Laplace integrals for quadratic Wiener functionals and moderate deviations for parameter estimators

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Outline

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   - Laplace integrals
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Model

- Model:
$$dX_t = \theta X_t dt + dV_t, \tag{1.1}$$
where $dV_t = \rho V_t dt + dW_t$, $\theta < 0$, $\rho < 0$ are unknown parameters and $W = \{W_t, t \in [0, \infty)\}$ is a standard Brownian motion, the initial values $X_0 = 0$ and $V_0 = 0$.
- Problem: asymptotic behaviors of the parameter estimators.

- The model (1.1) is the continuous-time version of second-order stable autoregressive process:

$$\tilde{X}_k = \tilde{\theta}\tilde{X}_{k-1} + \tilde{\varepsilon}_k, \qquad \tilde{\varepsilon}_k = \tilde{\rho}\tilde{\varepsilon}_{k-1} + \tilde{V}_k \tag{1.2}$$

  where the noise $(\tilde{V}_k)_{k\geq 1}$ is a sequence of independent and identically distributed random variables. The asymptotic properties of parameters in this model:
  - Bercu and Proïa (ESAIM Probab. Stat., 2013) ,
  - Bitseki Penda, Djellout and Proïa (ESAIM Probab. Stat., 2013).

Parameter estimators

- If V is observable, then from one-dimensional OU process,

$$\begin{aligned}\hat{\theta}_T =& \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \frac{X_T^2 - T}{2\int_0^T X_t^2 dt} \\ & \left(= \theta + \frac{\rho \int_0^T X_t V_t dt}{\int_0^T X_t^2 dt} + \frac{\int_0^T X_t dW_t}{\int_0^T X_t^2 dt} \right), \\ \hat{\rho}_{T,0} =& \frac{V_T^2 - T}{2\int_0^T V_t^2 dt} \\ & \left(= \rho + \frac{\int_0^T V_t dW_t}{\int_0^T V_t^2 dt} \right),\end{aligned} \tag{1.3}$$

are natural estimators of θ and ρ.

- When V is not observable, V can be estimated by

$$\hat{V}_t = X_t - \hat{\theta}_T \Sigma_t, \quad \text{where} \quad \Sigma_t = \int_0^t X_s ds. \tag{1.4}$$

Then

$$\hat{\theta}_T = \frac{X_T^2 - T}{2\int_0^T X_t^2 dt}, \quad \hat{\rho}_T = \frac{\hat{V}_T^2 - T}{2\int_0^T \hat{V}_t^2 dt} \tag{1.5}$$

are estimators of θ and ρ.

- The estimators are not asymptotic unbiased estimators of θ and ρ. In fact, when $T \to \infty$, almost surely,

$$\hat{\theta}_T \to \theta^*, \quad \hat{\rho}_T \to \rho^*,$$

where

$$\theta^* = \theta + \rho, \quad \rho^* = \frac{\theta\rho(\theta+\rho)}{(\theta+\rho)^2 + \theta\rho}. \tag{1.6}$$

- If $\theta > \rho$, then almost surely, $0 > \hat{\theta}_T > \hat{\rho}_T$ for T large enough, and $(\tilde{\theta}_T, \tilde{\rho}_T) := \varphi(\hat{\theta}_T, \hat{\rho}_T)$ is an asymptotic unbiased estimator of (θ, ρ), where, for $0 > x_1 > x_2$,

$$\varphi(x) = \frac{1}{2}\left(x_1 - x_1\sqrt{1 - \frac{4x_2}{x_1 - x_2}}, x_1 + x_1\sqrt{1 - \frac{4x_2}{x_1 - x_2}}\right).$$

- By delta method, it is sufficient to study asymptotic behaviors of $(\bar{\theta}_T, \bar{\rho}_T)$.

