Comparison Theorems of Spectral Gaps of Schrödinger Operators and Diffusion Operators

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Outline

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Spectral Gap of Semigroups

Fundamental Gap Conjecture

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Comparison on Wiener Space

Feynman-Kac semigroup

Let $(\Omega, (X_t)_{t\in\mathbb{R}^+}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, (\mathbb{P}_x)_{x\in E})$ be a càdlàg Markov process on the state space E. Assume the transition Markov semigroup P_t is symmetric in some $L^2(\mu)$ and essentially irreducible.

Given a potential $V : E \to \mathbb{R}$, define the Feynman-Kac semigroup:

$$P_t^V f(x) \; := \; \mathbb{E}^x f(X_t) \exp\left(\int_0^t V(X_s) ds\right), \; \forall f \geqslant 0.$$

Let $-\mathcal{L}^V$ be the lower-bounded self-adjoint Schrödinger operator generated by P_t^V. Define the lowest spectral point

$$\lambda_0(V) \; = \; \inf\left\{\int V f^2 \mathrm{d}\mu + \int f \cdot (-\mathcal{L}^V f) \mathrm{d}\mu; \right. \\ \left. f \in D(\mathcal{L}^V) \cap L^2(V^+\mu), \int f^2 \mathrm{d}\mu = 1\right\}.$$

People are concerned with:

- When is $\lambda_0(V)$ isolated in the spectrum $\sigma(-\mathcal{L}^V)$?
- How to estimate the gap between $\lambda_0(V)$ and the bottom of essential spectrum of $-\mathcal{L}^V$?
- How to characterize the ground state ϕ_0 corresponding to $\lambda_0(V)$? For example, is ϕ_0 non-negative and "concave"?
- Does it hold Logarithmic Sobolev inequality with respect to $\phi_0^2 \mathrm{d}\mu$? ...

Girsanov semigroup

As a counterpart, we can also consider a Girsanov semigroup as follows. Assume further that X_t is conservative.

Let $\nu \ll \mu$, and $(L_t)_{t\geqslant 0}$ is an additive $\mathbb{P}_\mu$-local martingale. Define a perturbation of $\mathbb{P}_\mu$ by the Girsanov's formula:

$$Q_{\nu|\mathcal{F}_t} := \exp\left(L_t - \frac{1}{2}\langle L\rangle_t\right)\mathbb{P}_{\mu|\mathcal{F}_t},$$

$$Q_t f(x) := \mathbb{E}^{\mathbb{P}_\nu}\left[f(X_t)\exp\left(L_t - \frac{1}{2}\langle L\rangle_t\right)\Big|X_0 = x\right].$$

The conversation implies that 0 is just the lowest eigenvalue for the generator of Q_t. We can ask the same questions as previous.

Existence

Indeed, we have found some criterions to yield the existence of the spectral gap, for example see

Simon B., Hoegh-Krohn R., *Hypercontractive semigroups and two dimensional self-coupled Bose fields*. J. Funct. Anal.9 (1972), 121–180.

Fuzhou Gong, Liming Wu, *Spectral gap of positive operators and applications*. J. Math. Pures Appl. 85 (2006), 151–191.

For simplicity, here we just give an application of Girsanov semigroup on abstract Wiener space $(\mathbb{W}, \mathbb{H}, \mu)$ endowed with the Ornstein–Uhlenbeck operator $\mathcal{L}$.

Given $b : \mathbb{W} \to \mathbb{H}$, consider the Girsanov semigroup associated to the diffusion operator $\mathcal{L}_b := \mathcal{L} + b \cdot \nabla$. We have

Theorem 1

If for some $\lambda > 1$ *holds*

$$\int \exp(\lambda \cdot |b|_{\mathbb{H}}^2) \mathrm{d}\mu < +\infty,$$

then $\mathcal{L}_b$ *has a spectral gap in* $L^p(\mu)$ *for any* $p \gg 1$.

Note that, the above integrability condition is sharp. However, there was no nice estimates on the spectral gap or ground state.

Roughly speaking, we have to make some control on the "derivative" of b, otherwise a high-frequency vibration on b will impact heavily on the spectral gap, but make no difference to the integrability.

Outline

Some notations:

- $\Omega \subset \mathbb{R}^n$: a bounded convex domain of diameter $D = \mathrm{diam}(\Omega)$;
- $V : \Omega \to \mathbb{R}$ a potential;
- $L = -\Delta + V$: the Schrödinger operator on Ω with Dirichlet boundary condition;
- Eigenvalues of L: $\lambda_0 < \lambda_1 \le \lambda_2 \le \ldots$, $\lim_{i\to\infty} \lambda_i = +\infty$;
- Eigenfunctions of L: $\phi_0, \phi_1, \phi_2, \ldots$, $\phi_i|_{\partial\Omega} \equiv 0$.

ϕ_0 and λ_0 are called the ground state and ground state energy, respectively. ϕ_0 is strictly positive in Ω.

