SCALING LIMITS OF INTERACTING DIFFUSIONS IN DOMAINS

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Joint work with Wai Tong (Louis) Fan

REFERENCES

- Z.-Q. Chen and W.-T. Fan, Hydrodynamic limits and propagation of chaos for interacting random walks in domains. arXiv:1311.2325.
- Z.-Q. Chen and W.-T. Fan, Systems of interacting diffusions with partial annihilations through membranes. arXiv:1403.5903.
- Z.-Q. Chen and W.-T. Fan, Functional central limit theorem for Brownian particles in domains with Robin boundary condition. arXiv:1404.1442.
- Z.-Q. Chen and W.-T. Fan, Fluctuation limit for systems of interacting diffusions with partial annihilations through membranes. In preparation.
- Z.-Q. Chen and W.-T. Fan, Scaling Limits of Interacting Diffusions in Domains. *Frontiers of Mathematics in China,* **9** (2014), 717-736.

HYDRODYNAMIC LIMIT

Rigorous derivation of macroscopic behavior of microscopic models .

Hilbert's 6th problem (1900)

Example: Reflected Brownian motion in (smooth) domain D:

$$dX_t = dB_t + \vec{n}(X_t)\, dL_t.$$

It is well-known that RBM is related to heat equation in domain D with insulated (Neumann) boundary condition.

• Transition density function $p(t, x, y)$ of X is the fundamental solution of the Neumann heat equation in D. In other words,

$$u(t, x) := \mathbb{E}_x[f(X_t)]$$

satisfies $\frac{\partial u}{\partial t} = \Delta u$ with $\frac{\partial}{\partial \vec{n}} u(t, x) = 0$ for $x \in \partial D$ and $u(0, x) = f(x)$.

HYDRODYNAMIC LIMIT AND FLUCTUATION

- A deeper connection is as follows. Let X_t^i, $1 \leq i \leq N$, be i.i.d. RBMs in D so that $\frac{1}{N}\sum_{j=1}^N \mathbf{1}_{X_0^i}(dx) \Rightarrow f(x)dx$, then

$$\frac{1}{N}\sum_{j=1}^N \mathbf{1}_{X_t^i}(dx) \Rightarrow \text{ deterministic } \mu_t(dx) = u(t,x)dx$$

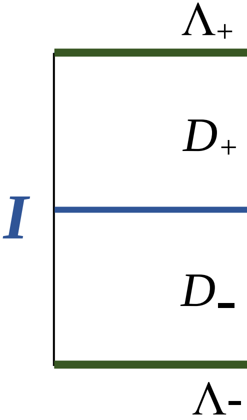
where $u(t,x)$ satisfies $\frac{\partial u}{\partial t} = \Delta u$ with $\frac{\partial}{\partial \vec{n}} u(t,x) = 0$ for $x \in \partial D$ and $u(0,x) = f(x)$. In fact, it is a functional convergence.

Fluctuation or functional CLT: Does there exist a_N so that

$$a_N \left(\frac{1}{N}\sum_{j=1}^N \mathbf{1}_{X_t^i}(dx) - u(t,x)dx \right)$$

converges to some non-trivial process?

INTERACTING PARTICLE SYSTEM



- Interface I
- Harvest sites $\Lambda_{\pm}$

MODELING

At microscopic level:

- We use reflecting Brownian motion (more generally, diffusion) with drift to model the movement of positive (and negative) charges. The drift models the electric potential these charges are subject to.
- These two types of reflecting Brownian motions with gradient drift $\nabla(\log \rho_\pm)$ are confined within their own media and annihilate each other at certain rate when they come close near the interface, where $\rho_\pm \in C^2(\overline{D}_\pm)$ is strictly positive.
- The interaction at the interface models the annihilation, trapping, recombination and separation phenomena of the charges.

One difficulty: Independent reflecting Brownian motions in dimension two and higher will not collide to each other.

