Quasi-stationarity and Quasi-ergodicity of Markov Processes

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Background

Let $X = \{X_t,\ t \geq 0\}$ be a homogeneous and strong Markov process on a Polish space E with the Borel σ-algebra $\mathcal{E}$, $P(t)$ be the corresponding semigroup and $\{P_x,\ x \in E\}$ the Markov family.

- Ergodicity: $\exists$ a unique stationary ν,s.t. starting from any initial μ,

$$\mu P(t) \to \nu \text{ weakly as } t \to \infty.$$

Furthermore, $\forall f \in L(\nu)$,

$$t^{-1} \int_0^t f(X_s)ds \to \nu(f) \ a.s. \text{ and in } L^1 \text{ as } t \to \infty.$$

Background

If $P(t)$ is sub-Markovian, by adding an extra point "$''0''$" into E, we can assume $P(t)$ transient on $E_0 = E \setminus \{0\}$, with Δ as a unique absorbing state. Define

$$\tau = \inf\{t \geq 0, \ X_t = 0\}.$$

• Assumption: $P_x(\tau < \infty) = 1 \ \ \forall x \in E$.

We are interested in the long-time behavior of the process conditional on $\{\tau > t\}$. The motivation comes from the study of conditional large deviations for non-ergodic Markov processes.

Quasi-stationary distribution

• Quasi-stationary distribution(QSD): A probability measure ν on E_0 satisfying that

$$P_\nu(X_t \in \cdot | \tau > t) = \nu. \tag{1}$$

• Quasi-limiting distribution(QLD): A probability measure ν on E_0 for which there is an initial distribution μ, s.t.

$$\lim_{t\to\infty} P_\mu(X_t \in B | \tau > t) = \nu(B) \;\; \forall B \in \mathcal{E} \cap E_0. \tag{2}$$

• Yaglom limit: A probability measure ν on E_0 satisfying

$$\lim_{t\to\infty} P_x(X_t \in B | \tau > t) = \nu(B) \;\; \forall B \in \mathcal{E} \cap E_0, \; \forall x. \tag{3}$$

Yaglom limit $\Rightarrow$ QSD $\Leftrightarrow$ QLD.

Proposition

If ν is a QSD, then there is a $\lambda \geq 0$, s.t.

$$P_\nu(\tau > t) = e^{-\lambda t}, \quad \forall t \geq 0. \tag{4}$$

• To study: The existence of a QSD, its description, the convergence to it of conditioned processes, domain of attraction, its role in the process conditioned on $\{\tau > t\}$, and the behavior of τ.

Notice that if $\mathcal{B}_b(E)$ denote the set of bounded and measurable functions on E, then (2)$\Leftrightarrow$

$$E_\mu[f(X_t)|\tau > t] \to \nu(f) \ \ \forall f \in \mathcal{B}_b(E). \tag{5}$$

• Question:

$$E_\mu[\frac{1}{t}\int_0^t f(X_s)ds|\tau > t] \to ? \ \ (t \to \infty).$$

Quasi-ergodicity

- Quasi-ergodic distribution: A probability measure ν on E_0 for which

$$E_x[\frac{1}{t}\int_0^t f(X_s)ds|\tau > t] \to \nu(f)(t \to \infty) \ \ \forall f \in \mathcal{B}_b(E_0), \ \forall x. \qquad (6)$$

- Fractional Yaglom limit: A probability measure ν on E_0 for which

$$P_x(X_{qt} \in B|\tau > t) \to \nu(B) \ \ \forall B \in \mathcal{E} \cap E_0, \ 0 < q < 1, \ x \in E_0. \qquad (7)$$

Quasi-ergodicity

- Irreducibility: $\exists$ a reference measure π on $\mathcal{E}$ s.t.$\forall h > 0,\ x \in E_0$,

$$\pi(B) > 0 \Rightarrow \sum_{n=0}^{\infty} P_x(X_{nh} \in B;\ \tau > t) > 0 \quad B \in \mathcal{E} \cap E_0.$$

- Decay parameter:

$$\lambda = \inf\{\rho \geq 0 : \int_0^{\infty} e^{\rho t} P_x(X_t \in B;\ \tau > t) dt = \infty \quad \pi - a.e.x\},$$

which is independent of B with $\pi(B) > 0$.

Quasi-ergodicity

The process is said to be • λ-recurrent: if

$$\int_0^\infty e^{\lambda t} P_x(X_t \in B;\ \tau > t)dt = \infty$$

$\forall B$ with $\pi(B) > 0$ and for $\pi - a.e.x$;

• λ-transient: otherwise.

