Fluctuation of Diffusions in Discontinuous Media

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Outline for section 1

Introduction and summary

Fluctuation of $u_t^{\epsilon+}$

Fluctuation of $X_\cdot^\epsilon$ around $\phi_\cdot$

References

Introduction and summary 1

- Consider the following diffusion

$$\begin{aligned} dX_t^\epsilon &= b(X_t^\epsilon)dt + \epsilon dW_t \\ X_0^\epsilon &= x \in H \subseteq R^d \end{aligned} \tag{1}$$

  where W_t is a d-dim Brownian motion, $d > 1$, and
- $b(x) = b^+(x)$ if $x_1 > 0$ and $b(x) = b^-(x)$ if the first component $x_1 \leq 0$,

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where W_t is a d-dim Brownian motion, $d > 1$, and

- $b(x) = b^{+}(x)$ if $x_1 > 0$ and $b(x) = b^{-}(x)$ if the first component $x_1 \leq 0$,
- $H = \{x \in R^d, x_1 = 0\}$,
- and $b^{+}(x)$ and $b^{-}(x)$ are two smooth and bounded vector fields on R^d.

To study the fluctuation of $X^{\epsilon}(\cdot)$, we need to find a deterministic function $\phi(\cdot)$ such that X^{ϵ} converges to ϕ weakly, i.e., $lim_{\epsilon \to 0} P(||X^{\epsilon}(\cdot) - \phi(\cdot)|| > \delta) = 0$ for every $\delta > 0$.

Introduction and summary 2

- We consider the case that $\phi(t) \in H$ for every t. For otherwise the fluctuation of X^ϵ is the same as Lipschitz coefficients. Thus,
- we require the stability condition of $b(x)$, i.e., there are two different non-negative constants c_1, c_2 and $\delta_0 > 0$ such that

$$b_1^+(x) \leq -c_2, x_1 \in (\delta_0, 0) \qquad (2)$$
$$b_1^-(x) \geq c_1, x_1 \in (-\delta_0, 0).$$

- Under the stability condition, there exists a unique weak limit ϕ of X^ϵ such that $\phi(t) \in H$ for all $t \geq 0$.

This result follows that of large deviation of X^ϵ and a quick review is as follows.

Introduction and summary 3

For the system (1), let $u_t^{\epsilon+} = \int_0^t \chi_{(0,\infty)X_1^\epsilon(s)} ds$. We have for $\phi \in C[0,T], \psi \in H^+(\phi)$,

- $$P(\| X^\epsilon(\cdot) - \phi(\cdot) \|_{[0,T]} < \delta, \| u^{\epsilon+}(\cdot) - \psi(\cdot) \|_{[0,T]} < \delta) \quad (3)$$
  $$\sim exp(-I(\phi,\psi)/\epsilon^2)$$

- where

  $$I(\phi,\psi) = 1/2 \int_0^1 |\dot\phi(t) - b_{\phi,\psi}(t)|^2 dt + 1/2 \int_{\phi_t \in H, b_1^-(\phi_t) < b_1^+(\phi_t)} (b_1^+(\phi_t) - b_1^-(\phi_t))^2 \dot\psi_t (1 - \dot\psi_t) dt.$$

  is the rate function. Here $H^+(\phi)$ is the set of all real-valued absolutely continuous functions on [0,T] starting from 0 satisfying $\dot\psi_t = 1$ if $\phi_1(t) > 0$, $\dot\psi_t = 0$ if $\phi_1(t) < 0$ and $\dot\psi_t \in [0,1]$ for $\phi_t = 0$.

Introduction and summary 4

- Here,

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$$b_{\phi,\psi}(t) = b^{+}(\phi_t)\dot{\psi}_t + b^{-}(\phi_t)(1 - \dot{\psi}_t).$$

- It hence follows from the contraction principle that

$$P(\| X^{\epsilon}(\cdot) - \phi(\cdot) \|_{[0,T]} < \delta) \sim exp(-I(\phi)/\epsilon^2)$$

where $I(\phi) = inf_{\psi \in H^{+}(\phi)} I(\phi, \psi)$.

