Flow superprocesses with spatially dependent branching

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I. Outline

II. Background

Superprocess:

1. WHAT is a superprocess?
2. WHY studying a superprocess?
3. HOW to study a superprocess?

Flow superprocess:

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WHAT is a superprocess and WHY studying it?

1. Heuristically and roughly, a superprocess is used to describe the *evolution of population* in an area, evolution of cells in a system, evolution of clouds, etc; it is a measure-valued Markov process.

2. Mathematically, a superprocess arises as the *high density limit of branching particle systems*; it is characterized by

$$\int_{M(E)} e^{-\nu(f)} Q_t(\mu, d\nu) = e^{-\mu(V_t f)},$$

$$V_t f(x) = P_t f(x) - \int_0^t ds \int_E \phi(y, V_s f(y)) P_{t-s}(x, dy),$$

$$\phi(x, z) = -\gamma(x) z + \sigma(x) z^2 + \int_{(0,\infty)} (e^{zu} - 1 + zu) m(x, du).$$

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HOW to study a superprocess?

particle systems approximation, Laplace functional, etc.

NOTATION:

E : Lusin topological space. Usually, $E = \mathbb{R}^d$

$M(E)$: space of finite Borel measures on E with weak topology

$\mu(f) \equiv \langle f, \mu \rangle = \int_E f(x) d\mu(x)$

Branching Particle Systems (BPSs)

(1) underlying motion ξ: e.g., Brownian motion, Feller process

(2) lifetime α: exponential

(3) branching mechanism $\{p_i(x)\}$: reproduction

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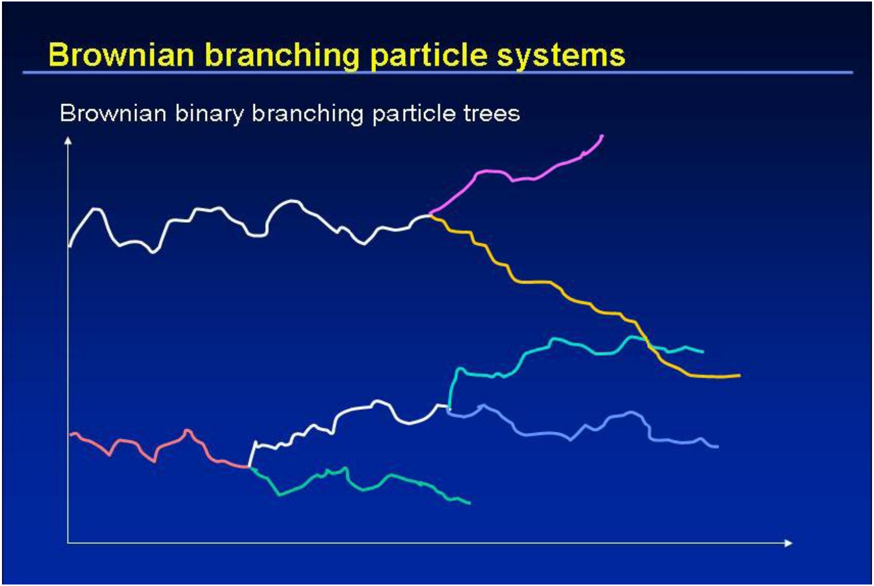
Brownian branching particle systems
Brownian binary branching particle trees

Define

$$X_t(A) = \# \text{ of particles alive in } A \text{ at time } t$$

- Basic Hypotheses:

(H1) the motions of particles are independent of one another; and

(H2) the branching and motions of particles are independent.

Under (H1) and (H2), $X = \{X_t : t \geq 0\}$ is an integer measure-valued Markov process, which is called a BPS.

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Canonical Representation

A superprocess $X = \{X_t : t \geq 0\}$ is determined by

$$\mathbf{E}\left[\mathrm{e}^{-X_t(f)} | X_0 = \mu\right] = \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, d\nu) = \mathrm{e}^{-\mu(V_t f)} \tag{1}$$

$\{Q_t\}$ transition semigroup, and $\{V_t\}$ cumulant semigroup of X, satisfying

$$V_t f(x) = P_t f(x) - \int_0^t ds \int_E \phi(y, V_s f(y)) P_{t-s}(x, dy),$$

with $P_t f(x) := \mathbf{E}_x f(\xi_t)$, and branching mechanism

$$\phi(x, z) = -\gamma(x) z + \sigma(x) z^2 + \int_{(0,\infty)} (\mathrm{e}^{zu} - 1 + zu) m(x, du)$$

$\gamma, 0 \leq \sigma \in B(E)$, and $(u \wedge u^2) m(x, du)$ bounded kernel from E to $(0, \infty)$.

