# The Critical Surfaces of Epidemic Spread on a Random Growth Network

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# Outline

1. **Motivation**

2. **Degree Distribution of a Random Growth Network**

3. **The Critical Surface of Epidemic Spread**

# 1. Motivation

Topological Structure of Networks.

- Classical Random Graphs (Erdös-Rényi (1959)): For $p_n = \frac{c}{n}$,

$$P_k = e^{-c}\frac{c^k}{k!}. \tag{1}$$

- Complex Networks

  Small-world Networks (Watts and Strogatz(1998))

  Scale-free Netorks (Barabási and Albert(1999)): For $p_i = \frac{d_i}{\sum_j d_j}$,

$$P_k = \frac{4}{k(k+1)(k+2)} \sim \frac{4}{k^\tau} \tag{2}$$

where $\tau = 3$.

Here $P_k$ is the limit probability that a node has $k$ degree in the random graphs when $n$ nodes goes to infinity.

There were many works on the random processes taking place on complex networks with power law degree distribution.

- Epidemic spreading in scale-free networks (Pastor and Vespignani (2001), $\lambda_c = \frac{\sum_{k=1}^{\infty} kP_k}{\sum_{k=1}^{\infty} k^2 P_k} = 0$ for $2 < \tau \leq 3$ and $\lambda_c > 0$ for $\tau > 3$ )
- Virtual Round Table on ten leading questions for network research ( Amaral, etc. (2004))
- Random walks on complex networks (Noh and Rieger (2006))
- Conservation laws for the voter model in complex networks (Suchecki, Eguíluz and Miguel (2006))
- Contact processes on random graphs with power law degree distributions have critical value $0$ (Chatterjee and Durrett (2009))
- Contact processes on scale-free networks (Chen and Liu (2010))

- Some features of the spread of epidemics and information on a random graph (Durrrett (2010))
- Epidemic spread in networks: Existing methods and current challenges (Miller and Kiss (2014))

The interacting graph-valued Markov processes can be used to describe the interaction between a random dynamic network and a random dynamic process taking place on the network.

- Let $x = (x_{ij})$ denote a network.
- Denoted by $D_k(x) = \sharp\{i : \; x_i = k\}$ the number of the nodes with degree $k$, where $x_i$ denotes the degree of node $i$ in the network $x$ .

*Network Growth:* At every one-step we add a new node which has no virus and one edge that links the new node to the node $i$ with probability proportional to a function $[\alpha(1-y_i)+\beta y_i](x_i+\theta)\wedge m$ which depends on the degree $x_i$ and the virus ($y_i = 1$ ) or no virus ( $y_i = 0$ ) at note $i$,where the two nonnegative numbers $\alpha$ and $\beta$ denote the intensity of connecting an edge to node without and with epidemic disease respectively, the nonnegative number $\theta$ represents initial attractiveness when the degree, $x_i$, of node $i$ is zero, $m$ denotes that the degree of node $i$ is at most $m$.

*Epidemic Dynamics:* The virus spreading on the evolving network considered here is the susceptible-infected-susceptible (SIS) model in which each susceptible node $i$ becomes infected and therefore has a virus with the rate of the epidemic spreading $\lambda > 0$ if at least one of neighbors $\{j : x_{ij} = 1\}$ has the virus. Infected nodes, on the other hand, recover and become susceptible again with the rate $\gamma > 0$.

## Two Problems

- **Degree distribution.**

- **The critical surface (value) of epidemic spread**

# 2. Degree distribution

Let $X(t) = \Big(X_{ij}(t)\Big)$ be adjacency matrix of the evolving network at time $t$, which describes the network growing in the environment of virus spreading.

$Y(t) = \Big(Y_i(t), i \geq 1\Big)$: describes the virus spreading in the growth network, where $Y_i(t) = 1$ means that the node $i$ has a virus, otherwise $Y_i(t) = 0$ at $t$ .