Gap Conjecture (van den Berg, 1983): If V is convex, then the spectral gap of L satisfies

$$\lambda_1 - \lambda_0 \ge \frac{3\pi^2}{D^2}. \tag{1}$$

Example 2

Consider the one dimensional case $\Omega = (-\frac{D}{2}, \frac{D}{2}) \subset \mathbb{R}^1$ and $V \equiv 0$. Then the operator is given by $L = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}$, and

	Eigenvalues λ_i	Eigenfunctions ϕ_i
$i = 0$	$\frac{\pi^2}{D^2}$	$\cos \frac{\pi t}{D}$
$i = 1$	$\frac{4\pi^2}{D^2}$	$\sin \frac{2\pi t}{D}$

Therefore the spectral gap is $\frac{3\pi^2}{D^2}$.

Known results

In one dimension:

- Ashbaugh & Benguria (1989): If V is symmetric and single-well (not necessarily convex), then the conjecture holds;
- Lavine (1994): The conjecture holds if V is convex.

In higher dimensions:

- Singer, Wong, Yau & Yau (1985): The gap $\lambda_1 - \lambda_0 \geq \frac{\pi^2}{4D^2}$;
- Qi Huang Yu & Jia Qing Zhong (1986): The gap $\lambda_1 - \lambda_0 \geq \frac{\pi^2}{D^2}$;
- $\ldots$;
- Andrews & Clutterbuck (2011): The gap conjecture holds. **Basic idea**: compare the spectral gap with one dimensional case.

Modulus of convexity

Let $\tilde{V} \in C^1([-\frac{D}{2}, \frac{D}{2}], \mathbb{R})$ be an even function, such that $\forall x, y \in \Omega, x \neq y$,

$$\left\langle \nabla V(x) - \nabla V(y), \frac{x-y}{|x-y|} \right\rangle \geq 2\tilde{V}'\left(\frac{|x-y|}{2}\right). \tag{2}$$

The function $\tilde{V}$ is called a <u>modulus of convexity</u> of V.

Remark 3

(i) If the sign $\geq$ is replaced by $\leq$, then $\tilde{V}$ is called a <u>modulus of concavity</u> of V.

(ii) If V is convex, then we can choose $\tilde{V} \equiv 0$.

Log-concavity estimate of ground state

Consider the one dimensional Schrödinger operator $\tilde{L} = -\frac{d^2}{dt^2} + \tilde{V}$ on the symmetric interval $[-\frac{D}{2}, \frac{D}{2}]$, satisfying the Dirichlet boundary condition.

Denote by the corresponding objects by adding a tilde, e.g. $\tilde{\lambda}_i$ and $\tilde{\phi}_i$, $i = 0, 1, 2, \ldots$.

Theorem 4 (Andrews & Clutterbuck, JAMS, 2011, Theorem 1.5)

Assume that $\tilde{V}$ is a modulus of convexity of V, i.e. (2) holds, then $\log \tilde{\phi}_0$ is a modulus of concavity of $\log \phi_0$.
More precisely, $\forall\, x, y \in \Omega, x \neq y$,

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \leq 2(\log \tilde{\phi}_0)' \left(\frac{|x-y|}{2} \right). \quad (3)$$

Remarks on Theorem 4

Remark 5

- *Recall that when V is convex, then $\tilde{V} \equiv 0$.*

  In this case, $\tilde{L} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ *has the ground state* $\tilde{\phi}_0(t) = \cos\frac{\pi t}{D}$*, thus* $(\log\tilde{\phi}_0)'(t) = -\frac{\pi}{D}\tan\frac{\pi t}{D}$*,* $t \in (-\frac{D}{2}, \frac{D}{2})$*.*

  The log-concavity estimate (3) *becomes*

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \le -\frac{2\pi}{D} \tan\left(\frac{|x-y|}{2D}\right). \tag{4}$$

- *Brascamp & Lieb (JFA, 1976) proved a weaker result: if V is convex, then the ground state ϕ_0 is log-concave.*

Spectral gap comparison theorem

Theorem 6 (Andrews & Clutterbuck, JAMS, 2011, Theorem 1.3)

If $\tilde{V}$ is a modulus of convexity of V, i.e. (2) *holds, then* $\lambda_1 - \lambda_0 \geq \tilde{\lambda}_1 - \tilde{\lambda}_0$.