- $\Phi : \mathcal{X} \to \mathcal{Y}$ Hadamard differentiable at θ. Let $r_n \to \infty$.
- Delta method:
  If $\{r_n(Z_n - \theta), n \geq 1\}$ converges weakly to Z, then $\{r_n(\Phi(Z_n) - \Phi(\theta)), n \geq 1\}$ converges weakly to $\Phi'_\theta(Z)$.
- Delta method in large deviations (Gao, Zhao (AOS,2011)) :
  If $\{r_n(Z_n - \theta), n \geq 1\}$ satisfies the large deviation principle with speed $\lambda(n)$ and rate function I, then $\{r_n(\Phi(Z_n) - \Phi(\theta)), n \geq 1\}$ satisfies the large deviation principle with speed $\lambda(n)$ and rate function $I_{\Phi'_\theta}$, where

$$I_{\Phi'_\theta}(y) = \inf\{I(x); \Phi'_\theta(x) = y\}.$$

- It is obvious that the estimator $\hat{\theta}_T$ is a function of the following quadratic functionals for the OU processes :

$$S_T = \int_0^T X_t^2 dt, \quad P_T = \int_0^T X_t V_t dt, \quad Q_T = \int_0^T \Sigma_t^2 dt. \tag{1.7}$$

- In particular, In order to study large deviations for these quadratic functionals, we need to estimate the Laplace integrals of these quadratic functionals.

- A powerful tool is the Log-Sobolev inequality on the Wiener space W:

$$Ent(f^2) \leq CE\left(\int_0^T |\nabla_t f|^2 dt\right), \quad f \in C_b^1(W/L^2).$$

where $C_b^1(W/L^2)$ is the space of all bounded function f on W, differentiable with respect to the L^2-norm, such that ∇f is also continuous and bounded from W equipped with L^2-norm to $L^2([0, T], \mathbb{R})$.

  - Gourcy, Wu(Potential Analysis, 2006),
  - Gao, Jiang (ECP,2009)).

- Here, we estimate Laplace integrals of the quadratic functionals by calculating the eigenvalues of the Hilbert-Schmidt operators associated with the quadratic functionals.

Multiple Wiener-Itô Integrals

- Set $\triangle = [0, T]$ and $\triangle_n = \{(t_1, \cdots, t_n) \in [0, T]^n; t_1 \le \cdots \le t_n \le T\}$ for every $n \ge 1$. For any symmetric function $f \in L^2([0, T]^n)$, the multiple stochastic integral $J_n(f)$ is defined by

$$J_n(f) = \int_{\triangle_n} f(s_1, \cdots, s_n) dW_{s_1} \cdots dW_{s_n}. \tag{1.8}$$

- For each symmetric function $f \in L^2([0, T]^2)$, define:

$$(A^f \varphi)(t) = \int_0^T f(s, t)\varphi(s) ds, \quad \varphi \in L^2([0, T]). \tag{1.9}$$

- A^f is a symmetric Hilbert-Schmidt operator on $L^2([0,T])$, i.e.,

$$\|A^f\|_{HS}^2 := \int_{[0,T]^2} |f(s,t)|^2 dsdt < \infty.$$

- The operator A^f is of trace class and can be written by

$$A^f \varphi(t) = \sum_{i=1}^{\infty} \lambda_i \langle h_i, \varphi \rangle_{L^2([0,T])} h_i(t)$$

where $\lambda_i, i = 1, \cdots$ are the eigenvalues of A^f counted repeatedly according to the multiplicity, and $h_i(t)$ is the eigenvector with eigenvalue $\lambda_i, i \geq 1$. Set $\sigma(A^f) = \sup\{|\lambda_i|, i \geq 1\}$.

Decompositions of the quadratic functionals

$$S_T = \frac{\theta}{\rho^2-\theta^2}J_2(f_1) + \frac{\rho}{\theta^2-\rho^2}J_2(f_2) + \frac{\theta}{(\rho-\theta)^2}J_2(f_3) + \frac{\rho}{(\rho-\theta)^2}J_2(f_4) - \frac{\theta\rho}{(\theta+\rho)(\rho-\theta)^2}J_2(f_5) - \frac{T}{2(\theta+\rho)} + \varepsilon_{1,T}, \tag{1.10}$$

$$Q_T = \frac{1}{\theta^2-\rho^2}J_2(f_1) - \frac{1}{\theta^2-\rho^2}J_2(f_2) + \frac{1}{\theta(\rho-\theta)^2}J_2(f_3) + \frac{1}{\rho(\rho-\theta)^2}J_2(f_4) - \frac{1}{(\theta+\rho)(\rho-\theta)^2}J_2(f_5) - \frac{T}{2\theta\rho(\theta+\rho)} + \varepsilon_{2,T} \tag{1.11}$$