Introduction and summary 5

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- Since $\dot{\phi}_1(t) = 0$, we must have
  $b_1^+(\phi_t)\dot{\psi}_t + b_1^-(\phi_t)(1 - \dot{\psi}_t) = 0$.

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- Since $\dot\phi_1(t) = 0$, we must have
$b_1^+(\phi_t)\dot\psi_t + b_1^-(\phi_t)(1-\dot\psi_t) = 0$.
- Therefore, $\dot\psi_t = b_1^-(\phi_t)/(b_1^-(\phi_t) - b_1^+(\phi_t))$. Let

$$p(x) = b_1^-(x)/(b_1^-(x) - b_1^+(x)) \text{ and } q(x) = 1 - p(x).$$

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Then the pair (ϕ, ψ) satisfies

$$
\begin{aligned}
\dot{\psi}_t &= p(\phi_t) \text{ and} \\
\dot{\phi}_t &= b_1^-(\phi_t)/(b_1^-(\phi_t) - b_1^+(\phi_t)) \cdot b^+(\phi_t) \\
&\quad -b_1^+(\phi_t)/(b_1^-(\phi_t) - b_1^+(\phi_t)) \cdot b^-(\phi_t) \\
&= p(\phi_t)b^+(\phi_t) + q(\phi_t)b^-(\phi_t).
\end{aligned}
$$

The large deviation principle then implies that

$$
lim_{\epsilon \to 0} P(\| X^\epsilon(\cdot) - \phi(\cdot) \|_{[0,T]} < \delta, \| u^{\epsilon+}(\cdot) - \int_0^T p(\phi_s) ds \|_{[0,T]} < \delta) = 1.
$$

We shall study the fluctuation of $(X^\epsilon(\cdot), u^{\epsilon+}(\cdot))$ around $(\phi(\cdot), \int_0^T p(\phi_s) ds)$.

Introduction and summary 7

- Let Y_t be the following Ornstein-Uhlenbeck process :

$$\begin{aligned} dY_t &= p(\phi_t)\nabla b^+(\phi_t)Y_t dt + q(\phi_t)\nabla b^-(\phi_t)Y_t dt \\ &+(\nabla p(\phi_t)\cdot Y_t)(b^+(\phi_t) - b^-(\phi_t))dt \\ &-(b^+(\phi_t) - b^-(\phi_t))/(b_1^+(\phi_t) - b_1^-(\phi_t))dW_1(t) + dW_t \\ Y_0 &= 0 \end{aligned}$$

where ∇ is the gradient operator.

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&+(\triangledown p(\phi_t)\cdot Y_t)(b^+(\phi_t) - b^-(\phi_t))dt \\
&-(b^+(\phi_t) - b^-(\phi_t))/(b_1^+(\phi_t) - b_1^-(\phi_t))dW_1(t) + dW_t \\
Y_0 &= 0
\end{aligned}
$$

  where $\triangledown$ is the gradient operator.

- The main theorem is $\{(X^\epsilon(\cdot) - \phi(\cdot))/\epsilon\}_{\epsilon>0}$ converges to an Ornstein-Uhlenbeck process $Y(\cdot)$ in probability in $C([0,T],R^d)$ and $(u^{\epsilon+}(\cdot) - \int_0^{\cdot} p(\phi_s)ds)/\epsilon$ converges to a Gaussian process in probability in $H^+(\phi)$ (the Cameron-Martin space), hence in distribution as $\epsilon \to 0$.

Outline for section 2

Fluctuation of $u_t^{\epsilon+}$ 8

Let $f : R^d \to R$ be a bounded smooth function and for $x \in R^d, |x_1| < \delta$, be defined as follows.

$$f(x) = f(x_1, \bar{x}) = \int_0^{x_1} -1/(b_1^-(s,\bar{x}) - b_1^+(s,\bar{x}))ds, \quad |x_1| < \delta_0. \tag{4}$$

By Ito's lemma, we have

$$df(X_t^\epsilon) \quad = \nabla f(X_t^\epsilon) \cdot dX_t^\epsilon + \epsilon^2/2 \triangle f(X_t^\epsilon)dt$$