Ingredients of the proof:

(i) the ground state transform: let $u_i(t,x) = e^{-\lambda_i t}\phi_i(x)$ and $v = \frac{u_1}{u_0} = e^{-(\lambda_1-\lambda_0)t}\frac{\phi_1}{\phi_0}$, then $v(t,\cdot) \in C^\infty(\bar{\Omega})$ and

$$\frac{\partial v}{\partial t} = \Delta v + 2\nabla \log \phi_0 \cdot \nabla v;$$

(ii) sharp log-concavity estimate of ground state ϕ_0 (Theorem 4);

(iii) estimate of the modulus of continuity:

$$v(t,x) - v(t,y) \le C\tilde{v}(t,|x-y|) = Ce^{-(\tilde{\lambda}_1-\tilde{\lambda}_0)t}\frac{\tilde{\phi}_1}{\tilde{\phi}_0}(|x-y|).$$

Recall that $v(t,x) - v(t,y) = e^{-(\lambda_1-\lambda_0)t}(\frac{\phi_1}{\phi_0}(x) - \frac{\phi_1}{\phi_0}(y))$, hence $\forall\, t \ge 0$ and $x, y \in \Omega$,

$$e^{-(\lambda_1-\lambda_0)t}\left(\frac{\phi_1}{\phi_0}(x) - \frac{\phi_1}{\phi_0}(y)\right) \le Ce^{-(\tilde{\lambda}_1-\tilde{\lambda}_0)t}\frac{\tilde{\phi}_1}{\tilde{\phi}_0}(|x-y|)$$

which implies $\lambda_1 - \lambda_0 \ge \tilde{\lambda}_1 - \tilde{\lambda}_0$.

Our purpose: extend the above spectral gap comparison theorem to the infinite dimensional setting.

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From Euclid to Wiener

In our opinion, the "modulus of convexity" performs uniformly as a lower bound of Hessian(V) in each direction and each interval.

And the most interesting thing is, this kind of control will be inherited by the logarithm of ground state. It is a big advantage arising from Andrews and Clutterbuck's work.

Now, recall the first section, we make an attempt to introduce the modulus of convexity to abstract Wiener space. It seems difficult to generalize the arguments of Andrews and Clutterbuck directly, due to the loss of compactness and regularity.

However, we still have similar results as follows.

Notation

Denote by $(\mathbb{W}, \mathbb{H}, \mu)$ an abstract Wiener space and $\mathcal{L}_*$ the Ornstein–Uhlenbeck operator on $\mathbb{W}$.

Let $V \in \mathcal{D}_1^p(\mathbb{W}, \mu)$ for some $p > 1$ satisfy the KLMN condition (see Reed and Simon: *Methods of modern mathematical physics, IV*). Define

$$-\mathcal{L} = -\mathcal{L}_* + V$$

to be a self-adjoint Schrödinger operator bounded from below.

Correspondingly, denote by $\tilde{\mathcal{L}}_*$ the Ornstein–Uhlenbeck operator on $\mathbb{R}^1$ with respect to the Gaussian measure.

Let $\tilde{V} \in C^1(\mathbb{R}^1) \cap L^1(\mathbb{R}^1, \gamma_1)$ be a symmetric potential satisfying the KLMN condition too. Define

$$-\tilde{\mathcal{L}} = -\tilde{\mathcal{L}}_* + \tilde{V}.$$

Variation formula

It is well known that, there are two equivalent min-max principles for any self-adjoint operator H bounded from below.

That is $\mu_i = \lambda_i$ for all $i \geqslant 0$, which are defined as

1. $\mu_i = \sup_{\varphi_0,\varphi_1,\ldots,\varphi_{i-1}} \inf_{\substack{\varphi\in\mathcal{D}[H],\|\varphi\|=1,\\ \varphi\in[\varphi_0,\varphi_1,\ldots,\varphi_i]^\perp}} (\varphi, H\varphi)$;

2. $\lambda_i = \inf_{\varphi_0,\varphi_1,\ldots,\varphi_i\in\mathcal{D}[H]} \sup_{\substack{\|\varphi\|=1,\\ \varphi\in\mathrm{span}\{\varphi_0,\varphi_1,\ldots,\varphi_i\}}} (\varphi, H\varphi)$.

By convention, $\varphi_0, \varphi_1, \ldots, \varphi_i$ are all linearly independent and $[\varphi_0, \varphi_1, \ldots, \varphi_i]^\perp$ denotes the orthogonal completion of $\mathrm{span}\{\varphi_0, \varphi_1, \ldots, \varphi_i\}$.

Main Theorems

Theorem 7

Suppose for almost all $w \in \mathbb{W}$ and every $h \in \mathbb{H}$ with $h \neq 0$

$$\left\langle \nabla V(w+h) - \nabla V(w), \frac{h}{\|h\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}} \geqslant 2\tilde{V}'\left(\frac{\|h\|_{\mathbb{H}}}{2}\right). \qquad (5)$$

Then there exists a comparison

$$\lambda_1 - \lambda_0 \geqslant \tilde{\lambda}_1 - \tilde{\lambda}_0.$$

Hence, the existence of the spectral gap of $-\mathcal{L}$ on Wiener space can sometimes be reduced to one dimensional case. Note that, V doesn't need to be convex at all.

We would like to list some examples and remarks to explain further what we have presented previously.

