

### The Critical Surfaces of Epidemic Spread on a Random Growth Network

Dong HAN

Tze Leung LAI

Shanghai Jiao Tong Univ.

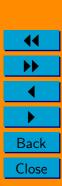
Stanford Univ.

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Outline

- 1. Motivation
- 2. Degree Distribution of a Random Growth Network
- 3. The Critical Surface of Epidemic Spread





## 1. Motivation

Topological Structure of Networks.

• Classical Random Graphs (Erdös-Rényi (1959)): For  $p_n = \frac{c}{n}$ ,

$$P_k = e^{-c} \frac{c^k}{k!}.$$
(1)

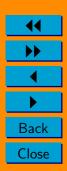
Complex Networks

Small-world Networks (Watts and Strogatz(1998)) Scale-free Netorks (Barabási and Albert(1999)): For  $p_i = \frac{d_i}{\sum_i d_i}$ ,

$$P_k = \frac{4}{k(k+1)(k+2)} \sim \frac{4}{k^{\tau}}$$
(2)

where  $\tau = 3$ .

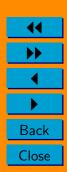
Here  $P_k$  is the limit probability that a node has k degree in the random graphs when n nodes goes to infinity.



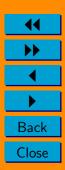
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There were many works on the random processes taking place on complex networks with power law degree distribution.

- Epidemic spreading in scale-free networks (Pastor and Vespignani (2001),  $\lambda_c = \frac{\sum_{k=1}^{\infty} kP_k}{\sum_{k=1}^{\infty} k^2 P_k} = 0$  for  $2 < \tau \leq 3$  and  $\lambda_c > 0$  for  $\tau > 3$ )
- Virtual Round Table on ten leading questions for network research ( Amaral, etc. (2004))
- Random walks on complex networks (Noh and Rieger (2006))
- Conservation laws for the voter model in complex networks (Suchecki, Eguíluz and Miguel (2006))
- Contact processes on random graphs with power law degree distributions have critical value 0 (Chatterjee and Durrett (2009))
- Contact processes on scale-free networks (Chen and Liu (2010))



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- Some features of the spread of epidemics and information on a random graph (Durrrett (2010))
- Epidemic spread in networks: Existing methods and current challenges (Miller and Kiss (2014))



The interacting graph-valued Markov processes can be used to describe the interaction between a random dynamic network and a random dynamic process taking place on the network.

• Let  $x = (x_{ij})$  denote a network.

• Denoted by  $D_k(x) = \sharp\{i : x_i = k\}$  the number of the nodes with degree k, where  $x_i$  denotes the degree of node i in the network x.

Back Close Network Growth: At every one-step we add a new node which has no virus and one edge that links the new node to the node i with probability proportional to a function  $[\alpha(1-y_i) + \beta y_i](x_i + \theta) \wedge m$  which depends on the degree  $x_i$  and the virus  $(y_i = 1)$  or no virus  $(y_i = 0)$ at note i, where the two nonnegative numbers  $\alpha$  and  $\beta$  denote the intensity of connecting an edge to node without and with epidemic disease respectively, the nonnegative number  $\theta$  represents initial attractiveness when the degree,  $x_i$ , of node i is zero, m denotes that the degree of node i is at most m.

*Epidemic Dynamics:* The virus spreading on the evolving network considered here is the susceptible-infected-susceptible (SIS) model in which each susceptible node i becomes infected and therefore has a virus with the rate of the epidemic spreading  $\lambda > 0$  if at least one of neighbors  $\{j : x_{ij} = 1\}$  has the virus. Infected nodes, on the other hand, recover and become susceptible again with the rate  $\gamma > 0$ .

Back Close

#### **Two Problems**

• Degree distribution.

• The critical surface (value) of epidemic spread

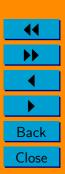




# 2. Degree distribution

Let  $X(t) = (X_{ij}(t))$  be adjacency matrix of the evolving network at time t, which describes the network growing in the environment of virus spreading.

 $Y(t) = \Big(Y_i(t), i \ge 1\Big): \text{ describes the virus spreading in the growth network, where } Y_i(t) = 1 \text{ means that the node } i \text{ has a virus, otherwise } Y_i(t) = 0 \text{ at } t$ .