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By Ito's lemma, we have

$$\begin{aligned}
df(X_t^\epsilon) &= \nabla f(X_t^\epsilon)\cdot dX_t^\epsilon + \epsilon^2/2\triangle f(X_t^\epsilon)dt \\
&= (-b_1(X_t^\epsilon)/(b_1^-(X_t^\epsilon) - b_1^+(X_t^\epsilon)) + \bar{\nabla} f(X_t^\epsilon)\cdot \bar{b}(X_t^\epsilon) \\
&+\epsilon^2/2\triangle f(X_t^\epsilon))dt + \epsilon\nabla f(X_t^\epsilon)\cdot dW_t \\
&= -b_1^+(X_t^\epsilon)/(b_1^-(X_t^\epsilon) - b_1^+(X_t^\epsilon))\chi_{X_1^\epsilon(t)>0}dt \\
&+(-b_1^-(X_t^\epsilon))/(b_1^-(X_t^\epsilon) - b_1^+(X_t^\epsilon))\chi_{X_1^\epsilon(t)<0}dt \\
&+\bar{\nabla} f(X_t^\epsilon)\cdot\bar{b}(X_t^\epsilon)dt + \epsilon^2/2\triangle f(X_t^\epsilon)dt + \epsilon\nabla f(X_t^\epsilon)\cdot dW_t \\
&= ((1 - p(X^\epsilon(t)))\chi_{(X_1^\epsilon(t)>0)}dt - p(X^\epsilon(t))\chi_{(X_1^\epsilon(t)<0)}dt \\
&+\bar{\nabla} f(X_t^\epsilon)dt\cdot\bar{b}(X_t^\epsilon)dt + \epsilon^2/2\triangle f(X_t^\epsilon)dt + \epsilon\nabla f(X_t^\epsilon)\cdot dW_t \\
&= du_t^{\epsilon+} - p(X_t^\epsilon)dt + \bar{\nabla} f(X_t^\epsilon)\cdot\bar{b}(X_t^\epsilon)dt + \epsilon^2/2\triangle f(X_t^\epsilon)dt \\
&+\epsilon\nabla f(X_t^\epsilon)\cdot dW_t.
\end{aligned}$$

Fluctuation of $u_t^{\epsilon+}$ 9

We conclude with

$$\begin{aligned} u_t^{\epsilon+} - \int_0^t p(X_s^\epsilon)ds &= f(X_t^\epsilon) - f(X_0) - \int_0^t \bar{b}(X_s^\epsilon) \cdot \bar{\nabla} f(X_s^\epsilon)ds \\ &-\epsilon^2/2 \int_0^t \triangle f(X_s^\epsilon)ds - \epsilon \int_0^t \nabla f(X_s^\epsilon) \cdot dW_s. \end{aligned}$$

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Our first claim is :
Lemma 2.1 $\int_0^\cdot \nabla f(X_s^\epsilon) \cdot dW_s \to \int_0^\cdot 1/(b_1^+(\phi_s) - b_1^-(\phi_s))dW_1(s)$ in probability in $C[0, T]$ as $\epsilon \to 0$.

This is true because of the following two observations.

- $lim_{\epsilon \to 0} \parallel X^\epsilon(\cdot) - \phi(\cdot) \parallel > \delta = 0$
- $\bar{\nabla} f = 0$ on H.

Fluctuation of $u_t^{\epsilon+}$ 10

The second claim is that $|X_1^\epsilon(t)|$ is small compared with ϵ.
Lemma 2.2 $X_1^\epsilon(\cdot)/\epsilon \to 0$ in probability in $C[0, T]$, i.e., for any $\delta > 0$,

$$lim_{\epsilon\to 0} P(sup_{0\le t\le T}|X_1^\epsilon(t)|/\epsilon \ge \delta) = 0$$

We outline the proof as follows.

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We outline the proof as follows.

- Let $y_t = X_1^\epsilon(\epsilon^2 t)/\epsilon^2$. Then y_t satisfies the following.

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We outline the proof as follows.