and

$$P_T = \frac{\theta}{\rho^2-\theta^2}J_2(f_1) - \frac{\rho}{\rho^2-\theta^2}J_2(f_2) + \frac{1}{\rho-\theta}J_2(f_4) - \frac{\theta}{\rho^2-\theta^2}J_2(f_5) - \frac{T}{2(\theta+\rho)} + \varepsilon_{3,T},$$

where

$$\begin{cases} f_1(s,t) = e^{\theta|t-s|}, \quad f_2(s,t) = e^{\rho|t-s|}, \quad f_3(s,t) = e^{2\theta T - \theta(t+s)}, \\ f_4(s,t) = e^{2\rho T - \rho(t+s)}, \quad f_5(s,t) = e^{(\theta+\rho)T}\left(e^{-\rho s - \theta t} + e^{-\theta s - \rho t}\right); \end{cases} \tag{1.13}$$

$$\begin{cases} \varepsilon_{1,T} = \dfrac{1}{(\rho-\theta)^2}\left(\dfrac{e^{2\theta T}-1}{4} + \dfrac{e^{2\rho T}-1}{4} - \dfrac{2\theta\rho\left(e^{(\theta+\rho)T}-1\right)}{(\theta+\rho)^2}\right), \\ \varepsilon_{2,T} = \dfrac{1}{(\rho-\theta)^2}\left(\dfrac{e^{2\theta T}-1}{4\theta^2} + \dfrac{e^{2\rho T}-1}{4\rho^2} - \dfrac{2\left(e^{(\theta+\rho)T}-1\right)}{(\theta+\rho)^2}\right), \\ \varepsilon_{3,T} = \dfrac{e^{2\rho T}-1}{4\rho(\rho-\theta)} - \dfrac{\theta\left(e^{(\theta+\rho)T}-1\right)}{(\theta+\rho)^2(\rho-\theta)}. \end{cases} \tag{1.14}$$

Laplace integrals for quadratic Wiener functionals

Theorem 1.1

(1). $\sigma(A^{f_i}) < -\frac{2}{\theta}$, *$i = 1, 2$, and for all* $\mu < -\frac{\theta}{2}$,

$$E\left(\exp\{\mu J_2(f_1)\}\right)$$
$$= \left(\cosh\left(T\sqrt{\theta^2+2\mu\theta}\right) - \frac{(\theta+\mu)\sinh\left(T\sqrt{\theta^2+2\mu\theta}\right)}{\sqrt{\theta^2+2\mu\theta}}\right)^{-\frac{1}{2}} \tag{1.15}$$
$$\times \exp\left\{-\frac{1}{2}(\theta+\mu)T\right\},$$

$$E\left(\exp\{\mu J_2(f_2)\}\right)$$
$$= \left(\cosh\left(T\sqrt{\rho^2+2\mu\rho}\right) - \frac{(\rho+\mu)\sinh\left(T\sqrt{\rho^2+2\mu\rho}\right)}{\sqrt{\rho^2+2\mu\rho}}\right)^{-\frac{1}{2}} \tag{1.16}$$
$$\times \exp\left\{-\frac{1}{2}(\rho+\mu)T\right\}$$

(2). The operators A^{f_3} and A^{f_4} have the eigenvalues $\frac{e^{2\theta T}-1}{\theta}$ and $\frac{e^{2\rho T}-1}{\rho}$, repectively, and for $\frac{\mu\left(e^{2\theta T}-1\right)}{\theta} < 1$

$$E\left(\exp\left\{\mu J_2(f_3)\right\}\right) = \left(1 - \frac{\mu}{\theta}\left(e^{2\theta T} - 1\right)\right)^{-\frac{1}{2}} \exp\left\{-\frac{\mu}{2\theta}\left(e^{2\theta T} - 1\right)\right\}, \tag{1.17}$$

for $\frac{\mu\left(e^{2\rho T}-1\right)}{\rho} < 1$

$$E\left(\exp\left\{\mu J_2(f_4)\right\}\right) = \left(1 - \frac{\mu}{\rho}\left(e^{2\rho T} - 1\right)\right)^{-\frac{1}{2}} \exp\left\{-\frac{\mu}{2\rho}\left(e^{2\rho T} - 1\right)\right\}. \tag{1.18}$$

