Some Questions Concerning Random Walks on Trees in a Random Environment

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Outline

1. The model
2. Primary results
3. A new result
4. Proof
5. Remarks & Observations

The Model

A Galton-Watson process $\{p_k\}$.

A realization can be identified as a rooted tree.

G-W measure: G-W process $\longleftrightarrow$ measure on the set of rooted trees.

$p_0 = 0$, a supercritical G-W process, or a G-W measure on the set of infinite rooted trees.

The Model

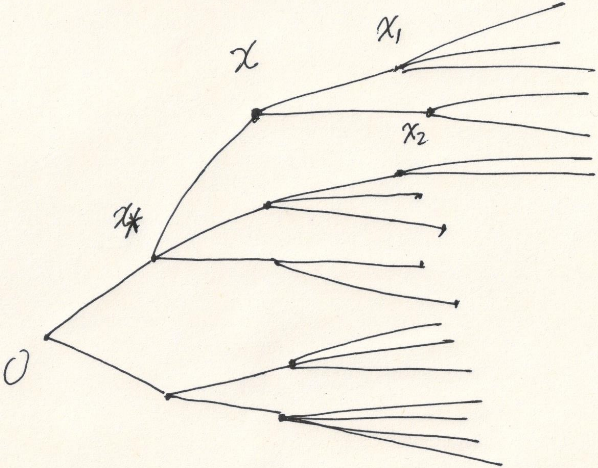
T = a rooted tree

o = the special vertex called the root

$|x|$ = distance from vertex x to the root o.

x_*, the parent of $x \neq o$, unique vertex which is next to x and is closer to the root, $|x_*| = |x| - 1$.
(Note there is no o_*, but can be added later on.)

x_i, the i-th child of x. There are several vertices which are next to x and are further away from the root.


x₁
x
x₂
x*
0

The Model

The λ-biased random walk on a rooted tree =

a Markov chain on the set of vertices with the following transition probabilities:

assuming $x \neq o$ has k cildren,

$$p(x, x_*) = \frac{\lambda}{\lambda + k}, \quad p(x, x_i) = \frac{1}{\lambda + k}.$$

If root o has k children, $p(o, o_i) = 1/k$, $i = 1, 2, \cdots, k$.

The Model

The λ-biased random walk on Glaton-Watson trees is a RWRE.

A rooted tree is chosen according to the Galton-Watson measure, and a λ-biased random walk is defined on the rooted tree.

Random environment: Galton-Watson trees, $\{p_k, k \geq 0\}$, $p_0 = 0$.

random walk: λ-biased random walk, with parameter λ

Primary Results

The λ-biased random walk on Glaton-Watson trees is transient iff $\lambda < m = \sum kp_k$.

The speed$(\lambda) = \lim_{n\to\infty} |X_n|/n$ exists a.s. in this case ($p_0 = 0$ is not required).

$$speed(1) = \sum_{k=0}^{\infty} p_k \frac{k-1}{k+1} \times \frac{1-\rho^{k+1}}{1-\rho^2} \qquad = \sum_{k=0}^{\infty} p_k \frac{k-1}{k+1} \text{ if } p_0 = 0.$$

where $\rho = \sum_k p_k \rho^k$ is the extinction probability.

$$speed(\lambda) = E\left(\frac{(k-\lambda)\beta_0}{\lambda - 1 + \sum_{i=0}^{k} \beta_i}\right) / E\left(\frac{(k+\lambda)\beta_0}{\lambda - 1 + \sum_{i=0}^{k} \beta_i}\right)$$

for $\lambda \in (\lambda_c, m)$, where $\beta_x = P_x(\tau(x_*) = \infty)$, due to Elie Aidekon.
$speed(\lambda) = 0$ for $\lambda < \lambda_c = \sum kq^{k-1}p_k$.

Primary Results

Question A: Is *speed*(λ) monotone in λ?