Example 8

On one-dimensional Gaussian space, there exist non-convex potential functions such that the associated Schrödinger operators have spectral gaps strictly greater than 1. More precisely, define $\bar{V}$ by

$$\bar{V}(x) = \log(1 + x^2) + ax^2,$$

where $a > 0$ and $a \approx 0$. The second-order derivative $\bar{V}'' = \frac{2(1-x^2)}{1+x^2} + 2a$, which contains a negative part. However, we have a modulus of convexity for $\bar{V}$ as

$$\tilde{V}(s) = -2s + as^2,$$

and then a modulus of convexity for $\tilde{V}$ as

$$\tilde{\tilde{V}}(s) = as^2.$$

Since the first two eigenvalues of $-\frac{d^2}{ds^2} + s\frac{d}{ds} + \tilde{\tilde{V}}(s)$ are

$$\tilde{\tilde{\lambda}}_0 = \frac{1}{2}(\sqrt{1+4a} - 1), \qquad \tilde{\tilde{\lambda}}_1 = 1 + \frac{3}{2}(\sqrt{1+4a} - 1),$$

we get a lower bound by comparison theorem

$$\bar{\lambda}_1 - \bar{\lambda}_0 \geqslant \tilde{\lambda}_1 - \tilde{\lambda}_0 \geqslant \tilde{\tilde{\lambda}}_1 - \tilde{\tilde{\lambda}}_0 = \sqrt{1+4a} > 1.$$

Example 9

To estimate a spectral gap, the exponential integrability in Simon and Hoegh-Krohn's result is not a necessary condition. On one-dimensional Gaussian space, define $\bar{V}$ by

$$\bar{V}(x) = \frac{a}{8}x^2 \sin(\log(1 + x^2)),$$

where $a > 0$ and $a \approx 0$. Due to $\bar{V}'' \geqslant -2a$, we have a modulus of convexity for $\bar{V}$ as $\tilde{V}(s) = -as^2$. Hence, we get a lower bound

$$\bar{\lambda}_1 - \bar{\lambda}_0 \geqslant \tilde{\lambda}_1 - \tilde{\lambda}_0 = \sqrt{1 - 4a}.$$

However, $\exp(-t\bar{V})$ is not integrable for big t.

Example 10

To get the existence of spectral gap, the exponential integrability in Gong-Wu's result is sharp but not necessary. Set $\phi_0 = \exp\left(-ae^{bx^2}\right)$ with $a > 0$ and $b < 1/4$ to be the ground state of operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x \cdot \frac{\mathrm{d}}{\mathrm{d}x} + \bar{V}$ for $\bar{V}(x) = 4a^2b^2x^2e^{2bx^2} + 2abe^{bx^2}\left((1-2b)x^2 - 1\right)$. Since $\bar{V}''$ is uniformly lower bounded, the gap exists for small a (similar as Example 9). Furthermore, it is known that the spectrum of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x \cdot \frac{\mathrm{d}}{\mathrm{d}x} + \bar{V}$ coincides with that of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x \cdot \frac{\mathrm{d}}{\mathrm{d}x} + 2\frac{\mathrm{d}}{\mathrm{d}x}\log\phi_0 \cdot \frac{\mathrm{d}}{\mathrm{d}x}$. However, let $\mathbf{b} = 2\frac{\mathrm{d}}{\mathrm{d}x}\log\phi_0$, $\exp\left((1+\delta)|\mathbf{b}|\right)$ is not integrable for $\delta > 0$.

Example 11

A modulus of convexity can exist even though $\bar{V}''$ is not uniformly lower bounded. Let $\bar{V}(x) = \mathrm{Sgn}(x)\sqrt{|x|^3}$, thus $\bar{V}' = \frac{3}{2}\sqrt{|x|}$ but $\bar{V}'' = \mathrm{Sgn}(x)\frac{3}{4\sqrt{|x|}}$ singular at 0. However, we have by the definition

$$\left(\bar{V}'(x) - \bar{V}'(y)\right) \cdot \frac{x-y}{|x-y|} \geqslant -\frac{3}{2}\sqrt{|x-y|}.$$

Example

Let $(\mathbb{W}, \mathbb{H}, \mu)$ be the classical Wiener space, i.e.

$$\mathbb{W} = C([0,1], \mathbb{R}), \quad \mathbb{H} = \{h \in L^2[0,1], h(0) = 0, \int_0^1 \dot{h}^2(s) \mathrm{d}s < \infty\}.$$

Set $h_0(s) = s$, $h_n(s) = \frac{\sqrt{2}}{\pi n} \sin \pi n s, n \geq 1$, then $\{h_n\}$ is an orthonormal basis in $\mathbb{H}$. It is well known that μ is the Wiener measure which can be introduced by
$w(s) = \sum_{n=0}^{\infty} \xi_n h_n(s), s \in [0,1]$, where ξ_n are i.i.d. random variables with distribution $\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \mathrm{d}t$. Denote
$\langle h, w \rangle_{\mathbb{H}} = \int_0^1 \dot{h}(s) \mathrm{d}w(s)$, $h \in \mathbb{H}$.