(3). The operator A^{f_5} has the eigenvalues

$$\frac{e^{(\theta+\rho)T}-1}{\theta+\rho} \pm \frac{1}{2}\sqrt{\frac{(1-e^{2\theta T})(1-e^{2\rho T})}{\theta\rho}}$$

and for $\mu\left(\frac{e^{(\theta+\rho)T}-1}{\theta+\rho} + \frac{1}{2}\sqrt{\frac{(1-e^{2\theta T})(1-e^{2\rho T})}{\theta\rho}}\right) < 1$

$$\begin{aligned} & E\left(\exp\left\{\mu J_2(f_5)\right\}\right) \\ & = \left(\left(1-\frac{\mu}{\theta+\rho}\left(e^{(\theta+\rho)T}-1\right)\right)^2 - \frac{\mu^2(1-e^{2\theta T})(1-e^{2\rho T})}{4\theta\rho}\right)^{-\frac{1}{2}} \\ & \quad \times \exp\left\{-\frac{\mu}{\theta+\rho}\left(e^{(\theta+\rho)T}-1\right)\right\}. \end{aligned} \tag{1.19}$$

Deviation inequalities

Theorem 1.2

There exist some positive constants C_1, C_2 depending only on θ, ρ, such that for all $r > 0$ and T large enough,

$$\left.\begin{array}{l} P\left(\left|S_T + \frac{T}{2(\theta+\rho)}\right| \geq rT\right) \\ P\left(\left|Q_T + \frac{T}{2\theta\rho(\theta+\rho)}\right| \geq rT\right) \\ P\left(\left|P_T + \frac{T}{2(\theta+\rho)}\right| \geq rT\right) \end{array}\right\} \leq 4e^{-C_1\psi(r)T} + 12e^{-C_2 rT}, \qquad (1.20)$$

where

$$\psi(r) = r^2 I_{[0,1/4)}(r) + \frac{8r-1}{16} I_{[1/4,\infty)}(r).$$

Moderate deviations for parameter estimators

Let $a(T)$ be positive numbers satisfying

$$\frac{a(T)}{T} \to 0, \quad \frac{a(T)}{\sqrt{T}} \to \infty, \quad T \to \infty.$$

Define

$$\Gamma = \begin{pmatrix} \sigma_\theta^2 & \iota \\ \iota & \sigma_\rho^2 \end{pmatrix},$$

where $\sigma_\theta^2 = -2\theta^*$,$\iota = \frac{2\rho^*\left((\theta^*)^2 - \theta\rho\right)}{(\theta^*)^2 + \theta\rho}$, and

$$\sigma_\rho^2 = \frac{2\rho^*\left((\theta^*)^6 + \theta\rho\left((\theta^*)^4 - \theta\rho(2(\theta^*)^2 - \theta\rho)\right)\right)}{\left((\theta^*)^2 + \theta\rho\right)^3}.$$

Theorem 1.3

The family $\left\{\frac{T}{a(T)}\left(\hat{\theta}_T-\theta^*,\hat{\rho}_T-\rho^*\right), T\geq 0\right\}$ *satisfies the large deviations with speed* $\frac{a^2(T)}{T}$ *and good rate function*

$$I(x)=\frac{1}{2}x\Gamma^{-1}x^{\tau},\quad x\in\mathbb{R}^2,$$

that is, for any closed set $F\in\mathcal{B}(\mathbb{R}^2)$*,*

$$\limsup_{T\to\infty}\frac{T}{a^2(T)}\log P\left(\frac{T}{a(T)}\left(\hat{\theta}_T-\theta^*,\hat{\rho}_T-\rho^*\right)\in F\right)\leq-\inf_{x\in F}I(x)$$

and for any open set $G\in\mathcal{B}(\mathbb{R}^2)$*,*

$$\liminf_{T\to\infty}\frac{T}{a^2(T)}\log P\left(\frac{T}{a(T)}\left(\hat{\theta}_T-\theta^*,\hat{\rho}_T-\rho^*\right)\in G\right)\geq-\inf_{x\in G}I(x).$$

A representation for Laplace integral of quadratic Wiener functionals.