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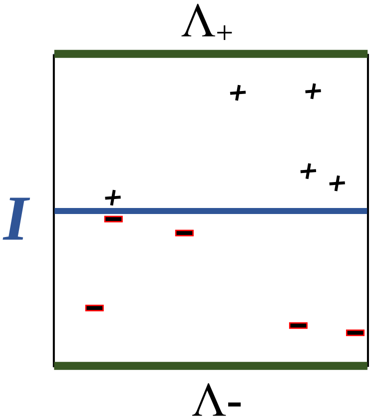
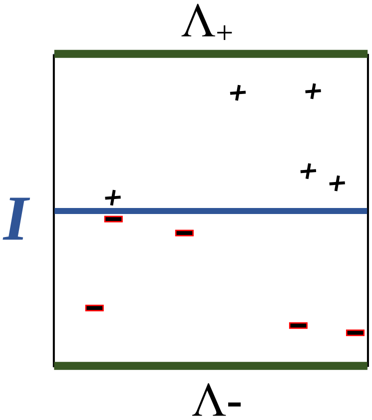


- $dX_t^{\pm} = dB_t^{\pm} + \nabla \log \rho_{\pm}(X_t^{\pm})dt + \vec{n}(X_t^{\pm})\, dL_t, \quad t < T_{\Lambda_{\pm}}$

INTERACTING PARTICLE SYSTEM



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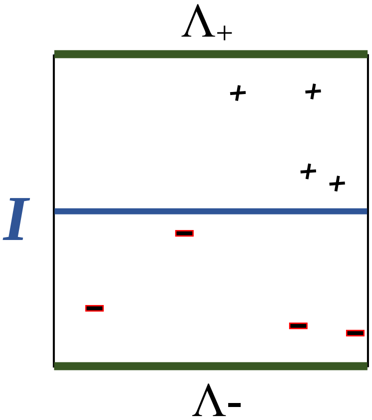

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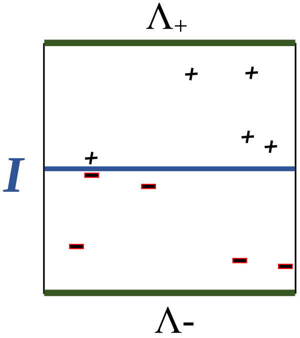
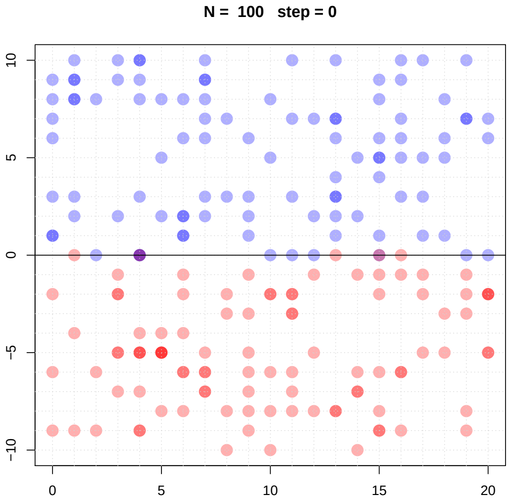
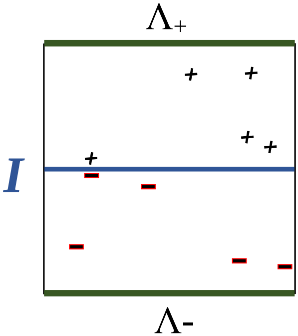


- N = initial number of particles
- annihilation distance $\epsilon \approx N^{-1/d}$ i.e. $N\epsilon^d = O(1)$
- per pair annihilation rate $\approx \lambda/\epsilon$



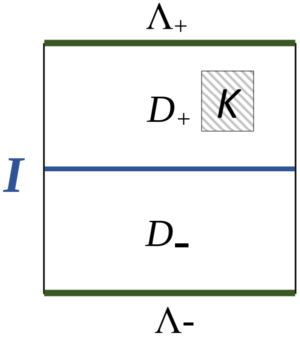
N = 100 step = 0
10
5
0
−5
−10
0
5
10
15
20

QUESTIONS



- How will the system evolve?

QUESTIONS

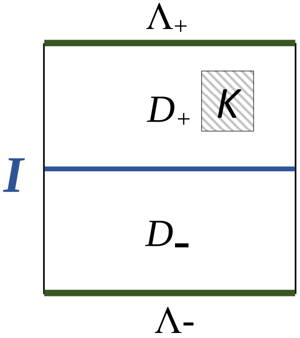


Q 1) How many particles are in K at time t?

Q 2) How many charges are harvested?

Q 3) How many charges are annihilated?