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Related Readings

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M.Jiřina (1958, 1964) (measure-valued branching processs)

S. Watanabe (1968) (limits of continuous state branching processes)

M. Silverstein (1969) (continuous state branching semigroups)

D. Dawson (1975,1977) (measure-valued diffusion)

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MONOGRAPHS

Measure-valued Markov processes (Dawson, 1993)

An introduction to branching measure-valued processes (Dynkin,1994)

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WHAT is a flow superprocess and WHY studying it?

Flow superprocesses are short for superprocesses over a stochastic flow

A flow superprocess was used to describe the *evolution of "red tide" phenomenon* by Skoulakis and Adler in 2001; it is a measure-valued Markov process in a random medium.

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Mathematically, a flow superprocess X arises as the *high density limit of branching particle systems over a stochastic flow* (FBPSs); it is usually characterized by

The Martingale Problem–[MP(2,3)']

$$Z_t(f) := X_t(f) - X_0(f) - \int_0^t X_s((G+\gamma)f)ds \tag{2}$$

is a continuous square integrable martingale with $Z_0(f) = 0$ and

$$\langle Z(f)\rangle_t = \int_0^t X_s(\sigma^2 f^2)ds + \int_0^t (X_s \times X_s)(\Lambda f)ds, \tag{3}$$

where $\Lambda f(x,y) = a_{ij}^{(m)}(x,y)f_i'(x)f_j'(y)$ and $a_{ij}^{(m)}(x,y) = c_{il}(x)c_{jl}(y)$.

γ is called the *drift function* and σ the *branching variance*.

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HOW to study a flow superprocess?

FBPSs approximation, martingale problem, duality, conditional log-Laplace functional (Xiong, 2004), etc.

NOTATION

$$E := \mathbb{R}^d, \quad b : \mathbb{R}^d \to \mathbb{R}^d, \quad c : \mathbb{R}^d \to \mathbb{R}^{d\times m}, \quad e : \mathbb{R}^d \to \mathbb{R}^{d\times d}$$

$W = \{W(t) : t \geq 0\}$: m-dimensional BM

$B = \{B(t) : t \geq 0\}$: d-dimensional BM, independent of W

- Stochastic flow

$$dY(t) = b(Y(t))dt + c(Y(t))dW(t), \quad Y(s) = y \in E \tag{4}$$

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The motion of particles over the above flow

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$Y = \{Y(t) : t \geq 0\}$ has generator

$$Gf(x) = \sum_{i=1}^{d} b_i(x) f_i'(x) + \frac{1}{2} \sum_{i,j=1}^{d} d_{ij}(x,x) f_{ij}''(x),$$

where $d_{ij}(x,y) = \sum_{k=1}^{d} e_{ik}(x)e_{jk}(y) + \sum_{l=1}^{m} c_{il}(x)c_{jl}(y)$.

- Branching Particle Systems over a Stochastic Flow (FBPSs)

Construction of FBPSs is postponed until next section.

A flow superprocess $X = \{X_t : t \geq 0\}$ arises as the scaling limit of FBPSs. Skoulakis and Adler (2001) showed that X is the unique solution to the martingale problem [MP(2,3)'] with γ and σ constant.

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III. Our Purpose

Construct flow superprocesses with location dependent branching. That is, solve the martingale problem [MP(2,3)'] with γ and σ generalized to functions, denoted by [MP(2,3)]

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IV. Main Results

1. Model description
2. Existence (approximation method)
3. Uniqueness (dual method)
4. Properties (moment formula)

IV.1 Model Description

- Construction of FBPSs

NOTATION:

$$I := \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k) : \alpha_i = 1, 2, \ldots, 0 \le i \le k\}$$

$$|\alpha| = |(\alpha_0, \alpha_1, \ldots, \alpha_k)| = k$$

$$\alpha - 1 = (\alpha_0, \ldots, \alpha_{|\alpha|-1})$$

$$\alpha|_i = (\alpha_0, \ldots, \alpha_i)$$

$$\alpha \sim_n t \Leftrightarrow |\alpha|/n \le t < (1 + |\alpha|)/n, t \ge 0$$

$$E = \mathbb{R}^d$$

K_n particles located separately at $x_1^n, \ldots, x_{K_n}^n \in E$ at $t = 0$. A particle α born at time $|\alpha|/n$ will die at $(1 + |\alpha|)/n$ with $N^{\alpha,n}$ offspring produced.