True when λ is very small, $\lambda \leq 1/717$. G. Ben Arous, A. Fribergh & V. Sidoravicius: A proof of the Lyons-Pementle-Peres monotonicity conjecture for high biases.

True when λ is very closed to m, $\lambda = me^{-\alpha}$,

$$\lim_{\alpha \searrow 0} \frac{speed(me^{-\alpha})}{\alpha} = \frac{D}{2} = \frac{m^2(m-1)}{\sum k^2 p_k - m}.$$

G. Ben Arous, Y. Hu, S. Olla & O. Zeitouni: Einstein relation for biased random walk on Galton-Watson tress, Ann. de Inst. Henri Poincare -Probab. & Stat, Vol 49, 698-721, 2013

Primary Results

Question B Is $speed(\lambda)$ differentiable in λ?

Question C. Is $speed(\lambda) \leq (m-\lambda)/(m+\lambda)$? Yes!

D. Chen, Average properties of random walks on Galton-Watson trees, Ann. Inst. H. Poincare, Vol.33, No.3, (1997), 359-369.

If m is an integer and $p_m = 1$, then the G-W measure is concentrated on the m-nary tree.
The speed of the λ-biased random walk on m-nary tree is $(m-\lambda)/(m+\lambda)$.

i.e. randomness slows down the speed.

Question D. Is speed monotone in the spread-out-ness of $\{p_k\}$?

Specifically, $p_1 = p_3 = \delta \le 1/2$ and $p_2 = 1 - 2\delta$. Is speed decreasing in δ?

$\{p_k\}$ is more spread-out than $\{q_k\}$ if $\{p_k\}$ can be derived from $\{p_k\}$ by a finite number of operations of

$$q_k \longrightarrow q_k - 2\delta, \quad q_{k-l} \longrightarrow q_{k-l} + \delta, \quad q_{k+l} \longrightarrow q_{k+l} + \delta,$$

for some $k > l \ge 1$ and $\delta < q_k/2$.
(Note that $\sum_k kp_k = \sum_k kq_k$, $\sum_k k^2 p_k \ge \sum_k k^2 q_k$.
more spread-out$\Longrightarrow$ larger variance.)

A New Result

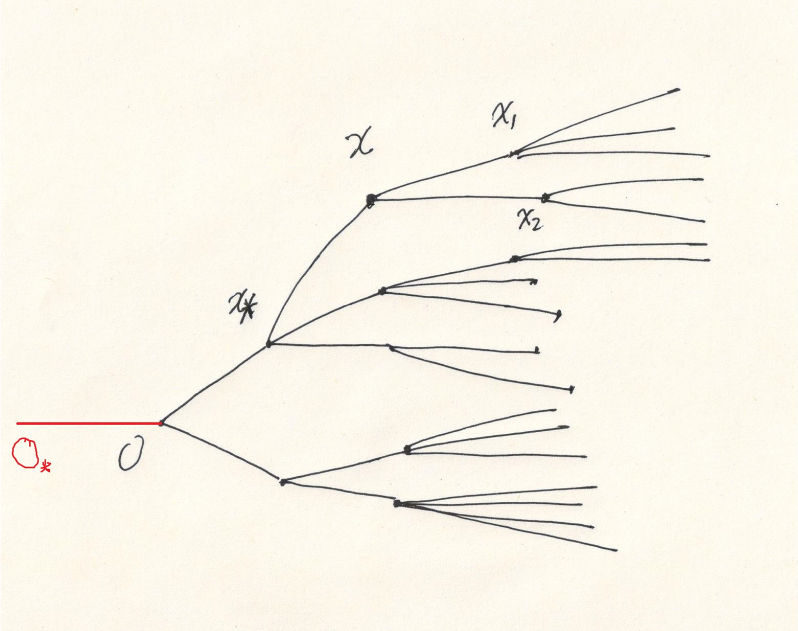
Define $\rho_x = P_x(\tau(x_*) < \infty)$.
Recall $\beta_x = P_x(\tau(x_*) = \infty)$, so $\rho_x + \beta_x = 1$.