Example

Now, we consider a potential function

$$V(w) = \sum_{n=0}^{N} a_n \langle h_n, w \rangle_{\mathbb{H}}^2 + \sum_{k=1}^{\infty} a_{N+k} \langle h_{N+k}, w \rangle_{\mathbb{H}}^2,$$

where $\frac{-1}{4} < a_0 < a_1 < \cdots < a_N < 0 < \cdots < a_{N+2} < a_{N+1}$, and $\sum_{k=1}^{\infty} a_{N+k} < \infty$. Set $\tilde{V}(s) = a_0 s^2$ to be a modulus of convexity for V. By direct computation, we have $\lambda_1 - \lambda_0 = \sqrt{1 + 4a_0}$, and $\tilde{\lambda}_1 - \tilde{\lambda}_0 = \sqrt{1 + 4a_0}$.

Example

This means that Theorem 7 is a sharp estimate. Moreover, there exists some $t > 0$ such that $\int e^{-tV(w)} \mathrm{d}\mu(w) = \infty$, which breaks the exponential integrability in Simon and Hoegh-Krohn's result. However, for Wiener spaces, we can not find an example to show the spectral gap can be strictly greater than 1.
Furthermore, we extend slightly the above idea to some other potential functions. Set $\hat{V} \in \mathbb{D}_1^p(\mathbb{W}, \mu)$ $(p > 1)$ to satisfy the KLMN condition and be $\bigvee_{n=1}^{\infty} \sigma(\langle h_n, w\rangle_{\mathbb{H}})$-measurable. If $\hat{V}$ has a modulus of convexity as bx^2 with $|b|$ small enough, let $V(w) = a_0\langle h_0, w\rangle_{\mathbb{H}} + c\hat{V}(w)$, then Theorem 7 gives a sharp estimate too. More precisely, for example, we can take $\hat{V}(w) = \bar{V}(\langle h_1, w\rangle_{\mathbb{H}})$, where $\bar{V}$ is given in Example 9 and 10.

The next result gives the modulus of log-concavity for ϕ_0.

Theorem 12

Assume the same condition as in Theorem 7 and the gap $\tilde{\lambda}_1 - \tilde{\lambda}_0 > 0$. Then $-\mathcal{L}$ and $-\tilde{\mathcal{L}}$ have a unique ground state respectively. Moreover, for almost all $w \in \mathbb{W}$ and every $h \in \mathbb{H}$ with $h \neq 0$,

$$\left\langle \nabla \log \phi_0(w+h) - \nabla \log \phi_0(w), \frac{h}{\|h\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}} \leqslant 2(\log \tilde{\phi}_0)' \left(\frac{\|h\|_{\mathbb{H}}}{2} \right).$$

Our proof relies on the approximation of eigenvalues and eigenfunctions, from bounded domains to n-dimensional Gaussian spaces and thus to Wiener space.

In other word, we can prove the followings:
Denote by $\bar{\mathcal{L}}_* = \Delta - x \cdot \nabla$ the Ornstein–Uhlenbeck operator on $\mathbb{R}^n$ with respect to the Gaussian measure $\mathrm{d}\gamma_n = (2\pi)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{2})\, \mathrm{d}x$. Let $\bar{V} \in \mathbb{D}_1^1(\mathbb{R}^n, \gamma_n)$ satisfy the KLMN condition. Then one can define $-\bar{\mathcal{L}} = -\bar{\mathcal{L}}_* + \bar{V}$ to be a self-adjoint Schrödinger operator bounded from below, which is associated to

$$\bar{\mathcal{E}}(f, f) = \int \left(|\nabla f|^2 + \bar{V} f^2\right) \mathrm{d}\gamma_n$$

with domain $\mathcal{D}[\bar{\mathcal{E}}] = \{f \in \mathbb{D}_1^2(\mathbb{R}^n, \gamma_n) : \bar{V} f^2 \text{ is } L^1\text{-integrable}\}$. For convenience, a bar will be added to all the relative notation to $\bar{\mathcal{L}}_*$ and $\bar{V}$. With a slight abuse of notation, we still denote by $(\cdot, \cdot)$ the inner product and $\| \cdot \|$ the norm of $L^2(\mathbb{R}^n, \gamma_n)$.

Proposition 13

Suppose $\bar{V} \in C^\infty(\mathbb{R}^n)$ *such that for any* $x \neq y$*,*

$$(\nabla \bar{V}(x) - \nabla \bar{V}(y)) \cdot \frac{x-y}{|x-y|} \geqslant 2\tilde{V}'\left(\frac{|x-y|}{2}\right). \tag{6}$$

Then the spectral gap of $-\bar{\mathcal{L}}$ *satisfies*

$$\bar{\lambda}_1 - \bar{\lambda}_0 \geqslant \tilde{\lambda}_1 - \tilde{\lambda}_0.$$

The strategy of proof is to get a comparison of spectral gaps for operator $\bar{\mathcal{L}}$ restricted on arbitrary ball and $\tilde{\mathcal{L}}$ on the interval with equal diameter, then prove the eigenvalues approximation when the diameter goes to infinity.