Let $\psi \in L^2([0,T])$ and let f be a symmetric function in $L^2([0,T]^2)$. Set $F = J_1(\psi) + J_2(f)$. Assume that $I + A^f$ has positive spectrum. Then for any $\mu \in \mathbb{R}$ with $\mu\sigma(A^f) < 1$,

$$E\left(e^{\mu F}\right) = \left(\det_2(I - \mu A^f)\right)^{-1/2} e^{\frac{\mu^2}{2}\int_0^T \psi(s)(I-\mu A^f)^{-1}\psi(s)ds}. \qquad (2.1)$$

where the Carleman-Fredholm determinant $\det_2(I + A^f)$ is defined by

$$\det_2(I + A^f) = \det(I + A^f)e^{-trA^f},$$

$$\det(I + A^f) = \prod_{i=0}^{\infty}(1 + \lambda_i), \quad trA^f = \sum_{i=0}^{\infty}\lambda_i,$$

and $\lambda_i, i \geq 0$ are the eigenvalues of A^f, counted with their multiplicities.

Therefore, in order to calculate the Laplace integrals

$$E\left(e^{\mu J_2(f_i)}\right),\ i=1,\cdots,5,$$

a key step is to calculate the eigenvalues of the operators A^{f_i}, $i=1,\cdots,5$.

- References for Laplace integrals of the quadratic functionals:
  - Lévy's stochastic area formula (Ikeda, Kusuoka and Manabe, Proc. Pure. Math. AMS, 1995),
  - Solutions of the KdV equation (Taniguchi, JFA, 2004).

Lemma 2.1

$\lambda \neq 0$ *is an eigenvalue of* A^{f_1} *if and only if*

$$\cosh\left(T\sqrt{\theta^2+2\theta\lambda^{-1}}\right) - \frac{\sinh\left(T\sqrt{\theta^2+2\theta\lambda^{-1}}\right)}{\sqrt{\theta^2+2\theta\lambda^{-1}}}\left(\theta+\lambda^{-1}\right) = 0, \quad (2.2)$$

where $\frac{\sinh(zt)}{z} = \frac{e^{zt}-e^{-zt}}{2z}$ *for* $z \in \mathbb{R}$ *and* $t \in [0, T]$.

Lemma 2.2

Let $\{\lambda_i\}$ *be the eigenvalues of* A^{f_1} *and*

$$\lambda^*(T) = \max\{\lambda_i, i \geq 1\}, \quad \lambda_*(T) = \min\{\lambda_i, i \geq 1\}.$$

Then

$$\lambda_* := \liminf_{T\to\infty} \lambda_*(T) = 0, \quad \lambda^* := \limsup_{T\to\infty} \lambda^*(T) \leq -\frac{2}{\theta}. \tag{2.3}$$

Lemma 2.3

For all $\mu < -\frac{\theta}{2}$,

$$\left(\det_2(I - \mu A^{f_1})\right)^{-\frac{1}{2}}$$
$$= \left(\cosh\left(T\sqrt{\theta^2 + 2\mu\theta}\right) - \frac{(\theta+\mu)\sinh\left(T\sqrt{\theta^2+2\mu\theta}\right)}{\sqrt{\theta^2+2\mu\theta}}\right)^{-\frac{1}{2}}$$
$$\times \exp\left\{-\frac{1}{2}(\theta+\mu)T\right\}.$$

Proof of Theorem 1.1

From Lemma 2.2, $\sigma(A^{f_1}) < -\frac{2}{\theta}$. For any $\mu < -\frac{\theta}{2}$, we can write by (2.1) that

$$E\left(\exp\{\mu J_2(f_1)\}\right) = \left(\det_2(I - \mu A^{f_1})\right)^{-\frac{1}{2}}.$$

Therefore, by Lemma 2.3,

$$\begin{aligned} & E\left(\exp\{\mu J_2(f_1)\}\right) \\ & = \left(\cosh\left(T\sqrt{\theta^2+2\mu\theta}\right) - \frac{(\theta+\mu)\sinh\left(T\sqrt{\theta^2+2\mu\theta}\right)}{\sqrt{\theta^2+2\mu\theta}}\right)^{-\frac{1}{2}} \\ & \quad \times \exp\left\{-\frac{1}{2}(\theta+\mu)T\right\}. \end{aligned}$$

Proof of Theorem 1.2

For all $0 < \mu < -\frac{\theta}{4}$,

$$\sqrt{\theta^2 + 2\mu\theta} \geq -(\theta + \mu) + \frac{\mu^2}{\theta}.$$

From (1.15), it follows that

$$\begin{aligned} & E\left(\exp\{\mu J_2(f_1)\}\right) \\ & = \left(\frac{1}{2} - \frac{\theta+\mu}{2\sqrt{\theta^2+2\mu\theta}} + \frac{1}{2}e^{-2T\sqrt{\theta^2+2\mu\theta}}\left(1 + \frac{\theta+\mu}{\sqrt{\theta^2+2\mu\theta}}\right)\right)^{-\frac{1}{2}} \\ & \quad \times e^{-\frac{T}{2}\left(\sqrt{\theta^2+2\mu\theta}+\theta+\mu\right)} \\ & \leq e^{-\frac{T\mu^2}{2\theta}}. \end{aligned}$$