- Let $y_t = X_1^\epsilon(\epsilon^2 t)/\epsilon^2$. Then y_t satisfies the following.
-

$$dy_t = b_1(X_{\epsilon^2 t}^\epsilon)dt + dB_t,$$
$$y_0 = 0$$

where $B_t = W_1(\epsilon^2 t)/\epsilon$ is a 1-dimensional Brownian motion.

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We outline the proof as follows.

- Let $y_t = X_1^\epsilon(\epsilon^2 t)/\epsilon^2$. Then y_t satisfies the following.
- $$dy_t = b_1(X^\epsilon_{\epsilon^2 t})dt + dB_t,$$
  $$y_0 = 0$$

  where $B_t = W_1(\epsilon^2 t)/\epsilon$ is a 1-dimensional Brownian motion.
- Let $\tau^\epsilon = inf(t : |y_t| \ge \delta/\epsilon)$. Then

  $$P(sup_{t\le T}|X_1^\epsilon(t)/\epsilon| \ge \delta) = P(sup_{t\le T}|y_{t/\epsilon^2}| \ge \delta/\epsilon) = P(\tau^\epsilon \le T/\epsilon^2).$$
- Ito's lemma then implies the result.

Fluctuation of $u_t^{\epsilon+}$ 11

The fluctuation of $u_t^{\epsilon+}$ is stated as follows.
Theorem 2.3 Let $u^{\epsilon+}(\cdot)$ be the occupation time of $X^{\epsilon}(\cdot)$ in H^+.
Then

$$1/\epsilon(u^{\epsilon+}(\cdot) - \int_0^{\cdot} p(X_s^{\epsilon})ds) \to \int_0^{\cdot} 1/(b_1^-(\phi_s) - b_1^+(\phi_s))dW_1(s)$$

in probability in $C[0, T]$ as $\epsilon \to 0$.

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$$1/\epsilon(u^{\epsilon +}(\cdot) - \int_0^{\cdot} p(X_s^{\epsilon})ds) \to \int_0^{\cdot} 1/(b_1^{-}(\phi_s) - b_1^{+}(\phi_s))dW_1(s)$$

in probability in $C[0, T]$ as $\epsilon \to 0$.

Outline of the proof.

$$\begin{aligned} &1/\epsilon(u_t^{\epsilon +} - \int_0^t p(X_s^{\epsilon})ds) \\ &= f(X_t^{\epsilon})/\epsilon - 1/\epsilon \int_0^t \bar{b}(X_s^{\epsilon}) \cdot \bar{\nabla} f(X_s^{\epsilon})ds - \epsilon/2 \int_0^t \triangle f(X_s^{\epsilon})ds \\ &- \int_0^t \nabla f(X_s^{\epsilon}) \cdot dW_s = I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Fluctuation of $u_t^{\epsilon+}$ 12

- $I_1, I_2 \to 0$ in probability follows from Lemma (2.2) because $|f(X_t^\epsilon)| \le |X_1^\epsilon(t)|, (4)$.
- $I_3 \to 0$ in probability is trivial.
  $I_4 \to \int_0^\cdot 1/(b_1^-(\phi_s) - b_1^+(\phi_s))dW_1(s)$ because of Lemma (2.1).

Outline for section 3

Introduction and summary

Fluctuation of $u_t^{\epsilon+}$

Fluctuation of $X_\cdot^\epsilon$ around $\phi_\cdot$

References

Fluctuation of $X^{\epsilon}_{\cdot}$ 13

Recall: X_t^ϵ and ϕ_t satisfy the following equations respectively.

$$\begin{aligned} dX_t^\epsilon &= b^+(X_t^\epsilon)du_t^{\epsilon+} + b^-(X_t^\epsilon)du_t^{\epsilon-} + \epsilon dW_t, \\ X_0 &= x \in H^+ \ \text{ and} \\ \dot{\phi}_t &= p(\phi_t)b^+(\phi_t) + q(\phi_t)b^-(\phi_t), \\ \phi_0 &= x \end{aligned}$$

where $q(x) = 1 - p(x)$ and $u_t^{\epsilon+} + u_t^{\epsilon-} = t$. Let

$$Y_t^\epsilon = (X_t^\epsilon - \phi_t)/\epsilon.$$

Fluctuation of $X^{\epsilon}_{\cdot}$ 14

$$dY^{\epsilon}_t = 1/\epsilon(b^+(X^{\epsilon}_t)du^{\epsilon+}_t + b^-(X^{\epsilon}_t)du^{\epsilon-}_t - (p(\phi_t)b^+(\phi_t)dt + q(\phi_t)b^-(\phi_t)dt)) + dW_t$$