The disturbed network growth process considered here is a continuous-time Markov chain $Z(t) = (X(t), Y(t))$ with the following one-step jump probabilities:

$$q(z, z') = \begin{cases} \frac{[\alpha(1-y_i)+\beta y_i](x_i+\theta)\wedge m}{S(z)} & \text{if } z' = z + (e_{i,n(x)+1},\ 0) \\ \frac{\sum_{j=1}^{n(x)} \lambda(1-y_i)x_{ij}y_j+\gamma y_i}{S(z)} & \text{if } z' = z + (0,\ 1-y_i) \\ 0 & \text{otherwise} \end{cases}$$

where $z = (x, y), z' = (x', y')$, both nonnegative numbers $\alpha$ and $\beta$ satisfying $\alpha + \beta > 0$, denote the rates of connecting the infected node and healthy node respectively, and $S(z) = S_1(z)+S_2(z)$ is normalization factor.

**(I).**

$$\lim_{t\to\infty}\frac{\mathbb{E}_z(S_1(Z(t)))}{t}=s_1>0,\qquad \lim_{t\to\infty}\frac{\mathbb{E}_z(S_2(Z(t)))}{t}=s_2\geq 0$$

**(II).** For every $k \geq 1$,

$$\lim_{t\to\infty}\mathbb{E}_z\Big(\sum_{j=1}^{n(X(t))}\frac{X_{ij}(t)}{X_i(t)}Y_j(t)\Big)=\rho_m$$

The probability that a link from a node to an infected node.

**(III).** The degrees of any two nodes are asymptoticly independent $(t \to \infty)$

**Theorem 1** Let $s = s_1 + s_2$ and $p = s_1/s$. Then

$$\lim_{t\to\infty} \frac{S_1(Z(t))}{t} = s_1, \quad \lim_{t\to\infty} \frac{S_2(Z(t))}{t} = s_2$$

$$\lim_{t\to\infty} \frac{n(X(t))}{t} = p, \quad \text{a.s. -}\mathbb{P}_z$$

**Proof of Theorem 1.** For any fixed $t > 0$, the stochastic process $M(s) = \mathbb{E}_z[S_1(Z(t))|\sigma_s]$ is a martingale for $0 \leqslant s \leqslant t$, and therefore, $\mathbb{E}_z(M(s_4) - M(s_3))(M(s_2) - M(s_1)) = 0$ for $0 \leq s_1 < s_2 \leq s_3 < s_4 \leq t$. Then, we have

$$\begin{aligned}\mathbb{E}_z[S_1(Z(t)) \;-\; & \mathbb{E}_z(S_1(Z(t)))]^2 \\ = \; & \mathbb{E}_z[M(t) - M(\lceil t \rceil) - \sum_{k=1}^{\lceil t \rceil}(M(k) - M(k-1))]^2 \\ = \; & \mathbb{E}_z(M(t) - M(\lceil t \rceil))^2 + \sum_{k=1}^{\lceil t \rceil}\mathbb{E}_z(M(k) - M(k-1))^2.\end{aligned}$$

Since

$$|M(k) - M(k-1)| \leq 2\max\{\alpha, \beta, m\}N(1)$$

for $1 \leq k \leq \lceil t \rceil$, it follows that

$$\begin{aligned} & \mathbb{E}_z[S_1(Z(t)) - \mathbb{E}_z(S_1(Z(t)))]^2 \\ \leq\ & 4(\max\{\alpha, \beta, m\})^2\mathbb{E}_z N^2(t - \lceil t \rceil) + \lceil t \rceil \mathbb{E}_z N^2(1) \\ =\ & 8(\max\{\alpha, \beta, m\})^2 t. \end{aligned}$$

and therefore

$$\lim_{t\to\infty} \frac{S_1(Z(t))}{t} = s_1, \quad \text{a.s. -}\mathbb{P}_z. \qquad (3)$$

$D_k(x) = \sum_{i=1}^{n(x)} I_k(x_i)$ , number of nodes with degree $k$.

$E_k(z) = \sum_{i=1}^{n(x)} y_i I_k(x_i)$ , number of infected nodes with degree $k$.

$P_k$ , probability that a node has degree $k$.

$Q_k$ , probability that a node has degree $k$ and is infected.