Proposition 1. $E\rho_x$ is monotone in the spread-out-ness of $\{p_k\}$.

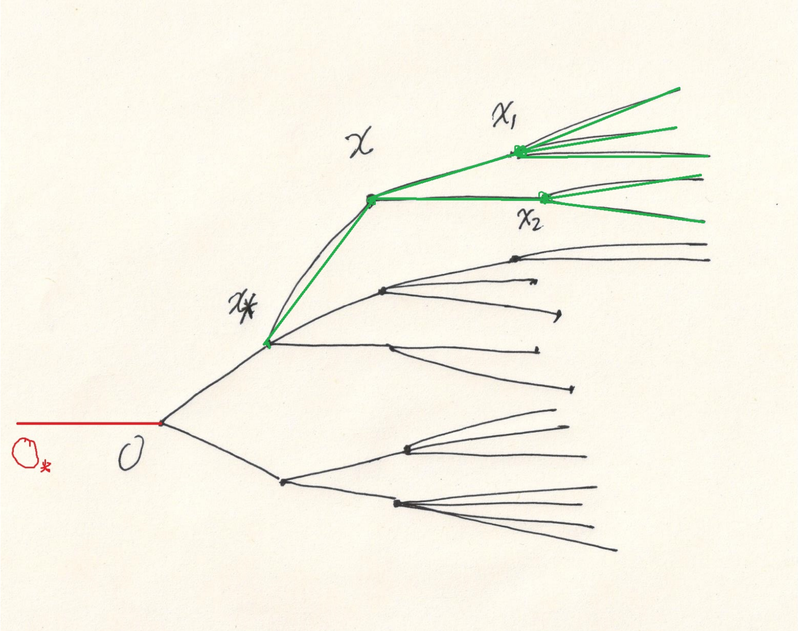
As a function of the G-W tree rooted at x, the distribution of random variable ρ_x is independent of $x \neq o$.

The degree of root o is stochastically less.
Add o_* to G-W tree, $o \sim o_*$, o_* is the parent of o.
Then the degree of o is the same distributed as all other vertices.
Take o as a typical choice.

A New Result



A New Result



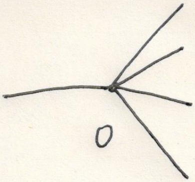
Let $\tau_n = \min\{m, |X_m| = n\}$, and
$\rho_n = P_o(\tau(o_*) < \tau_n) = P_o(X(s_n) = o_*)$.
Note $\lim_n \rho_n = \rho_o$.

Proposition 2. $E\rho_n$ is monotonely increasing in the spread-out-ness of $\{p_k\}$.

Compare two families of G-W trees generated by $\{p_k\}$ and $\{q_k\}$.
E and $\mathbb{E}$ are expectations with different G-W measures.
ρ_n= returning probability of the λ-biased random walk on G-W trees generated by $\{p_k\}$,
r_n the counterpart pf the λ-biased random walk on G-W trees generated by $\{q_k\}$
Assume that $\{p_k\}$ is more spread-out than $\{q_k\}$, then $E\rho_n \geq \mathbb{E}r_n$.

Proposition 3. $E(\rho_n)^m \geq \mathbb{E}(r_n)^m$ for integer $m \geq 1$.

Proof



$$\rho_1 = P_0(\tau_* \le \tau_1) = \frac{\lambda}{\lambda + k}$$

Proof

$$E\rho_1^m = \sum_k p_k(\frac{\lambda}{\lambda+k})^m \geq \sum_k q_k(\frac{\lambda}{\lambda+k})^m = \mathbb{E}r_1^m.$$