Fluctuation of $X_\cdot^\epsilon$ 14

$$
\begin{aligned}
dY_t^\epsilon &= 1/\epsilon(b^+(X_t^\epsilon)du_t^{\epsilon+} + b^-(X_t^\epsilon)du_t^{\epsilon-} - (p(\phi_t)b^+(\phi_t)dt \\
&\quad + q(\phi_t)b^-(\phi_t)dt)) + dW_t \\
&= 1/\epsilon((b^+(X_t^\epsilon) - b^+(\phi_t))du_t^{\epsilon+} + b^+(\phi_t)(du_t^{\epsilon+} - p(\phi_t)dt)) \\
&\quad + 1/\epsilon((b^-(X_t^\epsilon) - b^-(\phi_t))du_t^{\epsilon-} + b^-(\phi_t)(du_t^{\epsilon-} - q(\phi_t)dt)) + dW_t \\
&= 1/\epsilon(b^+(X_t^\epsilon) - b^+(\phi_t))du_t^{\epsilon+} + 1/\epsilon(b^-(X_t^\epsilon) - b^-(\phi_t))du_t^{\epsilon-} \\
&\quad + 1/\epsilon(b^+(\phi_t) - b^-(\phi_t))(du_t^{\epsilon+} - p(\phi_t)dt) + dW_t \\
&= 1/\epsilon(b^+(X_t^\epsilon) - b^+(\phi_t))du_t^{\epsilon+} + 1/\epsilon(b^-(X_t^\epsilon) - b^-(\phi_t))du_t^{\epsilon-} \\
&\quad + 1/\epsilon(b^+(\phi_t) - b^-(\phi_t))(du_t^{\epsilon+} - p(X_t^\epsilon)dt) \\
&\quad + 1/\epsilon(b^+(\phi_t) - b^-(\phi_t))(p(X_t^\epsilon) - p(\phi_t))dt + dW_t.
\end{aligned}
$$

Fluctuation of $X^{\epsilon}_{\cdot}$15

Let Y_t be the solution of the following linear stochastic differential equation.

$$\begin{aligned} dY_t &= p(\phi_t)\nabla b^{+}(\phi_t) Y_t dt + q(\phi_t)\nabla b^{-}(\phi_t) Y_t dt \\ &\quad + (\nabla p(\phi_t)\cdot Y_t)(b^{+}(\phi_t) - b^{-}(\phi_t))dt \\ &\quad - (b^{+}(\phi_t) - b^{-}(\phi_t))/(b_1^{+}(\phi_t) - b_1^{-}(\phi_t))dW_1(t) + dW_t \\ &= a_t Y_t dt + \sigma_t dW_t \\ Y_0 &= 0 \end{aligned}$$

where $(\nabla b)_{i,j} = \partial b_i/\partial x_j$ is the gradient of $b(x)$.

Fluctuation of $X_\cdot^\epsilon$ 16

The main result of this paper is the following.

Theorem 3.1 The processes $Y_\cdot^\epsilon$ converges to $Y_\cdot$ in probability in $C([0,1], R^d)$, hence in distribution as $\epsilon \to 0$.