Proposition 14

Suppose that $\bar{V} \in \mathbb{D}_1^1(\mathbb{R}^n, \gamma_n)$ *satisfies for almost every* $x, y \in \mathbb{R}^n$ *with* $x \neq y$,

$$(\nabla \bar{V}(x) - \nabla \bar{V}(y)) \cdot \frac{x-y}{|x-y|} \geqslant 2\tilde{V}'\left(\frac{|x-y|}{2}\right).$$

Then the spectral gap of $-\bar{\mathcal{L}}$ *satisfies*

$$\bar{\lambda}_1 - \bar{\lambda}_0 \geqslant \tilde{\lambda}_1 - \tilde{\lambda}_0.$$

Since $\nabla \bar{V}$ exists in the sense of distribution, we have to use certain mollifier. Here we choose the Ornstein–Uhlenbeck semigroup $(P_t)_{t>0}$.

We can also prove the modulus of log-concavity for the ground state of $-\bar{\mathcal{L}}$.

Proposition 15

Suppose $\tilde{\lambda}_1 - \tilde{\lambda}_0 > 0$. Then $-\bar{\mathcal{L}}$ (resp. $-\tilde{\mathcal{L}}$) has a unique ground state $\bar{\phi}_0$ (resp. $\tilde{\phi}_0$). Moreover, for almost every $x \neq y$,

$$(\nabla \log \bar{\phi}_0(x) - \nabla \log \bar{\phi}_0(y)) \cdot \frac{x-y}{|x-y|} \leqslant 2(\log \tilde{\phi}_0)' \left(\frac{|x-y|}{2} \right).$$

Let $\mathbb{H}$ be a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and norm $\| \cdot \|_{\mathbb{H}}$, which is called the Cameron–Martin space, and denote by

$$\mathcal{F}_{\mathbb{H}} = \{X \subset \mathbb{H} : X \text{ is a finite dimensional linear subspace}\}.$$

$\mathbb{W}$ is a completion of $\mathbb{H}$ under a *radonifying norm* $\| \cdot \|_{\mathbb{W}}$ satisfying that $\| \cdot \|_{\mathbb{W}} \leqslant C \| \cdot \|_{\mathbb{H}}$ and for any $\varepsilon > 0$. There exists $X \in \mathcal{F}_{\mathbb{H}}$ such that for any $Y \in \mathcal{F}_{\mathbb{H}}$ orthogonal to X,

$$\gamma_Y(y \in Y : \|y\|_{\mathbb{W}} \geqslant \varepsilon) \leqslant \varepsilon,$$

where γ_Y is the standard Gaussian measure on Y. Then there is an inclusion relation $\mathbb{W}^* \subset \mathbb{H}^* = \mathbb{H} \subset \mathbb{W}$, and we can take an orthogonal basis of $\mathbb{H}$ as $\{e_i \in \mathbb{W}^*\}_{i \geqslant 1}$.

For $n \geqslant 1$, let $X_n = \mathrm{span}\{e_1, e_2, \ldots, e_n\}$ be a n-dimensional linear subspace such that $\mathbb{H} = \overline{\cup_n X_n}$. For every X_n, there exist the direct sum $\mathbb{W} = X_n \oplus Y_n$ and measure decomposition $\mu = \gamma_n \otimes \mu_n$. Let P_{X_n} be the orthogonal projection from $\mathbb{H}$ onto X_n, and $\pi_n : \mathbb{W} \to X_n$ its extension to $\mathbb{W}$, that is, $\pi_n(w) = \sum_{i=1}^n e_i(w) e_i$. Then

$$\pi_n(w + h) - \pi_n(w) = P_{X_n} h. \tag{7}$$

For any $F \in L^1(\mathbb{W}, \mu)$, define $\mathbb{E}^{X_n}(F)$ to be the L^1 conditional expectation of F on the sub-Borel algebra generated by π_n, and there exists $f : X_n \to \mathbb{R}$ such that $f \circ \pi_n = \mathbb{E}^{X_n}(F)$. Furthermore, $\mathbb{E}^{X_n}(F)$ converges to F in the L^1-norm.