For $\alpha \sim_n t$, the motion of α is determined by

$$dY^{\alpha,n}(t) = b(Y^{\alpha,n}(t))dt + e(Y^{\alpha,n}(t))dB^{\alpha,n}(t) + c(Y^{\alpha,n}(t))dW^n(t), \quad (6)$$

where

$Y^{\alpha,n}(0) = x_{\alpha_0}^n$

W^n: m-dimensional BM

$B^{\alpha,n}$: d-dimensional BM stopped at $t = (|\alpha| + 1)/n$

$B^{\alpha,n}(t) = B^{\alpha-1,n}(t)$ for $t \leq |\alpha|/n$

Let $k_n = k/n$ and $a_n = 1/n$. Define for $t \in [k_n, k_n + a_n)$

$$\mathscr{F}_t^n = \sigma(B^{\alpha,n}, N^{\alpha,n} : |\alpha| < k) \bigvee \bigcap_{r>t} \sigma(W_s^n, B_s^{\alpha,n} : s \leq r, |\alpha| = k)$$

and

$$\overline{\mathscr{F}}_{k_n}^n = \mathscr{F}_{k_n}^n \bigvee \sigma(W_s^n, B_s^{\alpha,n} : s \leq k_n + a_n, |\alpha| = k).$$

Assume that $\{N^{\alpha,n} : |\alpha| = k\}$ are conditionally independent given $\overline{\mathscr{F}}^n_{k_n}$, and that for such $N^{\alpha,n}$

$$\begin{cases} \mathbf{E}\left(N^{\alpha,n}|\overline{\mathscr{F}}^n_{k_n}\right) = 1 + \gamma_n(Y^{\alpha,n}_{k_n+a_n})/n =: \beta_n(Y^{\alpha,n}_{k_n+a_n}) \\ \text{Var}\left(N^{\alpha,n}|\overline{\mathscr{F}}^n_{k_n}\right) = \sigma_n(Y^{\alpha,n}_{k_n+a_n})^2, \end{cases} \tag{7}$$

where $\gamma_n \in C_l(E)$ and $\sigma_n \in C_l(E)^+$. Now define

$$X^n_t(B) = \frac{\text{number of particles alive in } B \text{ at time } t}{n}.$$

Clearly $X^n_t = \frac{1}{n}\sum_{\alpha\sim t}\delta_{Y^{\alpha,n}(t)}$ (empirical measure).

Assume there exist $p > 2$ and $C > 0$ independent of α and n such that

$$\mathbf{E}[(N^{\alpha,n})^p] \leq C, \gamma_n \rightrightarrows \gamma \in C_l(E) \text{ and } \sigma_n \rightrightarrows \sigma \in C_l(E)^+. \tag{8}$$

Hypotheses (LU)

(L) $|b(x) - b(y)| + \|c(x) - c(y)\| + \|e(x) - e(y)\| \leq K|x - y|, \quad x, y \in E.$

(U) $b_i, c_{il}, e_{ik} \in C_l^2(E)$, $i, k = 1, \ldots, d, l = 1, \ldots, m$, and for any $N \geq 1$ there exists $\lambda_N > 0$ such that

$$\sum_{p,q=1}^{N} \sum_{i,j=1}^{d} \xi_i^p d_{ij}(x_p, x_q) \xi_j^q \geq \lambda_N \sum_{p=1}^{N} \sum_{i=1}^{d} (\xi_i^p)^2$$

with $d_{ij}(x, y) = \sum_{k=1}^{d} e_{ik}(x) e_{jk}(y) + \sum_{l=1}^{m} c_{il}(x) c_{jl}(y)$.

Assume there exist $p > 2$ and $C > 0$ independent of α and n such that

$$\mathbf{E}[(N^{\alpha,n})^p] \leq C, \gamma_n \rightrightarrows \gamma \in C_l(E) \text{ and } \sigma_n \rightrightarrows \sigma \in C_l(E)^+. \tag{8}$$

Hypotheses (LU)

(L) $|b(x) - b(y)| + \|c(x) - c(y)\| + \|e(x) - e(y)\| \leq K|x - y|, \quad x, y \in E.$

(U) $b_i, c_{il}, e_{ik} \in C_l^2(E)$, $i, k = 1, \ldots, d, l = 1, \ldots, m$, and for any $N \geq 1$ there exists $\lambda_N > 0$ such that

$$\sum_{p,q=1}^{N} \sum_{i,j=1}^{d} \xi_i^p d_{ij}(x_p, x_q) \xi_j^q \geq \lambda_N \sum_{p=1}^{N} \sum_{i=1}^{d} (\xi_i^p)^2$$

with $d_{ij}(x, y) = \sum_{k=1}^{d} e_{ik}(x) e_{jk}(y) + \sum_{l=1}^{m} c_{il}(x) c_{jl}(y)$.