The disturbed network growth process considered here is a continuoustime Markov chain Z(t) = (X(t), Y(t)) with the following one-step jump probabilities:

 $q(z, z') = \begin{cases} \frac{[\alpha(1-y_i)+\beta y_i](x_i+\theta)\wedge m}{S(z)} & \text{if } z' = z + (e_{i,n(x)+1}, \ 0) \\ \frac{\sum_{j=1}^{n(x)} \lambda(1-y_i)x_{ij}y_j+\gamma y_i}{S(z)} & \text{if } z' = z + (0, \ 1-y_i) \\ 0 & \text{otherwise} \end{cases}$ 

where z = (x, y), z' = (x', y'), both nonnegative numbers  $\alpha$  and  $\beta$  satisfying  $\alpha + \beta > 0$ , denote the rates of connecting the infected node and healthy node respectively, and  $S(z) = S_1(z) + S_2(z)$  is normalization factor.

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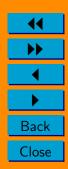
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(I).  
$$\lim_{t \to \infty} \frac{\mathbb{E}_z(S_1(Z(t)))}{t} = s_1 > 0, \quad \lim_{t \to \infty} \frac{\mathbb{E}_z(S_2(Z(t)))}{t} = s_2 \ge 0$$

(II). For every 
$$k \ge 1$$
,  
$$\lim_{t \to \infty} \mathbb{E}_z \Big( \sum_{j=1}^{n(X(t))} \frac{X_{ij}(t)}{X_i(t)} Y_j(t) \Big) = \rho_m$$

The probability that a link from a node to an infected node.

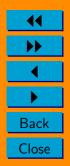
(III). The degrees of any two nodes are asymptoticly independent  $(t 
ightarrow \infty)$ 





**Theorem 1** Let  $s = s_1 + s_2$  and  $p = s_1/s$ . Then

$$\begin{split} &\lim_{t\to\infty}\frac{S_1(Z(t))}{t}=s_1, \quad \lim_{t\to\infty}\frac{S_2(Z(t))}{t}=s_2\\ &\lim_{t\to\infty}\frac{n(X(t))}{t}=p, \quad \text{ a.s. } -\mathbb{P}_z \end{split}$$



**Proof of Theorem 1.** For any fixed t > 0, the stochastic process  $M(s) = \mathbb{E}_{z}[S_{1}(Z(t))|\sigma_{s}]$  is a martingale for  $0 \leq s \leq t$ , and therefore,  $\mathbb{E}_{z}(M(s_{4}) - M(s_{3}))(M(s_{2}) - M(s_{1})) = 0$  for  $0 \leq s_{1} < s_{2} \leq s_{3} < s_{4} \leq t$ . Then, we have

$$\begin{split} \mathbb{E}_{z}[S_{1}(Z(t)) &- \mathbb{E}_{z}(S_{1}(Z(t)))]^{2} \\ &= \mathbb{E}_{z}[M(t) - M(\lceil t \rceil) - \sum_{k=1}^{\lceil t \rceil} (M(k) - M(k-1))]^{2} \\ &= \mathbb{E}_{z}(M(t) - M(\lceil t \rceil))^{2} + \sum_{k=1}^{\lceil t \rceil} \mathbb{E}_{z}(M(k) - M(k-1))^{2}. \end{split}$$



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#### Since

 $|M(k) - M(k-1)| \le 2 \max\{\alpha, \beta, m\} N(1)$ 

for  $1 \leq k \leq \lceil t \rceil$ , it follows that

$$\begin{split} & \mathbb{E}_{z}[S_{1}(Z(t)) - \mathbb{E}_{z}(S_{1}(Z(t)))]^{2} \\ & \leq 4(\max\{\alpha, \beta, m\})^{2} \mathbb{E}_{z} N^{2}(t - \lceil t \rceil) + \lceil t \rceil \mathbb{E}_{z} N^{2}(1) \\ & = 8(\max\{\alpha, \beta, m\})^{2} t. \end{split}$$

and therefore

$$\lim_{t\to\infty}\frac{S_1(Z(t))}{t}=s_1, \ \text{a.s. -}\mathbb{P}_z.$$



(3)

 $D_k(x) = \sum_{i=1}^{n(x)} I_k(x_i)$ , number of nodes with degree k.

 $E_k(z) = \sum_{i=1}^{n(x)} y_i I_k(x_i)$ , number of infected nodes with degree k.

 $P_k$ , probability that a node has degree k.

 $Q_k$ , probability that a node has degree k and is infected.