Proposition

If the process is λ-recurrent, then there is a measure α on E_0 and a non-negative and measurable function β on E_0, such that

$$(\alpha P_t)(B) \triangleq \int P_x(X_t \in B; \tau > t)\alpha(dx) = e^{-\lambda t}\alpha(B)$$

$$\forall t > 0,\ B \in \mathcal{E} \cap E_0,$$

and that

$$P_t\beta(x) \triangleq \int \beta(y)P_x(X_t \in dy, \tau > t) = e^{-\lambda t}\beta(x) \ \ \forall x \in E_0.$$

$$\left.\begin{array}{ll} \alpha: & \lambda\text{-invariant measure} \\ \beta: & \lambda\text{-invariant function} \end{array}\right\} \text{both unique}$$

Quasi-ergodicity

λ-positive recurrence: λ-recurrence and

$$\alpha(\beta) \triangleq \int \beta(x)\alpha(dx) < \infty.$$

Theorem

Suppose that $\{X_t,\ t \geq 0\}$ is λ-positive and α is finite. Normalize α and β so that

$$\alpha(1) = \alpha(\beta) = 1,$$

and define $dm = \beta d\alpha$. Then α is a QSD, whereas for any bounded, measurable f,

$$\lim_{t\to\infty} E_x[\frac{1}{t}\int_0^t f(X_s)ds|\tau > t] = m(f).$$

Quasi-ergodicity

Theorem

Under the same conditions as in the previous theorem, for any bounded,measurable f, g

$$\lim_{t\to\infty} E_x[f(X_{pt})g(X_{qt})|\tau > t] = \begin{cases} m(f)m(g), & \text{if } 0 < p,\ q < 1 \\ m(f)\alpha(g) & \text{if } 0 < p < q = 1 \end{cases}, \quad (8)$$

In particular,

$$\lim_{t\to\infty} E_x[g(X_{qt})|\tau > t] = \begin{cases} m(g), & \text{if } 0 < q < 1 \\ \alpha(g) & \text{if } q = 1 \end{cases}, \quad (9)$$

• Example 1. Linear birth-death process on Z_+, with birth and death rates given by

$$b_n = nb,\ \ n \geq 0,\ \ d_n = nd,\ \ n \geq 1,$$

where $b, d > 0$. 0 is the only absorbing state. τ is the absorbing time. The decay parameter $\lambda = |b - d|$. Then chain is always λ-positive for $b \neq d$. However the λ-invariant measure α is summable if and only if $b < d$.

• Example 2. Killed BM on R^d. Let $\{X_t,\ t \geq 0\}$ be a standard $d-$dimensional Brownian motion, $D \subset R^d$ is connected, bounded and open. Consider the Brownian motion killed outside D. Let $\{p^D(t,\ x,\ y),\ t \geq 0\}$ be the transition density of the killed BM with respect to the Lebesgue measure, then it is well known that it admits an eigen-expansion

$$p^D(t,\ x,\ y) = \sum_{n=1}^{\infty} \exp(-\lambda_n t)\varphi_n(x)\varphi_n(y),$$

where $0 < \lambda_1 < \lambda_2 \leq \cdots$ are the (nondecreasing) Dirichlet eigenvalues of $-\dfrac{\Delta}{2}$ counting multiplicity, φ_n are the corresponding eigenfunctions which form a complete orthonormal system of $L^2(D)$. Then $\lambda = \lambda_1$, and the conditions needed are fulfilled with $d\alpha = \varphi_1 dx$ and $dm = \varphi_1^2 dx$.

(Example 2 continued) Let $\{Y_t,\ 0 \le t < \infty\}$ be the diffusion on D with transition densities given by

$$Q(t; x, y) = \exp(\lambda_1 t)\frac{\varphi_1(y)}{\varphi_1(x)}P^D(t; x, y).$$

Corollary

Given $x \in D$, define

$$k = k(x) = \min\{n \ge 2 : \varphi_n(x) \ne 0\}$$

Then

$$\lim_{t\to\infty} e^{(\lambda_k - \lambda_1)t}||P_x^Q(Y_t \in \cdot) - m||_{Var} > 0.$$

A Variational representation of decay parameter

Now let $\{X_t,\ t \geq 0\}$ be a Markov chain on $E = E_0 \cup \{0\}$, irreducible on E_0, with the Q-matrix

$$Q = (q_{i,j}) = \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{q_0} & Q_1 \end{bmatrix}. \tag{10}$$

$q \triangleq \{q_i = \sum_{j \neq i} q_{i,j},\ i \in E\}$. Let L be the generator of the process, with domain $\mathcal{D}(L)$.

$$\mathcal{D}^+(L) = \{u \in \mathcal{D}(L),\ \inf u > 0\}.$$

For $A \in \mathcal{E}$, let $M_1(A)$ be the space of probability measures on A.

$$M_1^q(E) = \{\mu \in M_1(E) :\ \mu(q) < \infty\}.$$

A Variational representation of decay parameter

Define for $\mu \in M_1(E)$

$$I(\mu) = -\inf_{u \in \mathcal{D}^+(L)} \int \frac{Lu}{u} d\mu. \tag{11}$$

$$J(\mu) = \begin{cases} -\inf\limits_{u \in \mathcal{U}} \int \frac{Qu}{u} d\mu & \mu \in M_1^q(E) \\ +\infty & \textit{otherwise} \end{cases}, \tag{12}$$

where $\mathcal{U} = \{u : \text{ measurable on } E_0 \text{ with } \inf u > 0\}$.