By Chebychev's inequality, for any $r \geq 0$

$$P\left(|J_2(f_1)| \geq rT\right) \leq 2e^{\frac{\theta T}{2}\Psi(r)}. \tag{2.4}$$

Corollary 2.1

For any $\delta > 0$,

$$\lim_{T\to\infty} \frac{T}{a^2(T)} \log P\left(\left|\frac{S_T}{T} + \frac{1}{2(\theta+\rho)}\right| > \delta\right) = -\infty.$$

$$\lim_{T\to\infty} \frac{T}{a^2(T)} \log P\left(\left|\frac{Q_T}{T} + \frac{1}{2\theta\rho(\theta+\rho)}\right| > \delta\right) = -\infty$$

and

$$\lim_{T\to\infty} \frac{T}{a^2(T)} \log P\left(\left|\frac{P_T}{T} + \frac{1}{2(\theta+\rho)}\right| > \delta\right) = -\infty.$$

Exponential equivalence

Firstly, we show that $\frac{T}{a(T)}(\hat{\theta}_T - \theta^*, \hat{\rho}_T - \rho^*)$ is exponential equivalent to a linear transformation of the martingale $\frac{1}{a(T)}(M_T, M_T^V)$, where

$$M_T = \int_0^T X_t dW_t, \quad M_T^V = \int_0^T V_t dW_t.$$

Proposition 3.1

For any $\varepsilon > 0$,

$$\lim_{T\to\infty} \frac{T}{a^2(T)} \log P\left(\left|\frac{T}{a(T)}(\hat{\theta}_T - \theta^*) - \frac{-2\theta^* M_T}{a(T)}\right| > \varepsilon\right) = -\infty, \quad (3.1)$$

and

$$\lim_{T\to\infty} \frac{T}{a^2(T)} \log P\left(\left|\frac{T}{a(T)}(\hat{\rho}_T - \rho^*) - \frac{C_1 M_T + C_2 M_T^V}{a(T)}\right| > \varepsilon\right) = -\infty. \quad (3.2)$$

where

$$C_1 = -2\rho^*\left(-\frac{\rho\rho^*(2\theta+\rho)}{\theta^2\theta^*} - 2(\theta\rho)^{-1}\theta^*\rho^*\right), \quad C_2 = -\frac{2(\theta^*\rho^*)^2}{\theta^2\rho}.$$

We can write

$$\hat{\theta}_T - \theta^* = \frac{M_T}{S_T} + \frac{R_T}{S_T} = \frac{-2\theta^* M_T}{T} + \frac{M_T}{T}\left(\frac{T}{S_T} + 2(\theta+\rho)\right) + \frac{R_T}{S_T}, \quad (3.3)$$

where

$$R_T = -\frac{\theta\rho}{2}\left(\int_0^T X_t dt\right)^2 = -\frac{\theta\rho}{2}\Sigma_T^2.$$

Then, (3.1) is a conclusion fo the following lemma.

Lemma 3.1

For any $\delta > 0$, we have

$$\lim_{T\to\infty} \frac{T}{a^2(T)} \log P\left(\frac{T}{a(T)}\left|\frac{R_T}{S_T}\right| > \delta\right) = -\infty,$$

and

$$\lim_{T\to\infty} \frac{T}{a^2(T)} \log P\left(\left|\frac{M_T}{a(T)}\left(\frac{T}{S_T} + 2(\theta+\rho)\right)\right| > \delta\right) = -\infty. \quad (3.4)$$

Corollary 3.1

$$\limsup_{N\to\infty}\limsup_{T\to\infty}\frac{T}{a^2(T)}\log P\left(\left|\frac{T}{a(T)}(\hat{\theta}_T-\theta^*)\right|>N\right)=-\infty. \tag{3.5}$$

MDP

Proof of Theorem 1.3. Theorem 1.3 can be obtained from Proposition 3.1 .

Thank you for your attention

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