Optimal Investment in Defaultable Securities under Information Driven Default Contagion

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Talk Outline

Why Default Contagion?

- Recent crisis shows that contagion effects originated from systemically important financial institutions have strong impact on prices of credit derivatives
  - Most of the literature focused on markets consisting of default-free securities.
  - Defaultable security recently incorporated into portfolio optimization frameworks (Korn and Kraft (2003), Kraft and Steffensen (2005), Bo et al. (2010), Capponi and Lopez (2013), Jiao and Pham (2011))
  - Contagion effects ignored in above frameworks, as they only deal with one credit sensitive instrument.

Portfolio Allocation with Defaultable Securities

- Kraft and Steffensen (2008) consider investor allocating wealth across multiple defaultable bonds: constant default intensity prevents contagion
- Jeanblanc and Runggaldier (2010) consider several defaultable assets with discrete dynamics
- Jiao and Pham (2013) analyze portfolio framework under multiple jumps and default events using BSDE methods

Direct Default Contagion

- Direct and Causal Relationships between obligors' defaults
  - Shown to be empirically relevant for sectors such as commercial banks, where default likelihood of an entity increases if some of its major borrowers default. See South Korea Banking Crisis.
  - Natural model is the interacting intensity framework (Jarrow and Yu (2001)): default state of the system evolves as a continuous time Markov chain with transition rates depending on current default configuration (see also Davis and Lo (2001)).
- Optimal CDS Portfolio strategies based on the interacting intensity framework fully characterized in Bo and Capponi (2013) using HJB method.

Information Driven Default Contagion

- Default of firm or news of distress lead investors to update their valuations of related securities.
- Such informational effects arise when investors have incomplete information about actual creditworthiness of other obligors in the portfolio.
- Default risk depends on a number of correlated market variables which none of the market participants can directly observe.

Regime Switching Market Model

- The states of the economy are modeled by a continuous-time hidden Markov chain $\{\mathbf{X}_t\}$
- The process $\{\mathbf{X}_t\}$ has finite state space $\{1, 2, \ldots, K\}$ and generator $A(t) = [A_{i,j}(t)]_{i,j=1,\ldots,N}$:

$$A_{i,j}(t) = \lim_{h \to 0} \frac{1}{h} \{\mathbb{P}(\mathbf{X}_{t+h} = \mathbf{e}_j | \mathbf{X}_t = \mathbf{e}_i) - \delta_{i,j}\}$$

Default Time Specification

1. The default time τ_i is defined as

$$\tau_i := \inf\left\{ t > 0;\ \int_0^t \underbrace{h_i(u, \mathbf{X}(u))}_{\text{regime driven intensity}} \mathrm{d}u \geq \Theta_i \right\}$$

where $(\Theta_i;\ i = 1, \ldots, N)$ are independent unit mean exponentials.

The Market Securities

- Money Market Account: $dB(t) = rB(t)\mathrm{d}t, \qquad B_0 = 1.$
- Consider N defaultable stocks. Predefault stock price process of name i evolves according to diffusion given by

$$\frac{\mathrm{d}P_i(t)}{P_i(t-)} = \Big(b_i(t, \mathbf{X}(t)) + \underbrace{h_i(t, \mathbf{X}(t))}_{\text{default compensator of i}} \Big)\mathrm{d}t + \vartheta_i \mathrm{d}W_i(t)$$

$$W \perp \mathbf{X}$$

- Price of i-th defaultable stock given by $\tilde{P}_i(t) = \mathbf{1}_{\tau_i > t} P_i(t)$, with dynamics

$$\frac{\mathrm{d}\tilde{P}_i(t)}{\tilde{P}_i(t-)} = b_i(t, \mathbf{X}(t))\mathrm{d}t + \vartheta_i \mathrm{d}W_i(t) - \mathrm{d}\Xi_i(t)$$

- Model of similar type used by Linetsky (2006) and calibrated to CDS and equity prices by Carr and Madan (2010).