Theorem

(1) $\lambda = \inf_{\mu \in M_1(E_0)} I(\mu) = \inf_{\mu \in M_1(E_0)} J(\mu)$;
(2) the infimum in the above formula is attained at some μ iff $P(t)$ is λ-positive, in this case, the μ is unique and is given by $\mu = m$, the quasi-ergodic distribution.

A Variational representation of decay parameter

Corollary

For any irreducible transition function on E,

$$-\lambda = \sup_{\mu \in M_1^q(E)} \inf_{u \in \mathcal{U}} \int \frac{Qu}{u} d\mu \tag{13}$$

$$= \inf_{u \in \mathcal{U}} \sup_{\mu \in M_1^q(E)} \int \frac{Qu}{u} d\mu = \inf_{u \in \mathcal{U}} \sup_{j \in E} \frac{(Qu)_j}{u_j} \tag{14}$$

Applying it to a birth-death chain with birth rates $\{b_n\}$ and death rates $\{d_n\}$, we see that

$$\lambda = \sup_{u \in \mathcal{U}} \inf_n \{b_n + d_n - b_n \frac{u_{n+1}}{u_n} - d_n \frac{u_{n-1}}{u_n}\} \tag{15}$$

A Variational representation of decay parameter

The above approach can be generalized to give variational representation for decay parameter of Q-matrix with certain potential. Given $V \in C_b(E)$, let $Q + V = (q_{ij}^V)_{i,j \in E}$, with $q_{ij}^V = q_{ij} + \delta_{ij} V(i)$. $Q + V$ is a “quasi" q-matrix in the sense that for some constant C,

$$\sum_j q_{ij}^V \leq C \quad \forall i$$

A decay parameter $\lambda(V)$ can be defined for $Q + V$, and we have the following

$$-\lambda(V) = \sup_{\mu \in M_1(E_0)} \left\{ \int V d\mu - I(\mu) \right\}. \tag{16}$$

The limiting process

Let $\Omega = D([0, \infty), E)$. Given a path $\omega \in \Omega$, the corresponding empirical processes are defined by

$$R_t = R_t(\omega) = \frac{1}{t}\int_0^t \delta_{\theta_s(\omega^{(t)})} ds, \quad t > 0,$$

where θ_s is the usual shift operator defined by $(\theta_s \omega)_t = \omega_{t+s}$, and $\omega^{(t)}$ is the t-periodic version of ω given by

$$\omega^{(t)}_{kt+s} = \omega_s \quad \text{for integers } k \geq 0, \ \text{ and } 0 \leq s < t.$$

Denote by $M_s(\Omega)$ the space of all stationary probability measures on Ω, then $R_t \in M_s(\Omega)$ for each $t > 0$.

The limiting process

$H =$ D-V entropy functional for Markov process.

$$H_\tau(R) = \begin{cases} H(R) - \lambda_1 & \text{if } \mu_R(E_0) = 1, \\ +\infty & \text{if } \mu_R(E_0) < 1 \end{cases}$$

$\mu_R =$ the single time marginal of Q. For $i \in E_0$, we define for each $t > 0$

$$\rho_{t,i} = P_i(R_t \in \cdot | \tau > t)$$

and

$$\mu_{t,i} = E_i[R_t | \tau > t]$$

which are probability measures on $M_s(\Omega)$ and on Ω, respectively.

The limiting process

Theorem

Under the above definitions and notations, for $i \in E_0$,
(1) if $\{\rho_{t,i},\ t > 0\}$ are tight, any weak limit of them is supported on M_0;
(2) any weak limit R^* of $\{\mu_{t,i},\ t > 0\}$ admits a representation

$$R^* = \int_{M_0} R\rho(dR)$$

for some weak limit ρ of $\rho_{t,i}$. Hence $H_\tau(R^*) = 0$. If in addition, $M_0 = \{R^*\}$ is a singleton, then

$$\lim_{t\to\infty} \rho_{t,i} = \delta_{R^*} \quad \textit{weakly}$$

and

$$\lim_{t\to\infty} \mu_{t,i} = R^* \quad \textit{weakly.}$$

The limiting process

Next we connect H_τ with the entropy functional H^* for some unconditional Markov chain. To this end, we define a chain X^* on E_0 as follows:

$$P^*_{i,j}(t) = \frac{e^{\lambda_1 t} P_{i,j}(t)\beta_j}{\beta_i}, \quad i,j \in E_0.$$

Then X^* is irreducible on E_0. Note that the path space of X^* is $\Omega^* = D([0,\ \infty),\ E_0)$, thus $\tau = \infty$. For this chain, let H^* be defined in the same way as for H.

Theorem

5 If the eigenfunction $\{\beta_i,\ i \in E_0\}$ is bounded, then

$$H^* = H_\tau = H - \lambda_1. \tag{17}$$

Thank you !