For $F \in L^p(\mathbb{W}, \mu)$ with $p > 1$, it is called Malliavin differentiable, denoted by $F \in \mathbb{D}_1^p(\mathbb{W}, \mu)$, if there exists $\nabla F \in L^p(\mathbb{W}, \mathbb{H})$ such that for any $h \in \mathbb{H}$,

$$(\nabla F(w), h)_{\mathbb{H}} = D_h F(w) := \lim_{\varepsilon \to 0} \frac{F(w + \varepsilon h) - F(w)}{\varepsilon}.$$

Here the reason for $p > 1$ is that it is convenient to define the shift operator τ_h by the Cameron–Martin theorem (i.e. the integral transformation on $\mathbb{W}$)

$$\tau_h F(w) := F(w + h) \in L^{p-}(\mathbb{W}, \mu),$$

where $L^{p-} = \bigcap_{p' < p} L^{p'}$. Moreover, we have the formula

$$\nabla f(x) = P_{X_n} \left(\int_{Y_n} \nabla F(x, y) \, \mathrm{d}\mu_n(y) \right), \quad x = \pi_n(w). \qquad (8)$$

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz test functions on $\mathbb{R}^n$. We denote by

$$\mathrm{Cylin}(\mathbb{W}) = \{F : W \to \mathbb{R} \,| \text{there exist } n \geqslant 1 \text{ and } \ f \in \mathcal{S}(\mathbb{R}^n) \\ \text{such that } F = f \circ \pi_n\}$$

the set of *cylindrical Wiener functionals*, which is dense in $L^p(\mathbb{W}, \mu)$ and also the Sobolev spaces $\mathbb{D}_1^q(\mathbb{W}, \mu)$ for $q > 1$. Recall that we suppose the potential $V \in \mathbb{D}_1^p(\mathbb{W}, \mu)$ $(p > 1)$, and the Schrödinger operator $H = -\mathcal{L} = -\mathcal{L}_* + V$ is bounded from below, together with a sequence of λ_i defined by the min-max principle. For simplicity, we identify X_n with $\mathbb{R}^n$ and denote $V_n \circ \pi_n = \mathbb{E}^{X_n}(V)$.

Lemma 16

Suppose for almost all $w \in \mathbb{W}$ *and every* $h \in \mathbb{H}$ *with* $h \neq 0$*, it holds*

$$\left\langle \nabla V(w+h) - \nabla V(w), \frac{h}{\|h\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}} \geqslant 2\tilde{V}'\left(\frac{\|h\|_{\mathbb{H}}}{2}\right).$$

Then $\tilde{V}$ *is also a modulus of convexity of* V_n*, that is for a.e.* $x, y \in \mathbb{R}^n$ *with* $x \neq y$*,*

$$(\nabla V_n(x) - \nabla V_n(y)) \cdot \frac{x-y}{|x-y|} \geqslant 2\tilde{V}'\left(\frac{|x-y|}{2}\right).$$

Corresponding to λ_i, define by the min-max principle for $H_n = -\Delta + x \cdot \nabla + V_n$ on $L^2(\mathbb{R}^n, \gamma_n)$ that

$$\lambda_{i,n} = \inf_{\varphi_0,\varphi_1,\ldots,\varphi_i \in \mathcal{D}[H_n]} \sup_{\substack{\|\varphi\| = 1, \\ \varphi \in \mathrm{span}\{\varphi_0,\varphi_1,\ldots,\varphi_i\}}} (\varphi, H_n \varphi).$$

Since V_n is weakly differentiable and has a modulus of convexity as $\tilde{V}$, we can use Proposition 14 to get

$$\lambda_{1,n} - \lambda_{0,n} \geqslant \tilde{\lambda}_1 - \tilde{\lambda}_0.$$

Lemma 17

For every $i \geqslant 0$, $\lambda_{i,n}$ converges to λ_i as $n \to \infty$.

Let ϕ_0 be a ground state of $-\mathcal{L}$ with $-\mathcal{L}\phi_0 = \lambda_0\phi_0$.
Correspondingly, let $\phi_{0,n}$ be a ground state of
$H_n = -\Delta + x \cdot \nabla + V_n$ on $L^2(\mathbb{R}^n, \gamma_n)$ such that $H_n\phi_{0,n} = \lambda_{0,n}\phi_{0,n}$.

Lemma 18

Suppose that

1. $\lambda_1 - \lambda_0 > 0$;
2. ϕ_0 *and* $\phi_{0,n}$ *are all of multiplicity one;*
3. *there exists* $\kappa > 0$ *such that* $-\mathcal{L}_* + (1+\kappa)V$ *is bounded from below.*

Then $\phi_{0,n} \circ \pi_n$ *converges to* ϕ_0 *in the norm* $\|\cdot\|_{L^2(\mathbb{W},\mu)}$, *and* $\nabla(\phi_{0,n} \circ \pi_n)$ *to* $\nabla\phi$ *as well.*

As a counterpart, we also compare λ_1 of diffusion operator

$$-\mathcal{L} = -\mathcal{L}_* + \nabla F \cdot \nabla$$

with $\tilde{\lambda}_1$ of the one dimensional operator

$$-\tilde{\mathcal{L}} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (t + \omega'(t))\frac{\mathrm{d}}{\mathrm{d}t}.$$