Let $\overline{Y} = (Y^1, \ldots, Y^N)$ be the solution to

$$\begin{cases} dY^1(t) = b(Y^1(t))dt + e(Y^1(t))dB^1(t) + c(Y^1(t))dW(t) \\ \ldots\ldots \\ dY^N(t) = b(Y^N(t))dt + e(Y^N(t))dB^N(t) + c(Y^N(t))dW(t), \end{cases}$$

where

W : m-dimensional BM

$B^1, \ldots, B^N$: independent d-dimensional BMs, independent of W

$(S_t^N)_{t\geq 0}$: semigroup of $\overline{Y}$

G_N : generator of $\overline{Y}$

Then for $f \in \mathscr{D}(G_N)$

$$
\begin{aligned}
& G_N f(x_1, \ldots, x_N) \\
= & \sum_{p=1}^{N} \sum_{i=1}^{d} b_i(x_p) \frac{\partial f(x_1, \ldots, x_N)}{\partial x_{p,i}} \\
& + \frac{1}{2} \sum_{p=1}^{N} \sum_{i,j=1}^{d} d_{ij}(x_p, x_p) \frac{\partial^2 f(x_1, \ldots, x_N)}{\partial x_{p,i} \partial x_{p,j}} \\
& + \frac{1}{2} \sum_{\substack{p,q=1 \\ p \neq q}}^{N} \sum_{i,j=1}^{d} a_{ij}^{(m)}(x_p, x_q) \frac{\partial^2 f}{\partial x_{p,i} \partial x_{q,j}} (x_1, x_2, x_3, \ldots, x_N).
\end{aligned}
$$

Notice: $G_1 \equiv G$

IV.2 Existence of martingale problem

We are to solve the martingale problem [MP(2,3)]. Existence is established by branching particle systems approximation. Let $\overline{E} = E \cup \{\Delta\}$. Recall $E = \mathbb{R}^d$.

- Claim: *For each $N \geq 1$, $\exists$ a set $D(\overline{E}^N)$ dense in $C(\overline{E}^N)$ such that $D(E^N) := D(\overline{E}^N)|_{E^N} \subset C_l^2(E^N)$.*

For $t \in [k_n, k_n + a_n]$ and $f \in D(E) := D(E^1)$, by the construction of X^n

$$\begin{aligned} X_t^n(f) &= X_0^n(f) + [X_t^n(f) - X_{k_n}^n(f)] + \sum_{r<k} [X_{r_n+a_n}^n(f) - X_{r_n}^n(f)] \\ &= X_0^n(f) + M_t^{(n)}(f) + J_t^{(n)}(f) + N_t^{(n)}(f) \\ &\quad + Z_t^{(n)}(f) + C_t^{(n)}(f) + H_t^{(n)}(f), \end{aligned}$$

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- Claim: *For each* $N \geq 1$, $\exists$ *a set* $D(\overline{E}^N)$ *dense in* $C(\overline{E}^N)$ *such that* $D(E^N) := D(\overline{E}^N)|_{E^N} \subset C_l^2(E^N)$.

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where

$$M_t^{(n)}(f) = n^{-1} \sum_{r<k} \sum_{\alpha \sim_n r_n} M_{r_n+a_n}^{\alpha, r_n}(f)[N^{\alpha,n} - \beta_n(Y_{r_n+a_n}^{\alpha,n})],$$

$$J_t^{(n)}(f) = n^{-1} \sum_{\alpha \sim_n k_n} M_t^{\alpha, k_n}(f)$$

$$+ n^{-1} \sum_{r<k} \sum_{\alpha \sim_n r_n} \int_{r_n}^{r_n+a_n} \widehat{Gf}(X_u^{\alpha,n}) du [\beta_n(Y_{r_n+a_n}^{\alpha,n}) - 1]$$

$$+ n^{-1} \sum_{r<k} \sum_{\alpha \sim_n r_n} \hat{f}(X_{r_n}^{\alpha,n})[\beta_n(Y_{r_n+a_n}^{\alpha,n}) - \beta_n(Y_{r_n}^{\alpha,n})]$$