Assume $E(\rho_t)^m \geq \mathbb{E}(r_t)^m$ for all $t \leq n$ and for all $m \geq 1$.
Suppose there are k children of root o.
$\rho_{n,i} = P(\tau(o) < \tau_{n+1} | X_0 = o_i)$. Then

$$\rho_{n+1} = \frac{\lambda}{\lambda+k} + \sum_{i=1}^{k} \frac{\rho_{n,i}}{\lambda+k}\frac{\lambda}{\lambda+k} + \cdots + \left(\sum_{i=1}^{k}\frac{\rho_{n,i}}{\lambda+k}\right)^j \frac{\lambda}{\lambda+k} + \cdots.$$

$$= \frac{\lambda}{\lambda+k}\sum_{j=0}^{\infty}\left(\sum_{i=1}^{k}\frac{\rho_{n,i}}{\lambda+k}\right)^j = \frac{\lambda}{\lambda+k-\sum_{i=1}^{k}\rho_{n,i}}.$$

Proof

$$\rho_{n+1}^m = (\frac{\lambda}{\lambda + k - \sum_{i=1}^k \rho_{n,i}})^m = \left(\frac{\lambda}{\lambda + k} \sum_{j=0}^{\infty} \left(\sum_{i=1}^{k} \frac{\rho_{n,i}}{\lambda + k} \right)^j \right)^m =$$

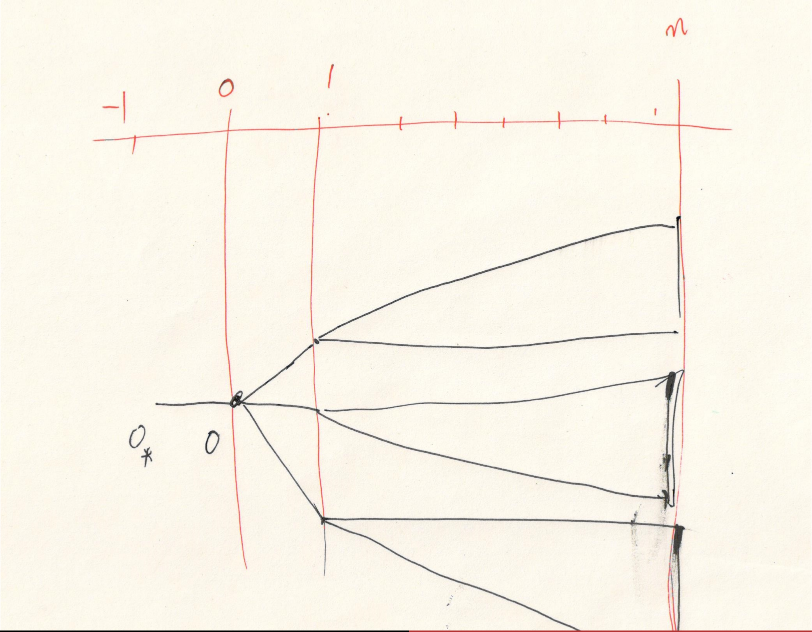
$$\frac{\lambda^m}{(\lambda + k)^m} \sum_{j=0}^{\infty} C_{m,j} (\sum_{i=1}^{k} \frac{\rho_{n,i}}{\lambda + k})^j = \frac{\lambda^m}{(\lambda + k)^m} \sum_{j=0}^{\infty} \frac{C_{m,j}}{(\lambda + k)^j} \sum \Pi_{i=1}^k (\rho_{n,i})^{m_i}$$

where $m_1 + m_2 + \cdots m_k = j$.

$$E\rho_{n+1} = \sum_k p_k E \frac{\lambda}{\lambda + k - \sum \rho_{n,i}} = \sum_k \frac{p_k \lambda}{\lambda + k} \sum_{j=0}^{\infty} \frac{1}{(\lambda + k)^j} E[\sum_{i=1}^{k} \rho_{n,i}]^j$$

$$= \sum_k p_k \frac{\lambda}{\lambda + k} \sum_{j=0}^{\infty} \frac{1}{(\lambda + k)^j} \Pi_{i=1}^k E[\rho_{n,i}]^{m_i}.$$

