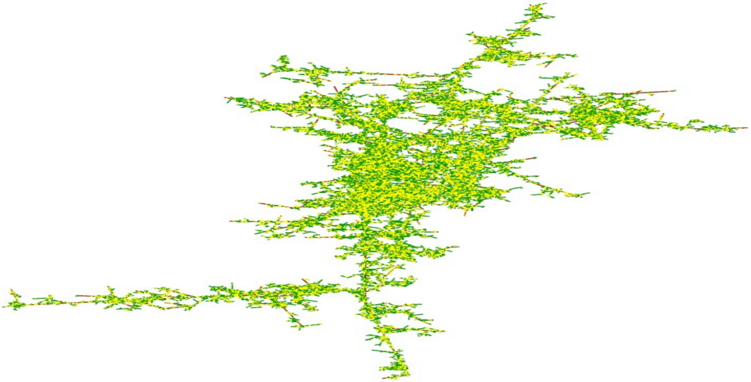

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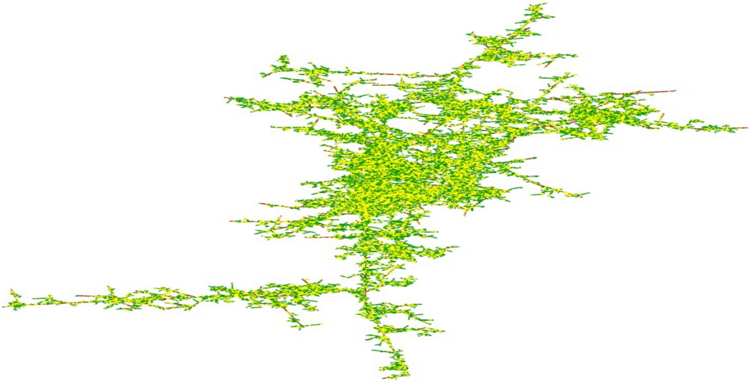

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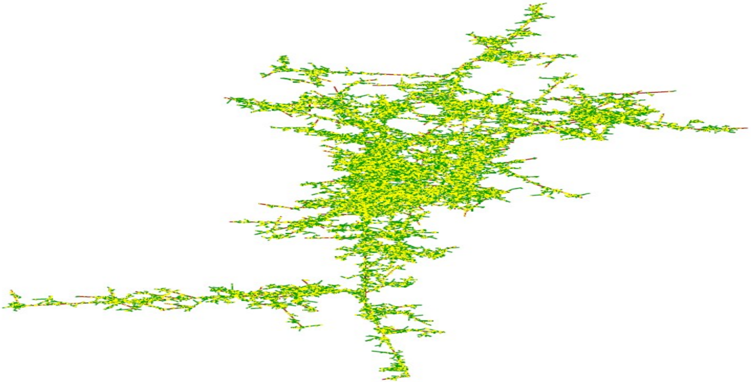

# Lévy trees: Scaling limits of GW trees

Aldous (90, 92, AOP); Le Gall and Le Jan (98a,b, AOP); Duquesne and Le Gall (02,Asterisque):

**Theorem**

*Under some regular conditions, for* $\gamma_k \to \infty$,

$$\frac{1}{\gamma_k}\mathcal{G}^{(k)} \longrightarrow a \text{ Lévy tree}.$$

$$\mathcal{G}^{(k)} = \bigcup k \text{ independent GW trees.}$$

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# Aldous and Pitman's question

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How to generalize Aldous and Pitman's pruning of GW trees to Lévy trees?

- In Aldous and Pitman (1998, AOP), it says that *"In the CRT, the analog of deleting randomly chosen edges is to cut the skeleton by a Poisson process of cuts with some rate per unit length."*
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# Poissonian marks

Given a Lévy tree $\mathcal{T}$

- At time $\theta_i$, there is a drop of sulfuric acid falling on the tree at $x_i \in \mathcal{T}$.
- We cut the tree at $x_i$.
- $\{\theta_1 < \theta_2 < \cdots,\}$ is a Poisson process and $\{x_i, i = 1, 2, \cdots\}$ are uniformly distributed on $\mathcal{T}$.
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# Answer to Aldous and Pitman's question

- Recall $\mathcal{G}^{(k)} = \bigcup k$ independent GW trees.
- $\{\mathcal{G}^{(k)}(u) : 0 \le u \le 1\}$: Aldous and Pitman's pruning process.