**Theorem 2** Let $s = s_1 + s_2$ and $W_m(x) = (x+\theta) \wedge m$. $P_k$ and $Q_k$ can be expressed in the following vector form:

$$
\begin{aligned}
(P_k,\ Q_k)^T &= \lim_{t\to\infty} \left( \frac{\mathbb{E}_z[D_k(X(t))]}{N(X(t))},\ \frac{\mathbb{E}_z[E_k(Z(t))]}{N(X(t))} \right)^T \\
&= (A(k)+I)^{-1} \left[\prod_{i=1}^{k-1} B(i)(A(i)+I)^{-1}\right] (s_1/s,\ 0)^T
\end{aligned}
$$

a.s. -$\mathbb{P}_z$, where

$$
A(k) = s^{-1} \begin{pmatrix} \alpha W_m(k) & (\beta-\alpha) W_m(k) \\ -\lambda(m\wedge k)\rho_m & \beta W_m(k) + \lambda(m\wedge k)\rho_m + \gamma \end{pmatrix}
$$

$$
B(k) = s^{-1} \begin{pmatrix} \alpha W_m(k) & (\beta-\alpha) W_m(k) \\ 0 & \beta W_m(k) \end{pmatrix}
$$

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**Corollary 1** If $\alpha = \beta$, then

$$P_k = \frac{\alpha W(k-1)}{\alpha W(k) + s} P_{k-1} = \frac{s}{\alpha W(k) + s} \prod_{i=1}^{k-1} \frac{\alpha W(i)}{\alpha W(i) + s} \sim C k^{-3}$$

**Corollary 2** If $\beta = 0$ and for large $\lambda$ such that $\rho_m(\lambda) \geq \rho > 0$ we have

$$P_k \sim C_1 A^k k^{-B}$$

where

$$A = \frac{\gamma + s}{\gamma + s + 2\lambda\rho}, \qquad B = \frac{2(\gamma + s)}{\gamma + s + 2\lambda\rho}$$

**Proof of Theorem 2** Let $D_k(t) = \mathbb{E}_z[D_k(X(t))]$ and $E_k(t) = \mathbb{E}_z[E_k(Z(t))]$. Both $D_k'(t)$ and $E_k'(t)$ can be written

$$\begin{aligned} D_k'(t) &= \alpha W_m(k-1)\frac{D_{k-1}(t)}{st} - \alpha W_m(k)\frac{D_k(t)}{st} \\ &\quad +(\beta-\alpha)W_m(k-1))\frac{E_{k-1}(t)}{st} - (\beta-\alpha)W_m(k))\frac{E_k(t)}{st} \\ &\quad +\delta_{k1}p + \epsilon_k(t) \end{aligned} \tag{4}$$

and

$$\begin{aligned} E_k'(t) &= \beta W_m(k-1)\frac{E_{k-1}(t)}{st} - \beta W_m(k)\frac{E_k(t)}{st} - \frac{\gamma E_k(t)}{st} \\ &\quad +\lambda(m\wedge k)\rho_m\frac{D_k(t)-E_k(t)}{st} + e_k(t) \end{aligned} \tag{5}$$

for large $t$, where $\epsilon_k(t) \to 0$ and $e_k(t) \to 0$ as $t \to \infty$ for all $k \geq 1$.

Let $U_k(t) = (D_k(t), E_k(t))^T$, $\Xi_k(t) = (\epsilon_k(t), e_k(t))^T$ and $P_{k1} = (\delta_{k1}p, 0)^T$. We can rewrite the above two equations in the matrix form

$$U_k'(t) = B(k-1)\frac{U_{k-1}(t)}{t} - A(k)\frac{U_k(t)}{t} + P_{k1} + \Xi_k(t) \qquad (6)$$

Note that

$$e^{A\log t} = \sum_{i=0}^{\infty} \frac{(A\log t)^i}{i!}, \quad e^{-I\log t} = \sum_{i=0}^{\infty} \frac{(-I\log t)^i}{i!} = \frac{1}{t} I$$

where $A$ is a matrix and $I$ is the unit matrix.

It follows that

$$\lim_{t\to\infty}\frac{U_k(t)}{t}=\prod_{j=1}^{k}[(A(j)+I)^{-1}[B(j-1)(A(j-1)+I)^{-1}]](p,0)^T$$

for $k\geq 1$. Thus, we have

$$\begin{aligned}(P_k,\ Q_k)^T &= \lim_{t\to\infty}(\frac{D_k(X(t))}{n(X(t))},\ \frac{E_k(Z(t))}{n(X(t))})^T\\ &= (A(k)+I)^{-1}[\prod_{i=1}^{k-1}B(i)(A(i)+I)^{-1}](1,\ 0)^T, \qquad \text{a.s. -}\mathbb{P}_z\end{aligned}$$

# 3. The Critical surface of Epidemic Spread



Note that the number $\rho_m = \rho_m(\alpha, \beta, \gamma, \theta, \lambda, )$ is dependent on the five parameters, $\alpha, \beta, \gamma, \theta$ and $\lambda$. Now we define a critical value $\lambda_c(m)$ for every $m \geq 1$ in the following.