QUESTIONS



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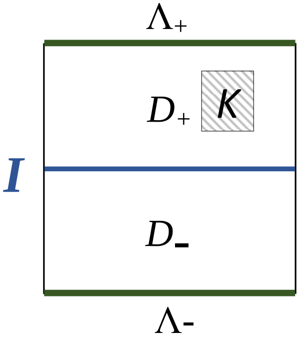

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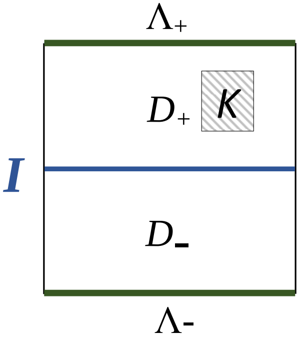

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MODELING

This interacting model can also be used to describe macroscopic phenomena with coupled boundary conditions, such as the population dynamics of two segregated species under competition.

Border war

RESEARCH TOPICS

- Random Walk Model and Continuous Diffusion Model
- Functional LLN for empirical measures $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$
- Functional CLT for empirical measures $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$

where

$$\mathfrak{X}_t^{N,+}(dx) \triangleq \frac{1}{N} \sum_{\alpha \sim t} \mathbf{1}_{X_\alpha^+(t)}(dx) \quad \text{on } \overline{D}_+$$

$$\mathfrak{X}_t^{N,-}(dy) \triangleq \frac{1}{N} \sum_{\beta \sim t} \mathbf{1}_{X_\beta^-(t)}(dy) \quad \text{on } \overline{D}_-$$

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HYDRODYNAMIC LIMIT

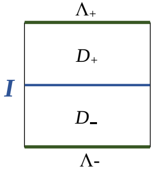
THEOREM (CHEN AND FAN, 2013)

$$(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-}) \xrightarrow{L} \big(u_+(t,x)\,\rho_+(x),\; u_-(t,y)\,\rho_-(y)\big)$$

satisfying

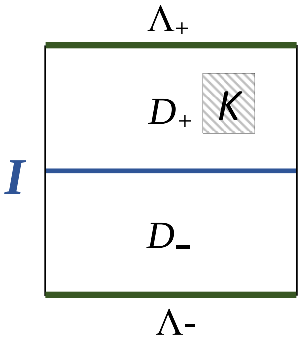
$$\begin{cases} \dfrac{\partial u_+}{\partial t} = \boxed{a_+\Delta u_+ + \nabla(\log \rho_+)\cdot\nabla u_+} & \text{on } D_+ \\ u_+ = 0 & \text{on } \Lambda_+ \\ \dfrac{\partial u_+}{\partial \vec{n}_+} = 0 & \text{on } \partial D_+ \setminus (\Lambda_+ \cup I) \\ \dfrac{\partial u_+}{\partial \vec{n}_+} = \dfrac{1}{\rho_+}\lambda\, u_+\, u_- & \text{on } I \end{cases}$$

with similar equations for u_- in D_-.



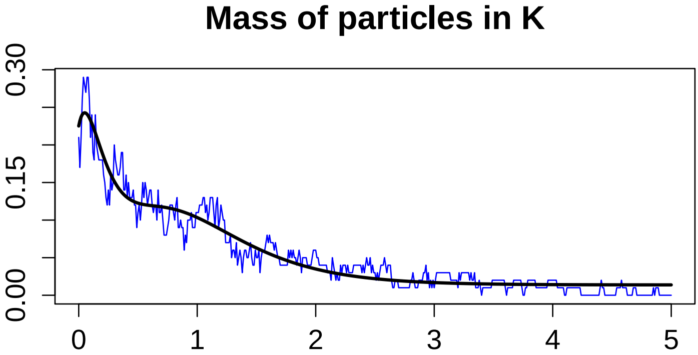
HYDRODYNAMIC LIMIT

Applications:



$$\frac{\text{Number of particles in } K}{N} \approx \int_K u_+(t,x)\,dx$$

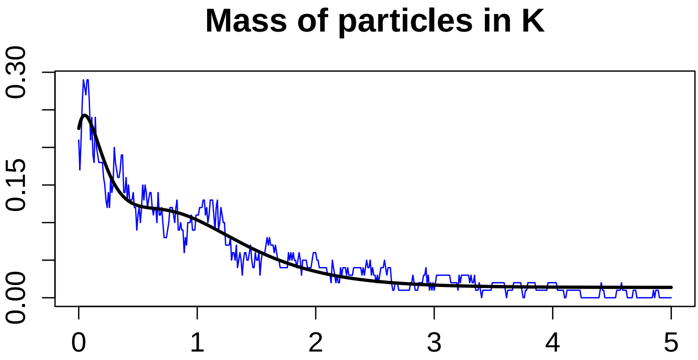
HYDRODYNAMIC LIMIT



$$\frac{\text{Number of particles in } K}{N} \quad \text{and} \quad \int_K u_+(t,x)\,dx$$

- What can we say about the error ?