The Filtrations

- $\mathcal{F}_t = \sigma(W_i(u); u \leq t)$: flow of information of the whole market, excluding default
- $\mathcal{H}_t$: flow of information generated by all default processes $\mathcal{H}_t := \bigvee_{i=1}^{N} \mathcal{H}_t^i$.
- $\mathcal{G}_t^I := \mathcal{F}_t \vee \mathcal{H}_t$: investor filtration
- $\mathcal{G}_t = \mathcal{F}_t^{\mathbf{X}} \vee \mathcal{G}_t^I$.
- $\mathbb{P}$: objective probability measure

$\mathbf{X} = (\mathbf{X}(t);\ t \geq 0)$ is $\mathbb{G}$-adapted, but is not $\mathbb{G}^I$-adapted.

The Stochastic Control Problem

- Investor needs to choose an admissible trading strategy $\boldsymbol{\pi}(t)$, which must be $\mathcal{G}_t^I$ adapted, so to maximize his expected utility from terminal wealth

$$J_T(v, \boldsymbol{\pi}) = \mathbb{E}^{\mathbb{P}}[U(V_T^{\pi})], \qquad U(v) = \frac{v^\gamma}{\gamma},$$

  where $\gamma \in (0,1)$ is a fixed constant, $v > 0$ is the initial wealth, and V_T^{π} the controlled wealth process.

- It is a partially observed stochastic control problem: economic factors $\mathbf{X} = (\mathbf{X}(t);\ t \geq 0)$ are not directly observable, and the strategies can only be based on past information of defaultable stock prices.

Summary of Contributions I

- Building on Nagai and Runggaldier (2008), develop two changes of measure technique to reduce partially observed problem to an equivalent fully observed risk-sensitive control problem
- Established the recursive Hamilton-Jacobi-Bellman (HJB) Equations for the value functions of the problem, based on different default states.

Summary of Contributions II

- Prove that each value function may be recovered as the weak solution for which we establish existence and uniqueness in suitably chosen Sobolev space
- Prove a verification theorem showing that weak solution of PDE corresponds to the value function of risk sensitive control problem.

The Wealth Dynamics

- Let $\bar{\boldsymbol{\pi}}(t) = (\pi_B(t), \boldsymbol{\pi}(t))$, $\boldsymbol{\pi}(t) = (\pi_1(t), \dots, \pi_N(t))^\top$. Here $\pi_i(t)$ is fraction invested in defaultable stock i at t.
- Dynamics of wealth process given by

$$\frac{\mathrm{d}V^{\bar{\boldsymbol{\pi}}}(t)}{V^{\bar{\boldsymbol{\pi}}}(t-)} = \pi_B(t)\frac{\mathrm{d}B(t)}{B(t)} + \sum_{i=1}^{N} \pi_i(t-)\frac{\mathrm{d}P_i(t)}{P_i(t-)},$$

- When i-th stock has defaulted, i.e. for $t > \tau_i$, $P_i(t) = 0$ and $\pi_i(t) = 0$.

The Filter Probabilities

- Filter probabilities of Markov chain $\mathbf{X}(t)$ denoted by by

$$p_k(t) := \mathbb{P}\left(\mathbf{X}(t) = \mathbf{e}_k | \mathcal{G}_t^I\right), \qquad k \in \{1, \ldots, K\}$$

- Consider pre-default log-price process $Y_i(t) := \log(P_i(t))$. Then $\mathbf{Y}(t) = (Y_1(t), \ldots, Y_N(t))^\top$ satisfies SDE:

$$\begin{aligned} \mathrm{d}\mathbf{Y}(t) &= \boldsymbol{\mu}(t, \mathbf{X}(t))\mathrm{d}t + \boldsymbol{\Sigma}\mathrm{d}\mathbf{W}(t) \\ &= \boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}(t, \mathbf{X}(t))\mathrm{d}t + \mathrm{d}\mathbf{W}(t)\right) \\ &:= \boldsymbol{\Sigma}\mathrm{d}\hat{\mathbf{W}}(t) \end{aligned}$$

The Filter dynamics

Proposition (Proposition 3.6 in Frey and Schmidt (2012))