**Definition.** For fixed $\alpha, \beta, \gamma$ and $\theta$, the epidemic critical value $\lambda_c(m) = \lambda_c(\alpha, \beta, \gamma, \theta, m)$ for $m \geq 1$ is defined by

$$\lambda_c(m) = \inf\{\lambda > 0 : \rho_m(\alpha, \beta, \gamma, \theta, \lambda) > 0\}.$$

The critical value means that if $\lambda(m) > \lambda_c(m)$, the infection spreads and becomes endemic. Below it, i.e., $\lambda(m) < \lambda_c(m)$, the infection dies out finally $(\rho_m = 0)$. The function $\lambda_c(\alpha, \beta, \gamma, \theta, m)$ on $\alpha, \beta, \gamma$ and $\theta$ can be seen as the critical surface for any fixed $m$.

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**Theorem 3** Let $\Lambda_m = (\alpha\mu_m + \gamma)(1 + \sigma_m) - \beta\delta_m$.
If $\lim_{\lambda \searrow \lambda_c(m)} \rho_m(\alpha, \beta, \gamma, \theta, \lambda) = 0$, then the critical value$\lambda_c(m)$ can be expressed as

$$\lambda_c(m) = \begin{cases} \frac{\Lambda_m \sum_{k=1}^{\infty} (m \wedge k) P_k}{\sum_{k=1}^{\infty} k(m \wedge k) P_k} & \text{if} \quad \Lambda_m > 0 \\ 0 & \text{if} \quad \Lambda_m \leq 0. \end{cases}$$

Here

$$\sigma_m = \frac{\sum_{k\geq m}^{\infty}(k-m)q_m(k)}{\sum_{k=1}^{\infty}(m\wedge k)q_m(k)}, \qquad \delta_m = \frac{\sum_{k=1}^{\infty}W_m(k)q_m(k)}{\sum_{k=1}^{\infty}(m\wedge k)q_m(k)}$$

and for large $k$, where

$$q_m(k) \sim \begin{cases} \frac{A_k(\nu)}{(k+\theta)^{1+\nu}} & \text{if } \ k \leq m \\ \frac{A_k(\nu)(1-\frac{\nu}{m})^{k-m}}{(m+\theta)^{1+\nu}} + \frac{P_{k-1}}{\beta}\sum_{j=m+1}^{k}\frac{P_{j-1}}{P_{k-1}}(1-\frac{\nu}{m})^{k-j} & \text{if } \ k > m \end{cases}$$

for large $k$, where

$$A_k(\nu) = \begin{cases} A(\nu) = \frac{1}{\alpha(2+\theta)f_1} + \frac{1}{\beta(1+\theta-\nu)} & \text{if } \ \nu < 1+\theta \\ \frac{1}{\alpha(2+\theta)f_1} + \frac{\ln k}{\beta} & \text{if } \ \nu = 1+\theta \\ \frac{1}{\alpha(2+\theta)f_1} + \frac{k^{\nu-1-\theta}}{\beta(\nu-1-\theta)} & \text{if } \ \nu > 1+\theta. \end{cases}$$

**Proof of Theorem 3.** Let $f(Z(t)) = \sum_{i=1}^{n(X(t))} Y_i(t)X_i(t)$ and $f_m(Z(t)) = \sum_{i=1}^{n(X(t))} Y_i(t)W_m(X_i(t))$. It follows that

$$\mathbb{E}_z(f(Z(t))) - f(z) = \int_0^t \Big(\beta \frac{\mathbb{E}_z(f_m(Z(u)))}{su} - \gamma \frac{\mathbb{E}_z(f(Z(u)))}{su} + \lambda\rho_m \frac{\mathbb{E}_z[\sum_{i=1}^{n(X(u))}(1 - Y_i(u))X_i(u)(X_i(u) \wedge m)]}{su} + \varepsilon(u)$$

where $\epsilon(t) \to 0$ as $t \to \infty$. We can further prove that

$$(s+\gamma)\sum_{k=1}^{\infty}(k \wedge m)Q_k + (s+\gamma)\sum_{k \geq m}^{\infty}(k-m)Q_k - \beta\sum_{k=1}^{\infty}W_m(k)Q_k = \lambda\rho_m \sum_{k=1}^{\infty} k(k \wedge m)(P_k - Q_k).$$