$$+ n^{-1} \sum_{r<k} \sum_{\alpha \sim_n r_n} M_{r_n+a_n}^{\alpha, r_n}(f)[\beta_n(Y_{r_n+a_n}^{\alpha,n}) - \beta_n(Y_{r_n}^{\alpha,n})],$$

$$N_t^{(n)}(f) = n^{-1} \sum_{r<k} \sum_{\alpha \sim_n r_n} \int_{r_n}^{r_n+a_n} \widehat{Gf}(X_u^{\alpha,n}) du [N^{\alpha,n} - \beta_n(Y_{r_n+a_n}^{\alpha,n})],$$

$$Z_t^{(n)}(f) = n^{-1}\sum_{r<k}\sum_{\alpha\sim_n r_n} \hat{f}(X_{r_n}^{\alpha,n})[N^{\alpha,n} - \beta_n(Y_{r_n+a_n}^{\alpha,n})]$$
$$+n^{-1}\sum_{r<k}\sum_{\alpha\sim_n r_n} M_{r_n+a_n}^{\alpha,r_n}(f)\beta_n(Y_{r_n}^{\alpha,n}),$$
$$C_t^{(n)}(f) = n^{-1}\sum_{r<k}\sum_{\alpha\sim_n r_n}\int_{r_n}^{r_n+a_n}\widehat{Gf}(X_u^{\alpha,n})du$$
$$+n^{-1}\sum_{\alpha\sim_n k_n}\int_{k_n}^{t}\widehat{Gf}(X_u^{\alpha,n})du = \int_0^t X_u^n(Gf)du,$$
$$H_t^{(n)}(f) = n^{-1}\sum_{r<k}\sum_{\alpha\sim_n r_n}\hat{f}(X_{r_n}^{\alpha,n})[\beta_n(Y_{r_n}^{\alpha,n}) - 1] = \int_0^{k_n} X_{[ns]_n}^n(f\gamma_n)ds,$$

where $\hat{h} := h$ on E and $\hat{h}(\Delta) := 0$.

Lemma (1, Moments of FBPSs)

Let p *be as in* (8) *and* $T \geq 0$*. If* $X_0^n \Rightarrow \nu \in M(E)$*, then*

$$C_T = \sup_{n\geq 1} \mathbf{E}\left(\sup_{0\leq t\leq T} X_t^n(1)^2 \right) < \infty \text{ and } C_T' = \sup_{n\geq 1} \mathbf{E}\left(\sup_{0\leq t\leq T} X_t^n(1)^p \right) < \infty.$$

Proof. Use martingale inequalities and Gronwall's inequality. □

Lemma (2, Tightness)

For every $f \in D(E)$,

(1) $\{M^{(n)}(f)\}, \{N^{(n)}(f)\}$ *and* $\{J^{(n)}(f)\} \Longrightarrow \overline{0}$ *in* $D_{\mathbb{R}}[0,\infty)$;

(2) $\{C^{(n)}(f)\}$ *and* $\{H^{(n)}(f)\}$ *are C-tight in* $D_{\mathbb{R}}[0,\infty)$;

(3) *for each* n, $\{(Z^{(n)}_{k_n}(f), \mathscr{F}^n_{k_n}) : k = 0, 1, \ldots\}$ *is a square integrable discrete martingale with quadratic variation process*

$$\langle Z^{(n)}(f)\rangle_{k_n} = \int_0^{k_n} X^n_{[\lambda ns]_n}(f^2 S^1_{a_n}(\delta_n^2))ds +$$

$$\int_0^{k_n} ds \int_{E^2} \left(\frac{1}{a_n}\int_0^{a_n} S^2_u(\Lambda f)(x,y)du\right)\beta_n(x)\beta_n(y)X^n_{[\lambda ns]_n}(dx)X^n_{[\lambda ns]_n}(dy)+$$

$$\frac{1}{n}\int_0^{k_n} ds \int_E \left(\frac{1}{a_n}\int_0^{a_n} \left[S^1_u(\Psi f)(x) - S^2_u(\Lambda f)(x,x)\right]du\right)\beta_n(x)^2 X^n_{[\lambda ns]_n}(dx)$$

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Lemma (2-continued, Tightness)

(4) *Moreover, let* $t \mapsto \langle Z^{(n)}(f)\rangle_t$ *be the càdlàg extension of* $k \mapsto \langle Z^{(n)}(f)\rangle_{k_n}$ *such that* $\langle Z^{(n)}(f)\rangle_t := \langle Z^{(n)}(f)\rangle_{k_n}$ *for* $t \in [k_n, k_n + a_n)$. *Then* $\{(\langle Z^{(n)}(f)\rangle_t)\}$ *is C-tight in* $D_{\mathbb{R}}[0,\infty)$;