Proof



Proof

$$E\rho_{n+1} = \sum_k p_k \frac{\lambda}{\lambda+k} \sum_{j=0}^{\infty} \frac{1}{(\lambda+k)^j} \Pi_{i=1}^{k} E_T[\rho_{n,i}]^{m_i}$$

$$\geq \sum_k p_k \frac{\lambda}{\lambda+k} \sum_{j=0}^{\infty} \frac{1}{(\lambda+k)^j} \Pi_{i=1}^{k} \mathbb{E}[r_{n,i}]^{m_i}$$

$$= \sum_k p_k \mathbb{E} \frac{\lambda}{\lambda+k-\sum_{i=1}^{k} r_{n,i}}$$

$$\geq \sum_k q_k \mathbb{E} \frac{\lambda}{\lambda+k-\sum_{i=1}^{k} r_{n,i}} = \mathbb{E} r_{n+1}$$

Proof

Likewise

$$E(\rho_{n+1})^m = \sum_k p_k E(\frac{\lambda}{\lambda + k - \sum_{i=1}^k \rho_{n,i}})^m$$

$$\geq \sum_k p_k \mathbb{E}(\frac{\lambda}{\lambda + k - \sum_{i=1}^k r_{n,i}})^m$$

$$\geq \sum_k q_k \mathbb{E}(\frac{\lambda}{\lambda + k - \sum_{i=1}^k r_{n,i}})^m = \mathbb{E}(r_{n+1})^m$$

By the assumption of spread-out-ness, the last inequality boils down to that for i.i.d positive random variables,

$$\mathbb{E}\frac{1}{(\lambda + \sum_{i=1}^{k-l} \eta_i)^m} + \mathbb{E}\frac{1}{(\lambda + \sum_{i=1}^{k+l} \eta_i)^m} \geq 2\mathbb{E}\frac{1}{(\lambda + \sum_{i=1}^{k} \eta_i)^m}.$$

Proof

Since the difference $1/A^m - 1/(A+B)^m \searrow$ as $A \nearrow$,

$$\frac{1}{(\lambda + \sum_{i=1}^{k-l} \eta_i)^m} - \frac{1}{(\lambda + \sum_{i=1}^{k} \eta_i)^m}$$

$$\geq \frac{1}{(\lambda + \sum_{i=1}^{k-l} \eta_i + \sum_{i=k+1}^{k+l} \eta_i)^m} - \frac{1}{(\lambda + \sum_{i=1}^{k+l} \eta_i)^m}.$$

$$\stackrel{d}{=} \frac{1}{(\lambda + \sum_{i=1}^{k} \eta_i)^m} - \frac{1}{(\lambda + \sum_{i=1}^{k+l} \eta_i)^m}.$$

Taking expectation we get the desired conclusion.

Remarks

1. $E\rho$ is monotone in λ.

2. more spread-out $\leftrightarrow$ slower speed $\leftrightarrow$ ρ is larger.
Recall $\tau_n = \min\{m, |X_m| = n\}$.

$$\lim_n \frac{\tau_n}{n} = \frac{1}{speed} \quad a.s.$$

More comfortable to study $E_o\tau_n$.

$$E_o\tau_{n+1} = (1 + E_o s_n) \sum_k p_k E \sum_{m=0}^{\infty} \left(\frac{\sum_{i=1}^k \rho_{n,i}}{k} \right)^m .$$

where $s_n = \tau(o_*) \wedge \tau_n = \min\{m, X_m = o_* or |X_m| = n\}$. Then $E_o s_1 \equiv 1$.

$$E_o s_{n+1} = \sum_k p_k(\lambda + k + kE_o s_n) E \frac{1}{\lambda + k - \sum_{i=1}^k \rho_{n,i}} .$$

3. A related problem.

Consider the Bernoulli bond percolation of a regular tree with retaining prob. p.
Take an infinite open cluster and run the SRW on the cluster.
The speed is monotone in p.