The vector $\mathbf{p}(t)$ *of filter probabilities satisfies*

$$\begin{aligned} \mathrm{d}p_k(t) &= \sum_{\ell=1}^{K} \varpi_{\ell,k}(t) p_\ell(t)\mathrm{d}t \\ &+ p_k(t)\mu_k(t)(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top)^{-1}(\mathrm{d}\mathbf{Y}(t) - \hat{\boldsymbol{\mu}}(t,\mathbf{p}(t))\mathrm{d}t) \\ &+ p_k(t-)\sum_{i=1}^{N}\left(\frac{h_i(t,\mathbf{e}_k)}{\hat{h}_i(t,\mathbf{p}(t-))} - 1\right)\mathrm{d}\Xi_i(t). \end{aligned}$$

with

$$\mu_k(t) = (\boldsymbol{\mu}(t,\mathbf{e}_k) - \hat{\boldsymbol{\mu}}(t,\mathbf{p}(t)))^\top$$

$$\Xi_i(t) := H_i(t) - \int_0^{t\wedge\tau_i} \hat{h}_i(s,\mathbf{p}(s))\mathrm{d}s, \qquad t \geq 0.$$

Partially Observed $\Longrightarrow$ Fully Observed

- Define the 1st change of measure:

$$\left.\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_t} = \Psi(t),$$

where $\Psi(t)$ is a density process satisfying the SDE:

$$\begin{aligned}\Psi(t) &= 1 - \int_0^t \Psi(s-)(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}(s,\mathbf{X}(s)))^\top \mathrm{d}\mathbf{W}(s) \\ &\quad + \int_0^t \Psi(s-)\sum_{i=1}^N \frac{1-h_i(s,\mathbf{X}(s-))}{h_i(s,\mathbf{X}(s-))}\mathrm{d}\Xi_i^{\mathbf{X}}(s)\end{aligned}$$

- Then

$$\begin{aligned}\mathrm{d}\hat{\mathbf{W}}(t) &:= \left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}(t,\mathbf{X}(t))\mathrm{d}t + \mathrm{d}\mathbf{W}(t)\right) \\ \hat{\Xi}_i(t) &:= H_i(t) - \int_0^{t\wedge\tau_i}\mathrm{d}u \qquad i=1,\dots,N\end{aligned}$$

are respective a $\mathbb{G}^I$ Brownian motion and a $\mathbb{G}^I$-martingale

Fully Observed Problem I

- Define the conditional estimates $\hat{g} : D_1 \times \mathcal{P} \times D_2 \longrightarrow \mathbb{R}$ by

$$\hat{g}(y, \mathbf{p}, v) := \sum_{k=1}^{K} g(y, \mathbf{e}_k, v) p_k,$$

- Consider the process

$$\begin{aligned} \hat{L}^{\pi}(t) \quad := \quad & \mathcal{E}_t \left(\int_0^{\cdot} \hat{\mathbf{q}}(s, \mathbf{p}(s), \boldsymbol{\pi}(s))^{\top} \boldsymbol{\Sigma} \mathrm{d}\hat{\mathbf{W}}(s) \right) \\ & \times \mathcal{E}_t \left(\sum_{i=1}^{N} \int_0^{\cdot} (\hat{h}_i(s, \mathbf{p}(s-)) - 1) \mathrm{d}\hat{\Xi}_i(s) \right) \\ & \times \exp \left(-\gamma \int_0^t \hat{\eta}(s, \mathbf{p}_s, \boldsymbol{\pi}(s)) \mathrm{d}s \right) \end{aligned}$$

Fully Observed Problem II

Proposition

The objective functional is given by

$$\underbrace{J_T(v,\pi)}_{\text{partially observed}} := \mathbb{E}^{\mathbb{P}}\left[U(V^{\pi}(T))\right]$$

$$= \frac{v^{\gamma}}{\gamma}\underbrace{\mathbb{E}^{\hat{\mathbb{P}}}\left[\hat{L}^{\pi}(T)\right]}_{\text{fully observed}}$$

The Risk Sensitive Control Problem

Define the 2nd change of measure:

$$\left.\frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}}\right|_{\mathcal{G}_t^I} := \mathcal{E}_t\left(\int_0^{\cdot} \hat{\mathbf{q}}(s,\mathbf{p}(s),\boldsymbol{\pi}(s))^\top \boldsymbol{\Sigma} d\hat{\mathbf{W}}(s)\right)$$
$$\times \mathcal{E}_t\left(\sum_{i=1}^N \int_0^{\cdot} (\hat{h}_i(s,\mathbf{p}(s-)) - 1) d\hat{\Xi}_i(s)\right)$$