Let $R_t^\epsilon = Y_t^\epsilon - Y_t$. Then

$$
\begin{aligned}
R_t^\epsilon &= (X_t^\epsilon - \phi_t)/\epsilon - Y_t \\
&= \int_0^t 1/\epsilon(b^+(X_s^\epsilon) - b^+(\phi_s))du_s^{\epsilon+} - \nabla b^+(\phi_s)Y_s p(\phi_s)ds \\
&+ \int_0^t 1/\epsilon(b^-(X_s^\epsilon) - b^-(\phi_s))du_s^{\epsilon-} - \nabla b^-(\phi_s)Y_s q(\phi_s)ds \\
&+ \int_0^t (b^+(\phi_s) - b^-(\phi_s))((du_s^{\epsilon+} - p(X_s^\epsilon)ds)/\epsilon \\
&\quad - dW_1(s)/(b_1^-(\phi_s) - b_1^+(\phi_s))) \\
&+ \int_0^t (b^+(\phi_s) - b^-(\phi_s))(1/\epsilon(p(X_s^\epsilon) - p(\phi_s)) - \nabla p(\phi_s) \cdot Y_s)ds \\
&= I_1(t) + I_2(t) + I_3(t) + I_4(t).
\end{aligned}
$$

Fluctuation of $X^{\epsilon}_{\cdot}$

We concentrate on I_1. First note that

$$
\begin{aligned}
&1/\epsilon(b^+(X_t^\epsilon) - b^+(\phi_t)) \\
&= 1/\epsilon \int_0^1 \nabla b^+(\phi_t + \theta(X_t^\epsilon - \phi_t))(X_t^\epsilon - \phi_t) d\theta \\
&= 1/\epsilon \int_0^1 (\nabla b^+(\phi_t + \theta(X_t^\epsilon - \phi_t)) - \nabla b^+(\phi_t))(X_t^\epsilon - \phi_t) d\theta \\
&+ 1/\epsilon \nabla b^+(\phi_t)(X_t^\epsilon - \phi_t),
\end{aligned}
$$

Fluctuation of $X^\epsilon_\cdot$

$$I_1(t) = \int_0^t 1/\epsilon(b^+(X^\epsilon_s) - b^+(\phi_s))du^{\epsilon+}_s - \nabla b^+(\phi_s)Y_s p(\phi_s)ds$$

Fluctuation of $X^\epsilon_\cdot$

$$
\begin{aligned}
I_1(t) &= \int_0^t 1/\epsilon(b^+(X_s^\epsilon) - b^+(\phi_s))du_s^{\epsilon+} - \nabla b^+(\phi_s) Y_s p(\phi_s) ds \\
&= \int_0^t \int_0^1 (\nabla b^+(\phi_s + \theta(X_s^\epsilon - \phi_s)) - \nabla b^+(\phi_s)) d\theta \\
&\quad ((X_s^\epsilon - \phi_s)/\epsilon - Y_s) du_s^{\epsilon+} \\
&+ \int_0^t \int_0^1 (\nabla b^+(\phi_s + \theta(X_s^\epsilon - \phi_s)) - \nabla b^+(\phi_s)) d\theta Y_s du_s^{\epsilon+} \\
&+ \int_0^t \nabla b^+(\phi_s)(X_s^\epsilon - \phi_s/\epsilon - Y_s) du_s^{\epsilon+} \\
&+ \int_0^t \nabla b^+(\phi_s) Y_s (du_s^{\epsilon+} - p(\phi_s) ds) \\
&= I_{1.1}(t) + I_{1.2}(t) + I_{1.3}(t) + I_{1.4}(t).
\end{aligned}
$$

Fluctuation of $X^\epsilon_\cdot$

Finally, we have $|R^\epsilon_t| = \int_0^t K|R^\epsilon_s|(\| X^\epsilon(\cdot) - \phi(\cdot) \| +1)ds + \bar{R}^\epsilon$ where $\bar{R}^\epsilon$ (independent of t)converges to 0 in $C[0, T]$ in probablility. Let

$$\Omega^\epsilon_\delta = \{\omega \in \Omega : \| \bar{R}^\epsilon \| \leq \delta\}.$$

Thus, $lim_{\epsilon \to 0} P(\Omega^\epsilon_\delta) = 1$ for any $\delta > 0$ But on Ω^ϵ_δ, we have

$$|R^\epsilon_t| \leq K \int_0^t |R^\epsilon_s| ds + \delta,$$

thus $|R^\epsilon_t| \leq \delta e^{Kt}$ by Gronwall inequality and this implies that $R^\epsilon_\cdot \to 0$ in probability.

Outline for section 4

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