HYDRODYNAMIC LIMIT



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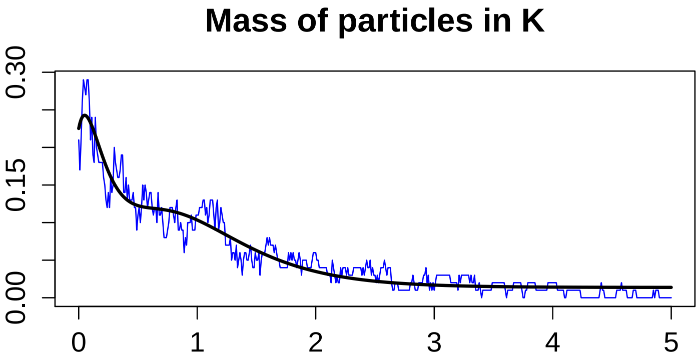

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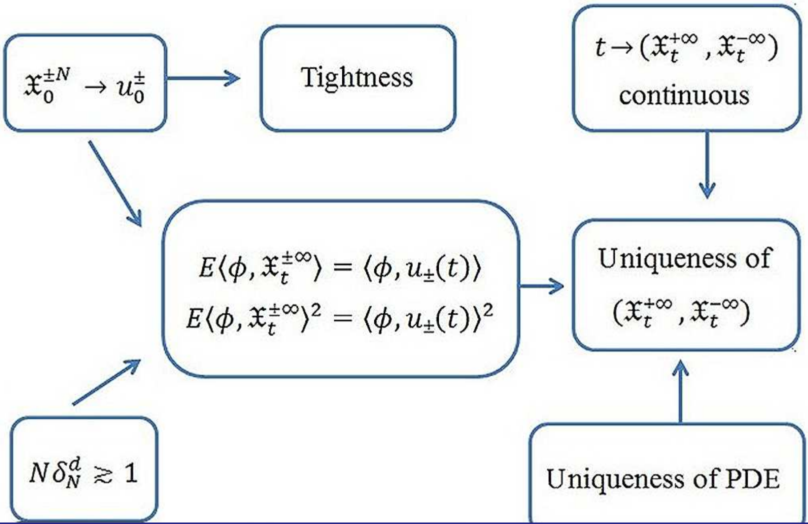
IDEA OF PROOFS