Proposition

The objective functional is given by

$$J_T(v,\boldsymbol{\pi}) := \mathbb{E}\left[U(V^{\boldsymbol{\pi}}(T))\right] = \frac{v^\gamma}{\gamma}\underbrace{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\exp\left(-\gamma\int_0^T \tilde{\eta}(s,\tilde{\mathbf{p}}(s),\boldsymbol{\pi}(s))ds\right)\right]}_{\textit{risk sensitive control}}$$

Filter Process and Admissible Strategies

- Under the new probability measure $\tilde{\mathbb{P}}$, there exist $\tilde{\mathbb{P}}$ Brownian motion $\tilde{\mathbf{W}}(t)$ and $\tilde{\mathbb{P}}$ default martingales $\tilde{\Xi}_i(t)$ so that the filter process becomes

$$
\begin{aligned}
\mathrm{d}p_k(t) &= \left(\sum_{\ell=1}^{K} \varpi_{\ell,k}(t) p_\ell(t) + \gamma \mu_k(t) \boldsymbol{\pi}(t) \right) \mathrm{d}t \\
&\quad + p_k(t) \mu_k(t) \boldsymbol{\Sigma}^{-1} \mathrm{d}\tilde{\mathbf{W}}(t) \\
&\quad + p_k(t-) \sum_{i=1}^{N} \frac{h_i(t, \mathbf{e}_k) - \hat{h}_i(t, \mathbf{p}(t-))}{\hat{h}_i(t, \mathbf{p}(t-))} \mathrm{d}\tilde{\Xi}_i(t)
\end{aligned}
$$

where

$$
\mu_k(t) = (\boldsymbol{\mu}(t, \mathbf{e}_k) - \hat{\boldsymbol{\mu}}(t, \mathbf{p}(t)))^{\top}
$$

Dynamic Programming Principle

- For a generic $0 \leq t \leq T$ such that $\tilde{\mathbf{p}}(t) = \boldsymbol{\lambda} \in \Delta_{K-1}$, and $\mathbf{H}(t) = \mathbf{z}$, define

$$G(t, \boldsymbol{\lambda}, \mathbf{z}, \boldsymbol{\pi}) := \mathbb{E}^{\tilde{\mathbb{P}}} \left[e^{-\gamma \int_t^T \tilde{\eta}(s, \tilde{\mathbf{p}}(s), \boldsymbol{\pi}(s)) \mathrm{d}s} \middle| \tilde{\mathbf{p}}(t) = \boldsymbol{\lambda}, \mathbf{H}(t) = \mathbf{z} \right],$$

- Define the value function:

$$w(t, \boldsymbol{\lambda}, \mathbf{z}) := \sup_{\boldsymbol{\pi} \in \tilde{\mathcal{U}}(t,T)} \log \left(G(t, \boldsymbol{\lambda}, \mathbf{z}, \boldsymbol{\pi}) \right).$$

The HJB Equation

- Under mild integrability assumption, we obtain the HJB equation

$$
\begin{aligned}
&\frac{\partial w}{\partial t}(t,\boldsymbol{\lambda},\mathbf{z}) + \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{\sigma}\boldsymbol{\sigma}^\top D^2 w(t,\boldsymbol{\lambda},\mathbf{z})\right] \\
&+\frac{1}{2}\left[(\nabla w)\boldsymbol{\sigma}\boldsymbol{\sigma}^\top(\nabla w)^\top\right](t,\boldsymbol{\lambda},\mathbf{z}) + \gamma r \\
&+\sum_{i=1}^{N}(1-z_i)\tilde{h}_i(t,\boldsymbol{\lambda})\left[e^{w\left(t,\frac{\boldsymbol{\lambda}\cdot\mathbf{h}_i^\perp(t)}{\tilde{h}_i(t,\boldsymbol{\lambda})},\mathbf{z}^i\right)-w(t,\boldsymbol{\lambda},\mathbf{z})}-1\right] \\
&+\sup_{\boldsymbol{\pi}\in\mathcal{U}(t,T,\boldsymbol{\lambda},\mathbf{z})}\Phi\left(w;t,\boldsymbol{\lambda},\mathbf{z},\boldsymbol{\pi}\right) = 0
\end{aligned}
$$