(5) *for each integer* $J \geq 1$,
$\lim_{n\to\infty} \mathbf{E}\left(\sup_{0\leq k\leq[\lambda nJ]} \left|Z^{(n)}_{(k+1)_n}(f) - Z^{(n)}_{k_n}(f)\right|^2\right) = 0$, *and the sequence* $\{\sup_{0\leq k\leq[\lambda nJ]} Z^{(n)}_{k_n}(f) : n \geq 1\}$ *is uniformly integrable;*

(6) $\{(Z^{(n)}_t(f))\}$ *is C-tight in* $D_{\mathbb{R}}[0,\infty)$.

Proof. Omitted. □

Tightness of $\{X^n\}$ and martingale characterization of its limits is given by

Theorem (3, Existence of MP(2,3))

$\{X^n\}$ is C-tight in $D_{M(\overline{E})}[0,\infty)$. Suppose that X is a weak limit of $\{X^n\}$. Then for any $f \in D(E)$,

$$Z_t(f) = X_t(\overline{f}) - \nu(f) - \int_0^t X_s((\overline{G+\gamma)f})ds \qquad (9)$$

is a continuous square integrable $(\mathscr{F}_t^X)$-martingale with $Z_0(f) = 0$ and quadratic variation process

$$\langle Z(f)\rangle_t = \int_0^t X_s((\overline{\sigma f})^2)ds + \int_0^t (X_s \times X_s)(\overline{\Lambda f})ds. \qquad (10)$$

Moreover, $\mathbf{P}\{X \in C_{M(E)}[0,\infty)\} = 1$.

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Moreover, $\mathbf{P}\{X \in C_{M(E)}[0,\infty)\} = 1$.

Proof.

- C-tightness of $\{X^n\}$:

  Lemma 2 $\Rightarrow \{X^n(f)\}$ tight $\Rightarrow \{X^n\}$ tight

- quadratic variation process:

  Use Skorokhod's Representation Theorem

- $X(\{\Delta\}) = 0$ $\mathbf{P}$-a.s.

  Use martingale equality □

Proof.

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  Lemma 2 $\Rightarrow \{X^n(f)\}$ tight $\Rightarrow \{X^n\}$ tight
- quadratic variation process:

  Use Skorokhod's Representation Theorem
- $X(\{\Delta\}) = 0$ $\mathbf{P}$-a.s.

  Use martingale equality □

- Equivalent martingale problems

The MP(2,3) will be equivalent to another martingale problem for $\mathscr{L}$, a diffusion operator on $C(M(E))$ defined by

$$\mathscr{L}F(\mu) = \int_E (G+\gamma)\left(\frac{dF(\mu)}{d\mu(x)}\right)\mu(dx) + \frac{1}{2}\int_E \sigma(x)^2 \frac{d^2F(\mu)}{d\mu(x)^2}\mu(dx)$$
$$+ \frac{1}{2}\sum_{i,j=1}^{d}\int_E\int_E a_{ij}^{(m)}(x,y)\frac{\partial^2}{\partial x_i \partial y_j}\left(\frac{d^2F(\mu)}{d\mu(x)d\mu(y)}\right)\mu(dx)\mu(dy)$$

Let $\mathscr{D}(\mathscr{L}) = \mathscr{D}_1(\mathscr{L}) \cup \mathscr{D}_2(\mathscr{L})$ with $\mathscr{D}_i(\mathscr{L}) \subset C(M(E))$ and

$\mathscr{D}_1(\mathscr{L}) = \{F : F_f(\mu) = \langle f, \mu^N \rangle, f \in D(E^N)\}$,

$\mathscr{D}_2(\mathscr{L}) = \{F : F_{f,\phi}(\mu) = f(\mu(\phi)), f \in C_b^2(\mathbb{R}^+), \phi \in D(E)^+\}$

$\bigcup \{F : F_{f,\phi}(\mu) = f(\mu(\phi_1), \ldots, \mu(\phi_N)), f \in C_b^2(\mathbb{R}^N), \{\phi_i\} \subset D(E)\}$.

Let X be a solution to MP(2,3). Equivalence is established by

Lemma (4, Equivalence)

(1) $\mathbf{E}_\nu[X_t(1)^n]$ *is locally bounded in* t *for each* $n \geq 1$.