**Theorem (H. and Winkel (2014+))**

*If*

$$\frac{1}{\gamma_k}\mathcal{G}^{(k)} \longrightarrow a \text{ Lévy tree } \mathcal{T},$$

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# Pruning at branching points

- A motivation of pruning trees is to study the fragmentation and coalescent; see Evans and Pitman (1998, AIHP); Aldous and Pitman (1998, AOP) for pruning Brownian trees (Poissonian marks).

- Miermont (2003, PTRF) introduced the pruning at branching points of stable trees to obtain some new fragmentations.

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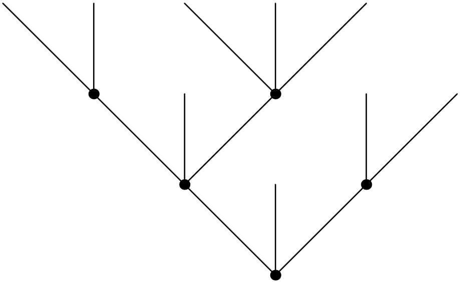
For a branching point $v$ in $\mathcal{G}$,

- associate an r.v. $\xi_\nu$ with $P(\xi_v \leq u) = u^{n(v)-1}, \quad 0 \leq u \leq 1$, $n(v)$ the number of children of $v$.

- For $0 \leq u \leq 1$, if $\xi_v > u$, then we mark it and remove all its offsprings (sons, grandsons,$\cdots$).



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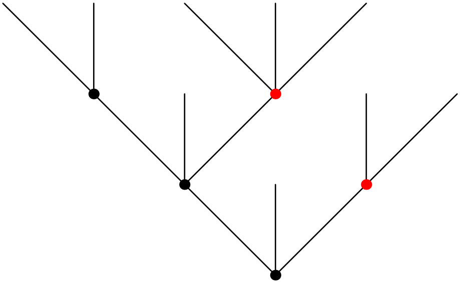

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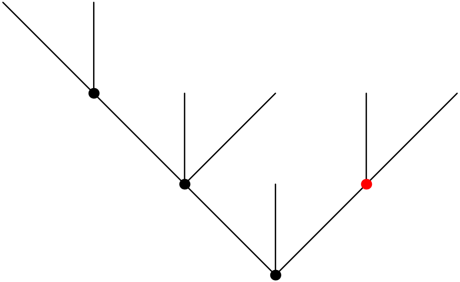

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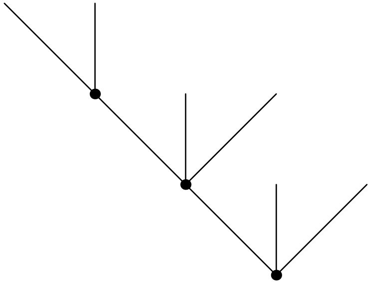

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# Abraham and Delmas (2012, AOP): Pruning process of Lévy trees

In Abraham and Delmas (2012, AOP), a general pruning= Poissonian marks+pruning at branching points.

It says that
*"The CRT-valued Markov process constructed here can be viewed as the continuous analog of the discrete models of [11] and [4] (or maybe a mixture of both constructions). However, no link is actually pointed out between the discrete and the continuous frameworks."*

[4] Abraham, Delmas and He (2012, AIHP);

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# Our answer

Let $\{\mathcal{G}^{(k)}(u) : 0 \le u \le 1\}$ be the pruning process obtained by pruning $\mathcal{G}^{(k)}$ at branching points.

**Theorem (H. and Winkel (2014+))**

*If* $\frac{1}{\gamma_k}\mathcal{G}^{(k)} \longrightarrow$ *a Lévy tree* $\mathcal{T}$*, then*

$$\{\frac{1}{\gamma_k}\mathcal{G}^{(k)}(e^{-\theta/k}) : 0 \le \theta < \infty\} \longrightarrow \{\mathcal{T}(\theta) : 0 \le \theta < \infty\},$$

*in* $D([0,\infty), \mathbb{T})$.

- $\mathcal{T}(\theta)$ is the pruning process constructed in Abraham and Delmas (2012, AOP).
- We give an answer to the problem in Abraham and Delmas (2012, AOP).

# References


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# Thanks!