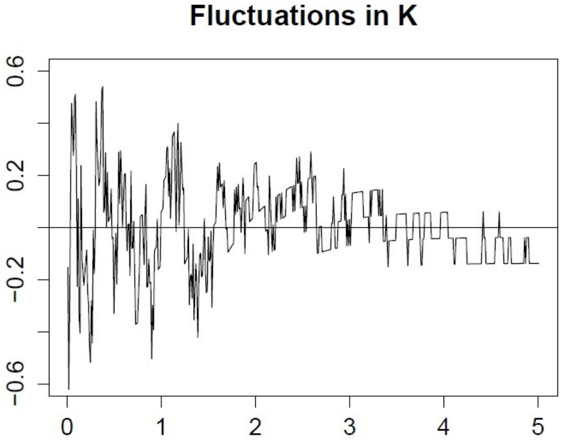


A BRIEF HISTORY

Rigorous derivation of macroscopic behavior of microscopic models

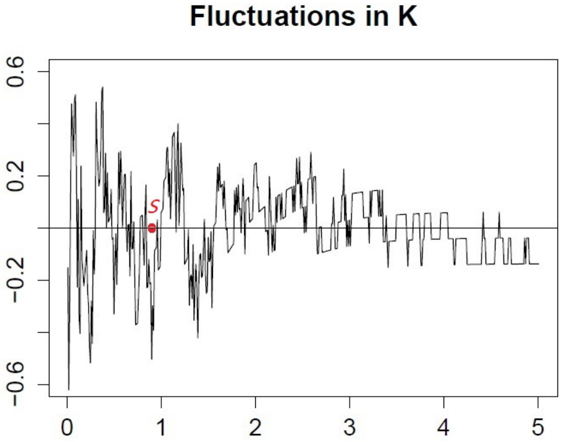
- Maxwell and Boltzmann's work on kinetic theory of gas
- Hilbert formulated it as a mathematical problem: his 6th problem (1900)
- Many work has been done on this subject by many people: simple exclusion process, reversible gradient system, . . . , including G.C. Papanicolaou, S.R.S. Varadhan, H.T. Yau, M.Z. Guo,
- Burdzy, Quastel (2006): N" $+$ " particles (X^i) and N" $-$ " particles performing independent RBWs. A pair of particles of opposite sign annihilates each other Simultaneously, 2 particles (one of each type) are chosen randomly to split.

FLUCTUATIONS



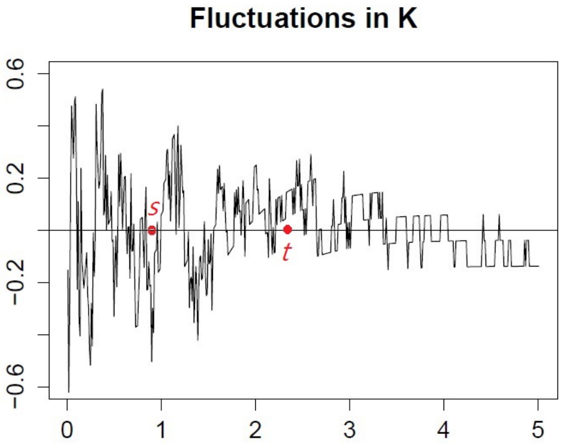
- $\mathcal{Y}_t^{N,+}(K) := \sqrt{N}\left(\frac{\text{Number of particles in } K}{N} - \int_K u_+(t,x)\,dx\right)$

FLUCTUATIONS



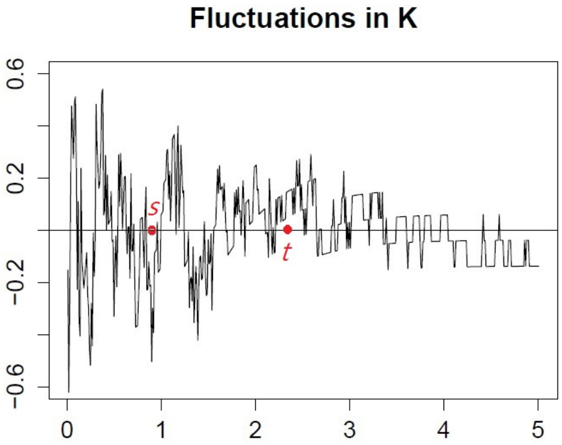
What is the probability distribution of $\mathcal{Y}_s^{N,+}(K)$ when N is large?

FLUCTUATIONS



- JOINT probability distribution of $\mathcal{Y}_s^{N,+}(K)$ and $\mathcal{Y}_t^{N,+}(K)$?
- What if we pick another region K?

FLUCTUATIONS



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FLUCTUATIONS

Consider $\mathcal{Z}_t^N(\phi_+, \phi_-) := \mathcal{Y}_t^{N,+}(\phi_+) + \mathcal{Y}_t^{N,-}(\phi_-)$.

THEOREM (CHEN AND FAN, 2014)

(CLT) $\mathcal{Z}^N \xrightarrow{L} \mathcal{Z}$ *in* $D([0, T_0], \mathbf{H})$*, where*

$$\mathcal{Z}_t = \mathbf{U}_{(t,0)}\mathcal{Z}_0 + \int_0^t \mathbf{U}_{(t,s)}\, dM_s.$$

M is a $\mathbf{H}$-valued Gaussian martingale with independent increments and covariance

$$\begin{aligned} &\langle M(\phi_+, \phi_-)\rangle_t \\ = &\int_0^t \Big(\int_{D_+} |\nabla \phi_+(x)|^2 u_+(s,x)\, dx + \int_{D_-} |\nabla \phi_-(y)|^2 u_-(s,y)\, dy \\ &+ \int_I (\phi_+(z) + \phi_-(z))^2 u_+(s,z) u_-(s,z)\, d\sigma(z) \Big) ds. \end{aligned}$$

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In the above, $u_{\pm}$ is the hydrodynamic limit of the interacting diffusion system.

$$(\mathbf{U}_{(t,s)}\mu)(\phi_+, \phi_-) := \mu(Q_{t,s}(\phi_+, \phi_-)),$$

where $Q_{t,s}(\phi_+, \phi_-)$ is the unique solution of a system of time-inhomogenous backward PDE with non-linear interacting term at the interface.