(2) X *is also a solution of the martingale problem for* $(\mathscr{L}, \mathscr{D}(\mathscr{L}), \nu)$. *That is, for all* $F \in \mathscr{D}(\mathscr{L})$

$$F(X_t) - F(\nu) - \int_0^t \mathscr{L}F(X_s)ds \qquad (11)$$

is a continuous martingale with $X_0 = \nu$.

(3) *MP(2,3) and the martingale problem (11) are equivalent.*

Note that for $f \in D(E^N)$

$$\begin{aligned}\mathscr{L} F_f(\mu) &= F_{G_N f}(\mu) + 1/2 \sum_{\substack{p,q=1 \\ p\neq q}}^{N} F_{\Phi_{p,q} f}(\mu) + 1/2 \sum_{p=1}^{N} F_{\Phi_p f}(\mu) \\ &= F_\mu(G_N f, N) + 1/2 \sum_{\substack{p,q=1 \\ p\neq q}}^{N} [F_\mu(\Phi_{p,q} f, N-1) - F_\mu(f,N)] \\ &\quad +1/2 \sum_{p=1}^{N} [F_\mu(\Phi_p f, N) - F_\mu(f,N)] + 1/2 N^2 F_\mu(f,N), \quad (12)\end{aligned}$$

where for $h \in B(E^n)$ and $x = (x_1, \ldots, x_n) \in E^n$, $F_\mu(h,n) \equiv F_h(\mu)$,

$$\Phi_{p,q} h(x_1, \ldots, x_{n-1}) := \sigma(x_{n-1})^2 h(x_1, \ldots, x_{n-1}, \ldots, x_{n-1}, \ldots, x_{n-2})$$

$$\Phi_p h(x) := 2\gamma(x_p) h(x).$$

- Construction of a dual process.

Let $\mathbf{B} := \cup_{n=1}^{\infty} B(E^n)$ be endowed with pointwise convergence on each $B(E^n)$ and $\mathbb{N} := \{1, 2, \ldots\}$. Assume $\{e_1, e_2, \ldots\}$ is a sequence of mutually independent unit exponential random variables with $e_0 := 0$. Define a sequence $\Gamma = \{\Gamma_k : k = 1, 2, \ldots\}$ of random operators on $\mathbf{B}$ and a $\mathbf{B}$-valued càdlàg process $L = \{L_t : t \geq 0\}$ as follows: Given a $\mathbf{B}$-valued random variable L_0, independent of $\{e_1, e_2, \ldots\}$, define recursively

$$\begin{cases} L_t = S_{t-\tau_k}^{N(L_{\tau_k})}\Gamma_k S_{\eta_k}^{N(L_{\tau_{k-1}})}\cdots\Gamma_2 S_{\eta_2}^{N(L_{\tau_1})}\Gamma_1 S_{\eta_1}^{N(L_{\tau_0})}L_{\eta_0}, & \text{if } \tau_k \le t < \tau_{k+1} \\ \mathbf{P}\{\Gamma_{k+1} = \Phi_{p,q} | N(L_{\tau_k}) = n_{k+1}\} = \frac{1}{n_{k+1}^2} \quad \text{for } 1 \le p \ne q \le n_{k+1} \\ \mathbf{P}\{\Gamma_{k+1} = \Phi_{p} | N(L_{\tau_k}) = n_{k+1}\} = \frac{1}{n_{k+1}^2} \quad \text{for } 1 \le p \ne q \le n_{k+1} \\ L_{\tau_{k+1}} = \Gamma_{k+1} S_{\eta_{k+1}}^{N(L_{\tau_k})}\Gamma_k S_{\eta_k}^{N(L_{\tau_{k-1}})}\cdots\Gamma_2 S_{\eta_2}^{N(L_{\tau_1})}\Gamma_1 S_{\eta_1}^{N(L_{\tau_0})}L_{\eta_0}, k = 0, 1, 2\,. \end{cases}$$

where $\eta_0 = 0, \eta_n = \frac{2e_n}{N(L_{\tau_{n-1}})^2}$, $\tau_k = \sum_{i=0}^{k}\eta_i$ and $N(h) := l$ if $h \in B(E^l)$.
Define $M_t = N(L_t)$.

Let X be a solution to the martingale problem (11).