with terminal condition $w(T,\boldsymbol{\lambda},\mathbf{z}) = 0$

The Optimal Strategy

- The optimal strategy $\boldsymbol{\pi}^*_{\mathbf{z}}$ is

$$\boldsymbol{\pi}^*_{\mathbf{z}} = \frac{(1-\mathbf{z})}{1-\gamma} \cdot \left[(\boldsymbol{\Sigma}^\top \boldsymbol{\Sigma})^{-1} \left(\boldsymbol{\Sigma} \boldsymbol{\sigma}(t, \boldsymbol{\lambda})^\top (\boldsymbol{\nabla} w)^\top (t, \boldsymbol{\lambda}, \mathbf{z}) - \boldsymbol{\Gamma}(t, \boldsymbol{\lambda}) \right) \right]$$

where $\boldsymbol{\Gamma}$ is the difference between risk-free rate and the vector of drifts of defaultable stocks

$$\boldsymbol{\Gamma}(t, \boldsymbol{\lambda}) := (1-\mathbf{z}) \cdot \left[\ldots, r - \tilde{b}_i(t, \boldsymbol{\lambda}) - \tilde{h}_i(t, \boldsymbol{\lambda}), \ldots) \right]^\top$$

Master HJB Equation

- Using the optimal strategy, we obtain the final form of the HJB equation: on $(t, \boldsymbol{\lambda}, \mathbf{z}) \in [0, T) \times \underbrace{\Delta_{K-1}}_{K-1 \text{ simplex}} \times \{0,1\}^N$

$$\frac{\partial w}{\partial t}(t, \boldsymbol{\lambda}, \mathbf{z}) + \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{\sigma}\boldsymbol{\sigma}^\top D^2 w\right](t, \boldsymbol{\lambda}, \mathbf{z}) + \frac{1}{2}\underbrace{\left[(\nabla w)\boldsymbol{\sigma}\boldsymbol{\sigma}^\top(\nabla w)^\top\right]}_{\text{quadratic gradient}}(t, \boldsymbol{\lambda}, \mathbf{z})$$

$$+\frac{\gamma}{2(1-\gamma)}\underbrace{\left[(\nabla w)\boldsymbol{\sigma}_{\mathbf{z}}\boldsymbol{\sigma}_{\mathbf{z}}^\top(\nabla w)^\top\right]}_{\text{quadratic gradient}}(t, \boldsymbol{\lambda}, \mathbf{z}) + (\nabla w)(t, \boldsymbol{\lambda}, \mathbf{z})\boldsymbol{\theta}(t, \boldsymbol{\lambda}, \mathbf{z})$$

$$+\sum_{i=1}^{N}(1-z_i)\tilde{h}_i(t, \boldsymbol{\lambda})e^{\underbrace{w\left(t, \frac{\boldsymbol{\lambda}\cdot h_i(t)}{\tilde{h}_i(t,\boldsymbol{\lambda})}, \mathbf{z}^i\right)}_{\text{contagion}} - \underbrace{w(t, \boldsymbol{\lambda}, \mathbf{z})}_{\text{nonlinear term}}} + \rho(t, \boldsymbol{\lambda}, \mathbf{z}) = 0$$

  with terminal condition $w(T, \boldsymbol{\lambda}, \mathbf{z}) = 0$.
- HJB equation is a nonlinear PDE with quadratic growth of gradient.

PDE Analysis

- Recursive system of PDE's: solution associated to portfolio state $\mathbf{z}$, $z_i = 0$, depends on solutions to the HJB equations associated to portfolio states $\mathbf{z}^i = (z_1, \ldots, 1 - z_i, z_{i+1}, \ldots)$ where name i defaults
- Optimal investment strategy $\boldsymbol{\pi}^*_{\mathbf{z}}$ depends on gradient of solution $w(t, \boldsymbol{\lambda}, \mathbf{z})$ leading to information driven contagion effects.
- Use $w_{j_1,\ldots,j_n}(t, \boldsymbol{\lambda}) := w(t, \boldsymbol{\lambda}, \mathbf{0}^{j_1,\cdots,j_n})$ to denote solution of HJB equation associated with the default state $\mathbf{z} = \mathbf{0}^{j_1,\cdots,j_n}$.
- Separately consider the following cases: (1) $n = N$, (2) $n = N - 1$, and (3) $2 \leq n \leq N - 1$