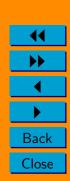


**Theorem 2** Let  $s = s_1 + s_2$  and  $W_m(x) = (x + \theta) \wedge m$ .  $P_k$  and  $Q_k$  can be expressed in the following vector form:

$$(P_k, Q_k)^T = \lim_{t \to \infty} \left(\frac{\mathbb{E}_z[D_k(X(t))]}{N(X(t))}, \frac{\mathbb{E}_z[E_k(Z(t))]}{N(X(t))}\right)^T$$
  
=  $(A(k) + I)^{-1} \left[\prod_{i=1}^{k-1} B(i)(A(i) + I)^{-1}\right] (s_1/s, 0)^T$ 

a.s. - $\mathbb{P}_z$ , where

 $A(k) = s^{-1} \begin{pmatrix} \alpha W_m(k) & (\beta - \alpha) W_m(k) \\ -\lambda(m \wedge k) \rho_m & \beta W_m(k) + \lambda(m \wedge k) \rho_m + \gamma \end{pmatrix}$  $B(k) = s^{-1} \begin{pmatrix} \alpha W_m(k) & (\beta - \alpha) W_m(k) \\ 0 & \beta W_m(k) \end{pmatrix}$ 



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**Corollary 1** If  $\alpha = \beta$ , then

$$P_k = \frac{\alpha W(k-1)}{\alpha W(k) + s} P_{k-1} = \frac{s}{\alpha W(k) + s} \prod_{i=1}^{k-1} \frac{\alpha W(i)}{\alpha W(i) + s} \sim Ck^{-3}$$



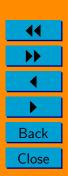


Corollary 2 If  $\beta=0$  and for large  $\lambda$  such that  $\rho_m(\lambda)\geq\rho>0$  we have

$$P_k \sim C_1 A^k k^{-B}$$

where

$$A = \frac{\gamma + s}{\gamma + s + 2\lambda\rho}, \qquad B = \frac{2(\gamma + s)}{\gamma + s + 2\lambda\rho}$$



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**Proof of Theorem 2** Let  $D_k(t) = \mathbb{E}_z[D_k(X(t))]$  and  $E_k(t) = \mathbb{E}_z[E_k(Z(t))]$ . Both  $D'_k(t)$  and  $E'_k(t)$  can be written

$$D'_{k}(t) = \alpha W_{m}(k-1) \frac{D_{k-1}(t)}{st} - \alpha W_{m}(k) \frac{D_{k}(t)}{st} + (\beta - \alpha) W_{m}(k-1) \frac{E_{k-1}(t)}{st} - (\beta - \alpha) W_{m}(k) \frac{E_{k}(t)}{st} + \delta_{k1} p + \epsilon_{k}(t)$$
(4)

and

$$E'_{k}(t) = \beta W_{m}(k-1) \frac{E_{k-1}(t)}{st} - \beta W_{m}(k) \frac{E_{k}(t)}{st} - \frac{\gamma E_{k}(t)}{st} + \lambda (m \wedge k) \rho_{m} \frac{D_{k}(t) - E_{k}(t)}{st} + e_{k}(t)$$

$$(5)$$

for large t, where  $\epsilon_k(t) \to 0$  and  $e_k(t) \to 0$  as  $t \to \infty$  for all  $k \ge 1$ .



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Let  $U_k(t) = (D_k(t), E_k(t))^T$ ,  $\Xi_k(t) = (\epsilon_k(t), e_k(t))^T$  and  $P_{k1} = (\delta_{k1}p, 0)^T$ . We can rewrite the above two equations in the matrix form

$$U_k'(t) = B(k-1)\frac{U_{k-1}(t)}{t} - A(k)\frac{U_k(t)}{t} + P_{k1} + \Xi_k(t)$$
(6)

Note that

$$e^{A\log t} = \sum_{i=0}^{\infty} \frac{(A\log t)^i}{i!}, \quad e^{-I\log t} = \sum_{i=0}^{\infty} \frac{(-I\log t)^i}{i!} = \frac{1}{t}I$$

where A is a matrix and I is the unit matrix.