Note that

$$\rho_m = \frac{\sum_{k=1}^{\infty} kQ_k}{\sum_{k=1}^{\infty} kP_k}.$$

Thus

$$[(s+\gamma)(1+\sigma_m) - \beta\delta_m] \sum_{k=1}^{\infty} (k \wedge m) P_k = \lambda \sum_{k=1}^{\infty} k(k \wedge m)(P_k - Q_k).$$

**Corollary 3** If $-1 < \theta \leq 0$ and $\lim_{\lambda \searrow \lambda_c(m)} \rho_m(\alpha, \beta, \gamma, \theta, \lambda) = 0$ for all $m \geq 1$, then

$$\lim_{m \to \infty} \lambda_c(m) = \lambda_c = 0$$

for any fixed $\alpha, \beta$ and $\gamma$. That is to say, the infection can spread and become endemic on the scale-free network with the power $\tau = 3 + \theta, 2 < \tau \leq 3$, as long as there is a small rate of the epidemic spreading when the maximum degree $m$ is large. This result was found first by Pastor-Satorras and Vespignani (2001).

**Corollary 4** If $\theta > 0, \alpha\beta > 0$, and $\lim_{\lambda \searrow \lambda_c(m)} \rho_m(\alpha, \beta, \gamma, \theta, \lambda) = 0$ for all $m \geq 1$, then

$$\lim_{m \to \infty} \lambda_c(m) = \begin{cases} 0 & \text{if } \beta \geq \alpha(2+\theta) + \gamma \\ \alpha(2+\theta) + \gamma - \beta & \text{if } \beta < \alpha(2+\theta) + \gamma. \end{cases}$$

Let

$$S_c = \{(\alpha, \beta, \gamma, \lambda) : \quad \alpha(2+\theta) + \gamma > \beta, \ \ \alpha\beta > 0, \gamma, \lambda \geq 0\}$$

Note that $\sum_{k=1}^{\infty} kP_k = 2$. Let $\lambda_c = \lim_{m\to\infty} \lambda_c(m)$,

$$A = -2(2+\theta), \qquad B = 2, \qquad C = -2, \qquad D = \sum_{k=1}^{\infty} k^2 P_k.$$

Then, we can write $A\alpha + B\beta + C\gamma + D\lambda_c = 0$ as $m \to \infty$ for $\theta > 0$.

**Remark.** For a fixed $\theta > 0$, we can define the critical hyperplane (surface) $\Gamma_c$ as follows

$$\Gamma_c = \{(\alpha, \beta, \gamma, \lambda) \in S_c : \; A\alpha + B\beta + C\gamma + D\lambda = 0\}$$

and

$$\begin{aligned}\Gamma^+ &= \{(\alpha, \beta, \gamma, \lambda) \in S_c : \; A\alpha + B\beta + C\gamma + D\lambda > 0\}, \\ \Gamma^- &= \{(\alpha, \beta, \gamma, \lambda) \in S_c : \; A\alpha + B\beta + C\gamma + D\lambda < 0\}.\end{aligned}$$

Thus, the infection spreads and becomes endemic for $(\alpha, \beta, \gamma, \lambda) \in \Gamma^+$, and the infection dies out finally for $(\alpha, \beta, \gamma, \lambda) \in \Gamma^-$.

# References

[1] Amaral, L.A.N., etc. (2004) **Virtual Round Table on ten leading questions for network research**, *Eur. Phys. J. B* 38, 143-145.

[2] Liggett, T. M. (1999) **Stochastic Interacting Systems: Contact, Voter and Exclusion Processes**. Springer, New York.

[3]Ganesh, A. G. and Massoulie, D. T. (2005) **The effect of network topology on the spread of epidemics**. http://ieeexplore.ieee.org/xpls

[4]Pastor-Satorras, R. and Vespignani, A. (2001) **Epidemic spreading in scale-free networks**. *Phys. Rev. Letters*, 86, 3200-3203.

[5]Chen, D. Y. and Liu, Q. (2007) **On the contact processes on scale-free networks**.

**Thank You !**