Case I: All names are defaulted

- $\mathbf{z} = \mathbf{0}^{j_1,\dots,j_N} = \mathbf{1} := (1,\dots,1)^\top$. Investor cannot invest in any stock, hence optimal strategy is $\boldsymbol{\pi}^* = \mathbf{0}$, $\pi_B = 1$.
- Value function $w_{\mathbf{1}}(t,\boldsymbol{\lambda})$ associated to this default state satisfies HJB equation

$$\frac{\partial w_{\mathbf{1}}}{\partial t}(t,\boldsymbol{\lambda}) + \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{\sigma}\boldsymbol{\sigma}^\top D^2 w_{\mathbf{1}}\right](t,\boldsymbol{\lambda}) + \frac{1}{2}\left[(\boldsymbol{\nabla} w_{\mathbf{1}})\boldsymbol{\sigma}\boldsymbol{\sigma}^\top(\boldsymbol{\nabla} w_1)^\top\right](t,\boldsymbol{\lambda}) + (\boldsymbol{\nabla} w_{\mathbf{1}})(t,\boldsymbol{\lambda})\beta_{\varpi}(t,\boldsymbol{\lambda}) + \gamma r = 0$$

  with $(t,\boldsymbol{\lambda}) \in [0,T) \times \Delta_{K-1}$ and terminal condition $w_{\mathbf{1}}(T,\boldsymbol{\lambda}) = 0$
- Solution is

$$w_{\mathbf{1}}(t,\boldsymbol{\lambda}) = \gamma r(T-t)$$

Case II: $n = N - 1$

- The HJB equation becomes

$$\frac{\partial w_{j_1,\dots,j_{N-1}}}{\partial t}(t,\boldsymbol{\lambda}) + \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{\sigma}\boldsymbol{\sigma}^\top D^2 w_{j_1,\dots,j_{N-1}}\right](t,\boldsymbol{\lambda})$$
$$+\frac{1}{2}\left[(\boldsymbol{\nabla} w_{j_1,\dots,j_{N-1}})\boldsymbol{\sigma}\boldsymbol{\sigma}^\top(\boldsymbol{\nabla} w_{j_1,\dots,j_{N-1}})^\top\right](t,\boldsymbol{\lambda})$$
$$+\frac{\gamma}{2(1-\gamma)}\left[(\boldsymbol{\nabla} w_{j_1,\dots,j_{N-1}})\boldsymbol{\sigma}_{\mathbf{z}}\boldsymbol{\sigma}_{\mathbf{z}}^\top(\boldsymbol{\nabla} w_{j_1,\dots,j_{N-1}})^\top\right](t,\boldsymbol{\lambda})$$
$$+(\boldsymbol{\nabla} w_{j_1,\dots,j_{N-1}})(t,\boldsymbol{\lambda})\boldsymbol{\theta}_{j_1,\dots,j_{N-1}}(t,\boldsymbol{\lambda})$$
$$+\xi_{j_1,\dots,j_{N-1}}(t,\boldsymbol{\lambda}, w_{j_1,\dots,j_{N-1}}(t,\boldsymbol{\lambda})) = 0$$

where

$$(t,\boldsymbol{\lambda}) \in [0,T] \times \Delta_{K-1}$$

and

$$\xi_{j_1,\dots,j_{N-1}}(t,\boldsymbol{\lambda},v) := \rho_{j_1,\dots,j_{N-1}}(t,\boldsymbol{\lambda}) + \tilde{h}_{j_N}(t,\boldsymbol{\lambda})\underbrace{e^{\gamma r(T-t)}}_{\text{contagion}} e^{-v}$$

The case $2 \neq n \leq N$

- The value function $w_{j_1,\dots,j_n}(t,\boldsymbol{\lambda})$ corresponding to names $j_1,\dots,j_n$ alive satisfies