FLUCTUATIONS

COROLLARY

*(**Properties of** M **and** $\mathcal{Z}$)*

$$M \overset{\mathcal{L}}{=} M^+ \oplus M^-,$$

where $M^\pm$ is a continuous $\mathcal{H}^\pm$-valued Gaussian martingale with known covariance and

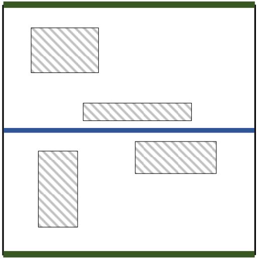
$$\mathbb{E}[M_s^+(\phi_+)\, M_t^-(\psi_-)] \quad = \quad \int_0^{s\wedge t} \int_I \phi_+\psi_-\, u_+(r)u_-(r)\, d\sigma\, dr$$

$\mathcal{Z}$ is a continuous Gaussian Markov process.

FLUCTUATIONS

Implications:

- $\{ Z_t(\phi_+, \phi_-) \; : \; t \geq 0, \; \phi_+, \; \phi_- \}$ is a Gaussian system.



- The curve $t \mapsto Z_t(\phi_+, \phi_-)$ can be simulated by 'transforming' a Brownian motion.
- Noise comes from 2 correlated sources: diffusive transport and interactions.

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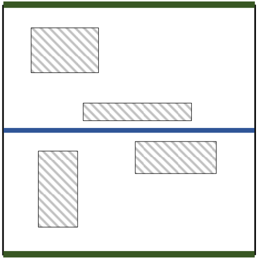

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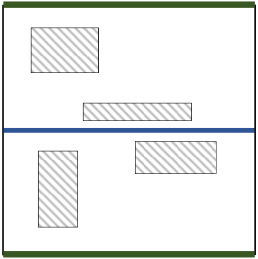

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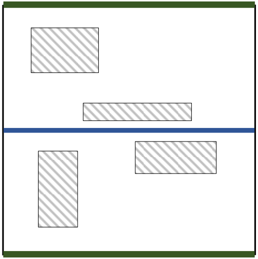

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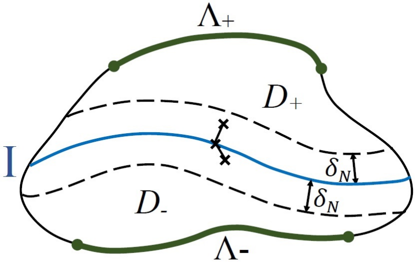
- Noise comes from 2 correlated sources: diffusive transport and interactions.

Fluctuations

- R.A. Holley, D.W. Stroock (1978)
- K. Itô (1980, 1983)
- A.S. Sznitman (1984)
- P. Kotelenez and P. Dittrich (1986, 1988)
- C. Boldrighini, A. De Masi, A. Pellegrinotti (1992)
- . . .

CONTINUOUS DIFFUSION MODEL

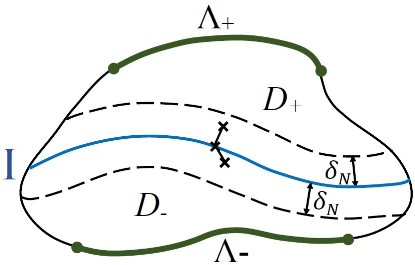


- $I^\delta \triangleq \{(x,y) : \sqrt{|x-z|^2 + |y-z|^2} < \delta \text{ for some } z \in I\}$
- per pair annihilation-rate

$$\frac{1}{N\,\delta_N^{d+1}}$$

- Goal: Prove functional CLT

CONTINUOUS DIFFUSION MODEL



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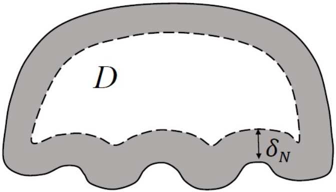
ROBIN BOUNDARY MODEL