Here, the two functions F and ω are related by the following inequality: for all $h \in \mathbb{H}$ and μ-a.e. $w \in \mathbb{W}$,

$$\left\langle \nabla F(w+h) - \nabla F(w), \frac{h}{\|h\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}} \geqslant 2\omega'\left(\frac{\|h\|_{\mathbb{H}}}{2}\right).$$

Theorem 19

Assume that $F \in \mathcal{D}_1^p(\mathbb{W}, \mathbb{R})$ satisfies $\int_{\mathbb{W}} e^{-F}\,\mathrm{d}\mu = 1$. Suppose also that $\omega \in C^1(\mathbb{R})$ is even, satisfying $\int_{\mathbb{R}} e^{-\omega}\,\mathrm{d}\gamma_1 = 1$ and $\lim_{t\to\infty}(\omega'(t) + t) = +\infty$. Then we have $\lambda_1 \geqslant \tilde{\lambda}_1$.

The proof is similar as Theorem 12.

We note that the spectrum of $-\mathcal{L} = -\mathcal{L}_* + \nabla F \cdot \nabla$ coincide with that of the Schrödinger type operator $-\hat{\mathcal{L}} = -\mathcal{L}_* + V$ where the functional $V = -\frac{1}{2}\mathcal{L}_* F + \frac{1}{4}\|\nabla F\|_{\mathbb{H}}^2$. Indeed, if ϕ is an eigenfunction of $-\mathcal{L}$ corresponding to the eigenvalue λ, then it is straightforward to check that $\psi := e^{-F/2}\phi$ is an eigenfunction of $-\hat{\mathcal{L}}$ corresponding to λ. Therefore, to compare the spectral gap of $-\mathcal{L}$ with the one-dimensional operator $-\tilde{\mathcal{L}}$, we can apply the spectral gap comparison theorem for the Schrödinger operator $-\hat{\mathcal{L}}$ by directly imposing conditions of modulus of convexity on the functional V. However, such conditions involve the third order derivatives of F which are difficult to check.

Example

Finally, we give a simple example to compare the condition (**??**) on the modulus of convexity with the exponential integrability in Gong-Wu's result.

We confine ourselves to the one dimensional case, i.e. $\mathbb{W} = \mathbb{H} = \mathbb{R}$ and $\mu = \gamma_1$ is the Gaussian distribution. Let $F(x) = \frac{x^2}{2} + Z_0$ where $Z_0 \in \mathbb{R}$ is a normalizing constant such that $\int_{\mathbb{R}} e^{-F} \, \mathrm{d}\gamma_1 = 1$. Clearly, F satisfies the conditions of Theorem 19 and it has a modulus of convexity, but $F'(x) = x$ does not satisfy the exponential integrability condition in Gong-Wu's result. In the following, we give an example in which F' verifies the condition in Gong-Wu's result, but F has no modulus of convexity. The basic idea is to construct a function whose second derivative is not bounded from below.

Example (cont.)

Let $l_1 : y = x/4$, $l_2 : y = -x/4\,(x \geqslant 0)$ be two radials. We shall define a function F such that the graph of $y = F'(x)$ oscillates between these two radials. More precisely, let

$$F'(2k) = (-1)^{k-1}k/2, \quad k \in \mathbb{Z}_+;$$

and for $x \in [2k, 2k+2]$, $y = F'(x)$ is the line segment linking the two points $(2k, (-1)^{k-1}k/2)$ and $(2k+2, (-1)^k(k+1)/2)$. Now the function F is given by $F(x) = \int_0^x F'(t)\,\mathrm{d}t$ for $x \geq 0$ and $F(x) = F(-x)$ for $x < 0$. Since $|F'(x)| \leq |x|/4$, it satisfies the exponential integrability in Gong-Wu's result.

Example (cont.)

Next for $h \in (0, 2]$, it is clear that

$$\frac{F'(4k+2+h) - F'(4k+2)}{h} = \frac{F'(4k+4) - F'(4k+2)}{2},$$

which leads to

$$F'(4k+2+h) - F'(4k+2) = -(k+3/4)h.$$

If F has ω as its modulus of convexity, then

$$2\omega'(h/2) \leqslant -(k+3/4)h, \quad \text{for all } k \in \mathbb{Z}_+.$$

Letting $k \to \infty$, we see that $\omega'|_{(0,1]} \equiv -\infty$ which is absurd.

In another paper, jointed with Huaiqian Li and Dejun Luo, we give a probabilistic proof of Andrews and Clutterbuck's spectral gap comparison theorem via the coupling by reflection of the diffusion processes. Moreover, we also present a simpler probabilistic proof of the fundamental gap conjecture.
The further research problem is following:

- How to get a complete new probabilistic proof of Andrews and Clutterbuck's spectral gap comparison theorem and the fundamental gap conjecture?
- How to extend the spectral gap comparison theorem to path and loop spaces over compact Riemannian manifolds?

Thank you for your attention!