Take an infinite G-W tree with the offspring distribution $\{p_k\}$.
$f(s) = \sum_k p_k s^k$ the generating function. $m = \sum_k kp_k$.
If

$$\frac{(1-s)f'(s)}{1-f(s)} \text{ is increasing in } s \text{ for } s \in (1/m, 1), \qquad (1)$$

Then the speed of the SRW on an infinite cluster of a G-W tree is $\nearrow$ in p, continuous and differentiable for $p \in (1/m, 1)$.

C. & F. Zhang, On the Monotonicity of the Speed of Random Walks on a Percolation Cluster of Trees, Acta Mathematica Sinica, English Series, 2007, Vol.23(11), 1949-1954.

Geometric, Poisson, Binomial distributions satisfy (1).

The conclusion could fail if the initial graph is not a G-W tree.

Question E: Is the speed of the SRW on an infinite cluster of a transitive graph increasing in p?

Question F: Is the anchored expansion constant of an infinite cluster of a transitive graph increasing in p? Is it continuous in p?

4. Dimension Drop

A random walk is slow down in a random environment
the random walk is confined in a smaller tree.

Recall that the speed is (d-1)/(d+1) for the d-regular tree.
$\log d$ is the dimension of the d-regular tree.
For a G-W tree, the dimension $=\log m$ where $m = \sum_k kp_k$.
But the speed $\leq (m-1)/(m+1)$.

Because random walk is confined in a smaller subtree.

Observations

the boundary ∂T of tree $T =$ the collection of rays.

μ is a measure on ∂T.

Dim $\mu = \min\{dim(E), E \subset \partial T, E$ is a support of $\mu\}$.

$dim(E)$ is the Hausdorff dimension.

Hölder exponent of μ

$$H_\mu(\xi) = \lim_n \frac{1}{n} \log \frac{1}{\mu(\xi_n)}.$$

Lemma: If the Hölder exponent of μ exists a.s. and is constant, then the constant is the Hausdorff dimension of μ.

Observations

For the d regular tree, μ is the uniform measure, then

$$\mu(\xi_n) = 1/d^n, \qquad \text{and} \qquad Dim(\mu) = \log d.$$

For the uniform measure of a Galton-Watson tree,

$$\mu(\xi_n) = \frac{1}{d_1 d_2 \cdots d_n}, \qquad \text{and} \qquad Dim(\mu) = \log m$$

where $m = \sum_k k p_k$.

A General Statement:
Let θ be the exiting distribution of a random walk on the G-W tree. Then $\mathrm{Dim}(\theta) < \log m$.

Observations

On the d-regular tree, $w(e)$ is assigned $i.i.d$ to every edge e.

Consider a random walk on the d-regular tree.

$$p(x, x_i) = \frac{w(e_i)}{w(e_*) + \sum_i w(e_i)}, \qquad p(x, x_*) = \frac{w(e_*)}{w(e_*) + \sum_i w(e_i)}.$$

Dimension drops!

Analogue phenomena: Critical G-W process $\{\xi_n\}$, $\lim_{n\to\infty} \xi_n = 0$ a.s. Yet $E\xi_n = 1$.

Observations

5. RWRE does NOT always slow down!

Consider a random walk on the d-regular tree.

$$p(x, x_i) = \frac{1}{d + \lambda(x)}, \qquad p(x, x_*) = \frac{\lambda(x)}{d + \lambda(x)}.$$

Let $\{\lambda(x)\}$ be $i.i.d.$ Then

$$\text{the speed } = \frac{d - E\lambda}{d + E\lambda}? \qquad \leq \frac{d - E\lambda}{d + E\lambda}?$$

Observations

5. RWRE does NOT always slow down!

Consider a random walk on the d-regular tree.

$$p(x, x_i) = \frac{1}{d + \lambda(x)}, \qquad p(x, x_i) = \frac{\lambda(x)}{d + \lambda(x)}.$$

Let $\{\lambda(x)\}$ be $i.i.d.$ Then

$$\text{the speed} \geq \frac{d - E\lambda}{d + E\lambda}.$$

Thank You!