- D^{δ_N} = shaded region
- RBMs killed by $q_N(t,x) := \frac{1}{\delta_N}\mathbf{1}_{D^{\delta_N}}(x)\, q(t,x)$
- Functional LLN:

$$\mathfrak{X}_t^N(dx) \triangleq \frac{1}{N}\sum_{\alpha\sim t}\mathbf{1}_{X_\alpha(t)}(dx) \to u(t,x)\,dx,$$

  u solves heat equation with $\frac{\partial u(t,x)}{\partial \vec{n}} = q(t,x)u(t,x)$.
- **Goal: Prove functional CLT**

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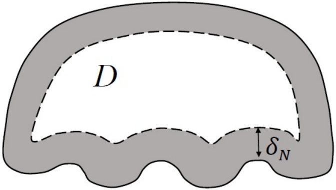

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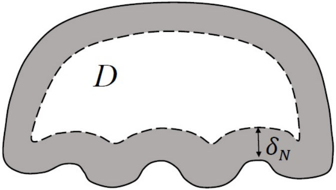
- **Goal: Prove functional CLT**

ROBIN BOUNDARY MODEL



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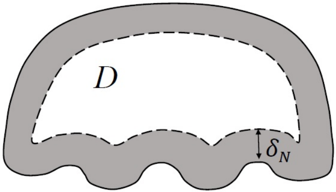

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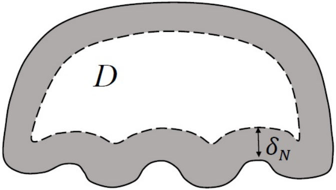

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IDEA OF PROOF FOR CLT

Fluctuation of $\mathfrak{X}_t^N$ is

$$\mathcal{Y}_t^N(\phi) := \sqrt{N}\left(\langle \mathfrak{X}_t^N, \phi\rangle - \mathbb{E}\langle \mathfrak{X}_t^N, \phi\rangle\right).$$

Question: What is the state space of $\mathcal{Y}^N$?

Consider $H_\alpha :=$ completion of linear span of $\{\phi_k\}$ with respect to

$$|\psi|_\alpha^2 = \sum_k (1+\lambda_k)^\alpha \langle \psi, \phi_k \rangle^2.$$

Then

$$\mathcal{H}_\alpha \subset \mathcal{H}_0 = \mathcal{H}_0' \subset \mathcal{H}_{-\alpha}, \quad \alpha \in [0, \infty).$$

LEMMA

When $\alpha > d/2$, we have $\mathcal{Y}_t^N \in \mathcal{H}_{-\alpha}$ for $t > 0$.

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EVOLUTION OPERATOR

The 'transportation part' is governed by the evolution operators $\{Q_{s,t}\}_{s\leq t}$ generated on $C(\overline{D})$ by the backward PDE $\frac{\partial v}{\partial s} = -Av$ on $(0,t) \times D$ with Robin boundary condition $\frac{\partial v}{\partial \vec{n}} = qv/\rho$ on $(0,t) \times \partial D$.

$$\mathbf{U}_{(t,s)}\mu(\phi) := \mu(Q_{s,t}\phi).$$

IDEA OF PROOF FOR CLT

LEMMA

Let $\{\mathbf{U}^N_{(t,s)}\}_{s\le t}$ be the evolution operator associated to $\frac{1}{2}\Delta - q_N(s)$.

$$\mathcal{Y}^N_t = \mathbf{U}^N_{(t,s)}\mathcal{Y}^N_s + \int_s^t \mathbf{U}^N_{(t,r)}\, dM^N_r,$$

where M^N be a $\mathcal{H}_{-\alpha}$-valued martingale with

$$\langle M^N(\phi)\rangle_t = \int_0^t \langle |\nabla\phi|^2 + q_N(s)\phi^2,\, \mathfrak{X}^N_s\rangle\, ds.$$

Reason:

$$M^\phi_t := \langle \phi, \mathfrak{X}^N_t\rangle - \langle \phi, \mathfrak{X}^N_0\rangle - \int_0^t \langle \frac{1}{2}\Delta\phi - q_N(s)\,\phi,\, \mathfrak{X}^N_s\rangle\, ds$$