Theorem (5, Dual Relationship and Uniqueness)

Suppose that hypotheses (LU) hold. Then for all $n \geq 1, t \geq 0$ and $h \in B(E^n)$ we have

$$\mathbf{E}\left[\langle h, X_t^n \rangle\right] = \mathbf{E}_{h,n}\left[\langle L_t, \mu^{M_t} \rangle \exp\left\{\frac{1}{2}\int_0^t M_s^2 ds\right\}\right], \tag{13}$$

where $X_t^n = X_t \times \cdots \times X_t \in M(E^n)$. Moreover, uniqueness holds for the martingale problem (11) and hence for MP(2,3).

Proof. Use martingale equalities, approximation, and moment problem. Techniques developed in Dawson *et al.* [1]. □

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Theorem (6, Weak Convergence and Martingale Characterization)

If $X_0^n = \frac{1}{n}\nu_n \Rightarrow \nu$ *in* $M(E)$*, then under the hypotheses (LU),* $X^n \Rightarrow X$ *in* $D_{M(E)}[0,\infty)$*, where* $X \in C_{M(E)}[0,\infty)$ *is the unique solution of the following martingale problem: for any* $f \in D(E)$*,*

$$Z_t(f) = X_t(f) - \nu(f) - \int_0^t X_s((G+\gamma)f)ds \tag{14}$$

is a continuous square integrable martingale with $Z_0(f) = 0$ *and quadratic variation process*

$$\langle Z(f)\rangle_t = \int_0^t X_s(\sigma^2 f^2)ds + \int_0^t (X_s \times X_s)(\Lambda f)ds. \tag{15}$$

Proof. This is immediate from Theorem 3 and Theorem 5. □

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IV.4 Moments of flow superprocesses

Definition (7, Flow Superprocess (G, γ, σ))

An adapted càdlàg process in $M(E)$ which satisfies [MP(2,3)] is called a superprocess over a stochastic flow, or simply flow superprocess (G, γ, σ).

For $h \in B(E^n)$, define operators $U^{(n)}$ and $V^{(n)}$ by

$$U^{(n)}h = \frac{1}{2}\sum_{p \neq q \in \{1,\ldots,n\}} \Phi_{p,q}h, \text{ and } V^{(n)}h = \frac{1}{2}\sum_{p=1}^{n} \Phi_p h.$$

Recall that S^N is the semigroup of $\overline{Y}$. Define a semigroup T^n as follows

$$T_t^n h(y) = \mathbf{E}_y\left[e^{\int_0^t V^{(n)}(\overline{Y}(s))ds} h(\overline{Y}(t))\right].$$

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Let X be a flow superproccess (G, γ, σ).

Proposition (8, Moments)

For $h \in B(E^n)$ *and each* $n \geq 1$

$$\mathbf{E}_\nu \langle h\,, X_t^n \rangle = \langle T_t^n h, \nu^n \rangle + \sum_{i=1}^{n-1} \langle \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{i-1}} T_{t_i}^{n-i} \Pi^{(n)}(i-1;t) h dt_i, \nu^{n-i} \rangle,$$

where $\Pi^{(n)}(i-1;t) = (U^{(n-(i-1))} T_{t_{i-1}-t_i}^{n-(i-1)} \cdots U^{(n-1)} T_{t_1-t_2}^{n-1}) U^{(n)} T_{t-t_1}^n$.

Proof. Use the dual relation (13), the following relation and the Markov property of X.

$$\begin{aligned}
&\mathbf{E}_{h,n}\left[\langle L_t, \mu^{M_t}\rangle \exp\left\{\frac{1}{2}\int_0^t M_s^2 ds\right\}\right] \\
=& \langle S_t^n h, \mu^n\rangle \\
&+\frac{1}{2}\sum_{\substack{p,q=1\\p\neq q}}^{n}\int_0^t \mathbf{E}_{\Phi_{p,q}S_s^n h, n-1}\left[\langle L_{t-s}, \mu^{M_{t-s}}\rangle \exp\left\{\frac{1}{2}\int_0^{t-s} M_u^2 du\right\}\right] ds \\
&+\frac{1}{2}\sum_{p=1}^{n}\int_0^t \mathbf{E}_{\Phi_p S_s^n h, n}\left[\langle L_{t-s}, \mu^{M_{t-s}}\rangle \exp\left\{\frac{1}{2}\int_0^{t-s} M_u^2 du\right\}\right] ds. \quad (16)
\end{aligned}$$

□

IV.5 Extension to bounded spatially dependent branching

In terms of the relations (13) and (16), flow superprocesses (G, γ, σ) with $\gamma \in B(E)$ and $\sigma \in B(E)^+$ can be constructed.

VI. References

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THANK YOU!