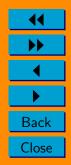


### It follows that

$$\lim_{t \to \infty} \frac{U_k(t)}{t} = \prod_{j=1}^k \left[ (A(j) + I)^{-1} [B(j-1)(A(j-1) + I)^{-1}] \right] (p,0)^T$$

for  $k \ge 1$ . Thus, we have

$$\begin{aligned} (P_k, \ Q_k)^T &= \lim_{t \to \infty} (\frac{D_k(X(t))}{n(X(t))}, \ \frac{E_k(Z(t))}{n(X(t))})^T \\ &= (A(k) + I)^{-1} [\prod_{i=1}^{k-1} B(i)(A(i) + I)^{-1}] (1, \ 0)^T, \quad \text{ a.s. } -\mathbb{P}_z \end{aligned}$$



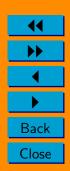
# 3. The Critical surface of Epidemic Spread

Note that the number  $\rho_m = \rho_m(\alpha, \beta, \gamma, \theta, \lambda, )$  is dependent on the five parameters,  $\alpha, \beta, \gamma, \theta$  and  $\lambda$ . Now we define a critical value  $\lambda_c(m)$  for every  $m \ge 1$  in the following.

**Definition.** For fixed  $\alpha, \beta, \gamma$  and  $\theta$ , the epidemic critical value  $\lambda_c(m) = \lambda_c(\alpha, \beta, \gamma, \theta, m)$  for  $m \ge 1$  is defined by

 $\lambda_c(m) = \inf\{\lambda > 0 : \rho_m(\alpha, \beta, \gamma, \theta, \lambda) > 0\}.$ 

The critical value means that if  $\lambda(m) > \lambda_c(m)$ , the infection spreads and becomes endemic. Below it, i.e.,  $\lambda(m) < \lambda_c(m)$ , the infection dies out finally ( $\rho_m = 0$ ). The function  $\lambda_c(\alpha, \beta, \gamma, \theta, m)$  on  $\alpha, \beta, \gamma$  and  $\theta$ can be seen as the critical surface for any fixed m.

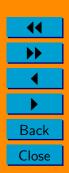


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**Theorem 3** Let  $\Lambda_m = (\alpha \mu_m + \gamma)(1 + \sigma_m) - \beta \delta_m$ . If  $\lim_{\lambda \searrow \lambda_c(m)} \rho_m(\alpha, \beta, \gamma, \theta, \lambda) = 0$ , then the critical value $\lambda_c(m)$  can be expressed as

$$\lambda_c(m) = \begin{cases} \frac{\Lambda_m \sum_{k=1}^{\infty} (m \wedge k) P_k}{\sum_{k=1}^{\infty} k (m \wedge k) P_k} & \text{if } \Lambda_m > 0\\ 0 & \text{if } \Lambda_m \le 0. \end{cases}$$



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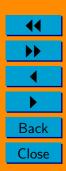
$$\sigma_m = \frac{\sum_{k\geq m}^{\infty} (k-m)q_m(k)}{\sum_{k=1}^{\infty} (m\wedge k)q_m(k)}, \qquad \delta_m = \frac{\sum_{k=1}^{\infty} W_m(k)q_m(k)}{\sum_{k=1}^{\infty} (m\wedge k)q_m(k)}$$

and for large k, where

$$q_m(k) \sim \begin{cases} \frac{A_k(\nu)}{(k+\theta)^{1+\nu}} & \text{if } k \le m \\ \frac{A_k(\nu)(1-\frac{\nu}{m})^{k-m}}{(m+\theta)^{1+\nu}} + \frac{P_{k-1}}{\beta} \sum_{j=m+1}^k \frac{P_{j-1}}{P_{k-1}} (1-\frac{\nu}{m})^{k-j} & \text{if } k > m \end{cases}$$

for large k, where

$$A_k(\nu) = \begin{cases} A(\nu) = \frac{1}{\alpha(2+\theta)f_1} + \frac{1}{\beta(1+\theta-\nu)} & \text{if } \nu < 1+\theta \\ \frac{1}{\alpha(2+\theta)f_1} + \frac{\ln k}{\beta} & \text{if } \nu = 1+\theta \\ \frac{1}{\alpha(2+\theta)f_1} + \frac{k^{\nu-1-\theta}}{\beta(\nu-1-\theta)} & \text{if } \nu > 1+\theta. \end{cases}$$