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IDEA OF PROOF FOR CLT

LEMMA

When $\alpha > d \vee (d/2+1)$*, we have*

$$M^N \xrightarrow{L} M \ \in D([0,T], \mathcal{H}_{-\alpha}),$$

where M *is a* $\mathcal{H}_{-\alpha}$*-valued Gaussian martingale with independent increments and covariance*

$$\langle M(\phi)\rangle_t \ = \ \int_0^t \left(\int_D |\nabla \phi|^2 \, u(s) dx + \int_{\partial D} \phi^2 \, q(s) \, u(s) \, d\sigma \right) ds.$$

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Idea of proof for CLT

Step 1: $\mathcal{Y}^N$ satisfies

$$\mathcal{Y}_t^N = \mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N + \int_0^t \mathbf{U}_{(t,r)}^N \, dM_s^N$$

Step 2: $M^N \xrightarrow{L} M$ in $D([0,T], \mathcal{H}_{-\alpha})$.

Step 3: $\mathcal{Y}^N$ is tight in $D([0,T], \mathcal{H}_{-\alpha})$.

Step 4: $\mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N \xrightarrow{L} \mathbf{U}_{(t,0)} \mathcal{Y}_0$.

Step 5: $\int_0^t \mathbf{U}_{(t,s)}^N \, dM_s^N \xrightarrow{L} \int_0^t \mathbf{U}_{(t,s)} \, dM_s$.

Step 6: $\mathcal{Y}$ is continuous Gaussian Markov. □

We just "proved"

THEOREM (CHEN AND FAN, 2014)

*(**CLT**) For $\alpha > d+2$ and $T > 0$, $\mathcal{Y}^N \xrightarrow{L} \mathcal{Y}$ in $D([0,\ T], \mathcal{H}_{-\alpha})$, where*

$$\mathcal{Y}_t = \mathbf{U}_{(t,0)}\mathcal{Y}_0 + \int_0^t \mathbf{U}_{(t,s)}\, dM_s.$$

Moreover, $\mathcal{Y}$ is a Gaussian Markov process which has a version in $C^\gamma([0,T], \mathcal{H}_{-\alpha})$ for any $\gamma \in (0, 1/2)$.

CLT FOR ANNIHILATION MODEL:

Recall

THEOREM (CHEN AND FAN, 2014)

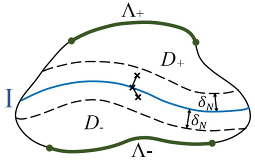
(CLT) $\mathcal{Z}^N \xrightarrow{L} \mathcal{Z}$ *in* $D([0, T_0], \mathbf{H})$, *where*

$$\mathcal{Z}_t = \mathbf{U}_{(t,0)}\mathcal{Z}_0 + \int_0^t \mathbf{U}_{(t,s)}\, dM_s.$$

M is a $\mathbf{H}$-valued Gaussian martingale with independent increments and covariance

$$\begin{aligned} &\langle M(\phi_+, \phi_-)\rangle_t \\ = &\int_0^t \Big(\int_{D_+} |\nabla\phi_+(x)|^2 u_+(s,x)\, dx + \int_{D_-} |\nabla\phi_-(y)|^2 u_-(s,y)\, dy \\ &+ \int_I (\phi_+(z) + \phi_-(z))^2 u_+(s,z) u_-(s,z)\, d\sigma(z) \Big) ds. \end{aligned}$$

IDOOF:



- We have

$$\mathcal{Z}_t^N = \mathbf{U}_{(t,0)}^N \mathcal{Z}_0^N + \int_0^t \mathbf{U}_{(t,s)}^N \, dM_s^N + \text{Error}^N(t)$$

- Asymptotic expansion of the (n, m)-correlation functions

$$\gamma_t^{N,(n,m)}(\vec{x}, \vec{y}) \approx \prod_{i=1}^{n} u_+(t, x_i) \prod_{j=1}^{m} u_-(t, y_j) + \frac{G_t^{N,(n,m)}(\vec{x}, \vec{y})}{N} + \frac{o(N)}{N}$$

- Boltzman-Gibbs principle

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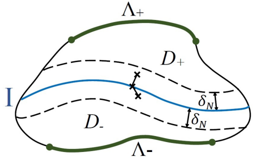

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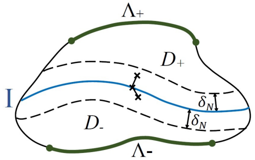

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- Generalized O-U processes
- Branching models
- Hard annihilation
- Multi-types, Multiple deletions

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Thank you!