$$\frac{\partial w_{j_1,\dots,j_n}}{\partial t}(t,\boldsymbol{\lambda}) + \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{\sigma}\boldsymbol{\sigma}^\top D^2 w_{j_1,\dots,j_n}\right](t,\boldsymbol{\lambda})$$
$$+\frac{1}{2}\left[(\nabla w_{j_1,\dots,j_n})\boldsymbol{\sigma}\boldsymbol{\sigma}^\top(\nabla w_{j_1,\dots,j_n})^\top\right](t,\boldsymbol{\lambda})$$
$$+\frac{\gamma}{2(1-\gamma)}\left[(\nabla w_{j_1,\dots,j_n})\boldsymbol{\sigma}_{\mathbf{z}}\boldsymbol{\sigma}_{\mathbf{z}}^\top(\nabla w_{j_1,\dots,j_n})^\top\right](t,\boldsymbol{\lambda})$$
$$+(\nabla w_{j_1,\dots,j_n})(t,\boldsymbol{\lambda})\boldsymbol{\theta}_{j_1,\dots,j_n}(t,\boldsymbol{\lambda})$$
$$+\xi_{j_1,\dots,j_n}(t,\boldsymbol{\lambda},w_{j_1,\dots,j_n}(t,\boldsymbol{\lambda}))=0$$

where $(t,\boldsymbol{\lambda}) \in [0,T) \times \Delta_{K-1}$ and nonlinear term is given by

$$\xi_{j_1,\dots,j_n}(t,\boldsymbol{\lambda},v) := \sum_{i\in\{j_{n+1},\dots,j_N\}} \underbrace{\tilde{h}_i(t,\boldsymbol{\lambda})\ e^{w_{j_1,\dots,j_n,i}\left(t,\frac{\boldsymbol{\lambda}\cdot\mathbf{h}_i(t)}{\tilde{h}_i(t,\boldsymbol{\lambda})}\right)}}_{\text{default contagion}}\ e^{-v} + \rho_{j_1,\dots,j_n}(t,\boldsymbol{\lambda})$$

The generalized solution

- Reverse flow of time, $t \longrightarrow T - t$, and rewrite PDE as

$$
\begin{aligned}
\frac{\partial \bar{u}}{\partial t}(t,\boldsymbol{\lambda}) = & \frac{1}{2}\mathrm{Tr}\left[\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^{\top} D^2 \bar{u}\right](t,\boldsymbol{\lambda}) + \frac{1}{2}\left[(\boldsymbol{\nabla}\bar{u})\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^{\top}(\boldsymbol{\nabla}\bar{u})^{\top}\right](t,\boldsymbol{\lambda}) \\
& + \frac{\gamma}{2(1-\gamma)}\left[(\boldsymbol{\nabla}\bar{u})\bar{\boldsymbol{\sigma}}_{\mathbf{z}}\bar{\boldsymbol{\sigma}}_{\mathbf{z}}^{\top}(\boldsymbol{\nabla}\bar{u})^{\top}\right](t,\boldsymbol{\lambda}) \\
& + (\boldsymbol{\nabla}\bar{u})(t,\boldsymbol{\lambda})\bar{\boldsymbol{\theta}}_{j_1,\ldots,j_n}(t,\boldsymbol{\lambda}) + \bar{\xi}_{j_1,\ldots,j_n}(t,\boldsymbol{\lambda},\bar{u}(t,\boldsymbol{\lambda})) \\
= & \;\; 0
\end{aligned}
$$

with $(t,\boldsymbol{\lambda}) \in (0,T] \times \Delta_{K-1}$ and initial condition $\bar{u}(0,\boldsymbol{\lambda}) = 0$

Sobolev Space

- Let $D \subseteq \mathbb{R}^{K-1}$. Then $\mathrm{H}_1(D)$ denotes the Sobolev space consisting of all functions $f \in L^1(D)$ such that

$$\|f\|_{\mathrm{H}_1} := \left(\int_D |Df(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}},$$

- The Sobolev space $\mathrm{W}_p^{1,2}(Q_T)$ with $Q_T = [0, T] \times \Delta_{K-1}$, is the set of all functions $f(t, \boldsymbol{\lambda}) : Q_T \longrightarrow \mathbb{R}$ belonging to $L^p(Q_T)$ which admit first-order weak derivative $\partial_t f$ w.r.t. time t and k-order weak derivative $D^k f$ w.r.t. $\boldsymbol{\lambda}$, for $1 \leq k \leq 2$.
- The norm of $f \in \mathrm{W}_p^{1,2}(Q_T)$ is defined as

$$\|f\|_{\mathrm{W}_p^{1,2}(Q_T)} := \left(\int_{Q_T} |\partial_t f(t, \boldsymbol{\lambda})|^p + |Df(t, \boldsymbol{\lambda})|^p + |D^2 f(t, \boldsymbol{\lambda})|^p \mathrm{d}\boldsymbol{\lambda} \mathrm{d}t \right)^{\frac{1}{p}}$$