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**Proof of Theorem 3.** Let  $f(Z(t)) = \sum_{i=1}^{n(X(t))} Y_i(t)X_i(t)$  and  $f_m(Z(t)) = \sum_{i=1}^{n(X(t))} Y_i(t)W_m(X_i(t))$ . It follows that

$$\begin{split} \mathbb{E}_{z}(f(Z(t))) - f(z) &= \int_{0}^{t} \left( \beta \frac{\mathbb{E}_{z}(f_{m}(Z(u)))}{su} - \gamma \frac{\mathbb{E}_{z}(f(Z(u)))}{su} - \gamma \frac{\mathbb{E}_{z}(f(Z(u)))}{su} + \lambda \rho_{m} \frac{\mathbb{E}_{z}[\sum_{i=1}^{n(X(u))} (1 - Y_{i}(u))X_{i}(u)(X_{i}(u) \wedge m)]}{su} \right] \end{split}$$

where  $\epsilon(t) \to 0$  as  $t \to \infty$ . We can further prove that

$$(s+\gamma)\sum_{k=1}^{\infty}(k\wedge m)Q_k + (s+\gamma)\sum_{k\geq m}^{\infty}(k-m)Q_k - \beta\sum_{k=1}^{\infty}W_m(k)Q_k$$
$$= \lambda\rho_m\sum_{k=1}^{\infty}k(k\wedge m)(P_k - Q_k).$$

Note that

$$\rho_m = \frac{\sum_{k=1}^{\infty} kQ_k}{\sum_{k=1}^{\infty} kP_k}.$$

Thus

$$[(s+\gamma)(1+\sigma_m)-\beta\delta_m]\sum_{k=1}^{\infty}(k\wedge m)P_k=\lambda\sum_{k=1}^{\infty}k(k\wedge m)(P_k-Q_k).$$

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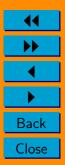
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**Corollary 3** If  $-1 < \theta \leq 0$  and  $\lim_{\lambda \searrow \lambda_c(m)} \rho_m(\alpha, \beta, \gamma, \theta, \lambda) = 0$  for all  $m \geq 1$ , then

$$\lim_{m \to \infty} \lambda_c(m) = \lambda_c = 0$$

for any fixed  $\alpha, \beta$  and  $\gamma$ . That is to say, the infection can spread and become endemic on the scale-free network with the power  $\tau = 3+\theta, 2 < \tau \leq 3$ , as long as there is a small rate of the epidemic spreading when the maximum degree m is large. This result was found first by Pastor-Satorras and Vespignani (2001).

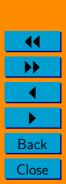


**Corollary 4** If  $\theta > 0, \alpha\beta > 0$ , and  $\lim_{\lambda \searrow \lambda_c(m)} \rho_m(\alpha, \beta, \gamma, \theta, \lambda) = 0$  for all  $m \ge 1$ , then

 $\lim_{m \to \infty} \lambda_c(m) = \begin{cases} 0 & \text{if } \beta \ge \alpha(2+\theta) + \gamma \\ \alpha(2+\theta) + \gamma - \beta & \text{if } \beta < \alpha(2+\theta) + \gamma. \end{cases}$ 

#### Let

 $S_c = \{ (\alpha, \beta, \gamma, \lambda) : \alpha(2 + \theta) + \gamma > \beta, \alpha\beta > 0, \gamma, \lambda \ge 0 \}$ 





Note that  $\sum_{k=1}^{\infty} kP_k = 2$ . Let  $\lambda_c = \lim_{m \to \infty} \lambda_c(m)$ ,

$$A = -2(2 + \theta), \quad B = 2, \quad C = -2, \quad D = \sum_{k=1}^{\infty} k^2 P_k.$$

 $\infty$ 

Then, we can write  $A\alpha + B\beta + C\gamma + D\lambda_c = 0$  as  $m \to \infty$  for  $\theta > 0$ . **Remark.** For a fixed  $\theta > 0$ , we can define the critical hyperplane (surface)  $\Gamma_c$  as follows

$$\Gamma_c = \{ (\alpha, \beta, \gamma, \lambda) \in S_c : A\alpha + B\beta + C\gamma + D\lambda = 0 \}$$

and

 $\Gamma^{+} = \{ (\alpha, \beta, \gamma, \lambda) \in S_{c} : A\alpha + B\beta + C\gamma + D\lambda > 0 \},$  $\Gamma^{-} = \{ (\alpha, \beta, \gamma, \lambda) \in S_{c} : A\alpha + B\beta + C\gamma + D\lambda < 0 \}.$ 

Thus, the infection spreads and becomes endemic for  $(\alpha, \beta, \gamma, \lambda) \in \Gamma^+$ , and the infection dies out finally for  $(\alpha, \beta, \gamma, \lambda) \in \Gamma^-$ .



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# Thank You !