The Generalized Solution

A function $\bar{u} : [0, T] \longrightarrow \mathrm{H}^1(\Delta_{K-1})$ is a generalized solution if

(I) $\bar{u} \in L^2([0, T]; \mathrm{H}^1(\Delta_{K-1}))$, and $\partial_t \bar{u} \in L^2([0, T]; \mathrm{H}^{-1}(\Delta_{K-1}))$. Here $\mathrm{H}^{-1}(\Delta_{K-1})$ denotes the dual space of $\mathrm{H}^1(\Delta_{K-1})$.

(II) For every test function $\phi \in \mathrm{H}^1_0(\Delta_{K-1})$, the following variational representation holds

$$
\begin{aligned}
&\int_0^T \langle \partial_t \bar{u}, \phi \rangle \,\mathrm{d}t + \frac{1}{2}\int_{Q_T} (\boldsymbol{\nabla}\phi)(\boldsymbol{\lambda}) \left[\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^\top (\boldsymbol{\nabla}\bar{u})^\top \right](t, \boldsymbol{\lambda}) \mathrm{d}\boldsymbol{\lambda}\mathrm{d}t \\
&= \frac{1}{2}\int_{Q_T} \left[(\boldsymbol{\nabla}\bar{u})\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^\top (\boldsymbol{\nabla}\bar{u})^\top \right](t, \boldsymbol{\lambda})\phi(t, \boldsymbol{\lambda}) \mathrm{d}\boldsymbol{\lambda}\mathrm{d}t \\
&+ \frac{\gamma}{2(1-\gamma)}\int_{Q_T} \left[(\boldsymbol{\nabla}\bar{u})\bar{\boldsymbol{\sigma}}_{\mathbf{z}}\bar{\boldsymbol{\sigma}}_{\mathbf{z}}^\top (\boldsymbol{\nabla}\bar{u})^\top \right](t, \boldsymbol{\lambda})\phi(\boldsymbol{\lambda}) \mathrm{d}\boldsymbol{\lambda}\mathrm{d}t \\
&+ \int_{Q_T} \left[(\boldsymbol{\nabla}\bar{u})(t, \boldsymbol{\lambda})\bar{\boldsymbol{\theta}}_{j_1,\ldots,j_n}(t, \boldsymbol{\lambda}) + \bar{\xi}_{j_1,\ldots,j_n}(t, \boldsymbol{\lambda}, \bar{u}(t, \boldsymbol{\lambda})) \right] \phi(\boldsymbol{\lambda}) \mathrm{d}\boldsymbol{\lambda}\mathrm{d}t \\
&- \frac{1}{2}\int_{Q_T} \mathbf{div}(\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^\top)(t, \boldsymbol{\lambda})(\boldsymbol{\nabla}\bar{u})^\top (t, \boldsymbol{\lambda})\phi(\boldsymbol{\lambda}) \mathrm{d}\boldsymbol{\lambda}\mathrm{d}t
\end{aligned}
$$

(III) $\bar{u}(0, \boldsymbol{\lambda}) = 0$ for all $\boldsymbol{\lambda} \in \Delta_{K-1}$

Sketch of Proof

- Consider an approximation problem to the variational representation.
- Prove uniform L^∞ bounds of the solutions to the approximation problem.
- Develop a priori estimates for solutions of the approximation problem in Sobolev space, and show that sequence of approximating solutions converges to the generalized solution of HJB PDE.
- Apply a one to one solution transformation technique to establish the uniqueness of the generalized solution to the HJB PDE.

The Uniform Boundedness of Approximating Solution I

- Challenge: The nonlinear term $\bar{\xi}$ cannot be guaranteed to be bounded from above. Need to develop analysis establishing both a lower bound and an upper bound for the approximate solution.
- Define

$$
\begin{aligned}
L_\xi &:= \inf_{(t,\boldsymbol{\lambda},v)\in Q_T\times\mathbb{R}} \bar{\xi}_{j_1,\ldots,j_n}(t,\boldsymbol{\lambda},v) \\
U_\xi &:= C_N e^{B_n - L_\xi(T)} + \sup_{(t,\boldsymbol{\lambda})\in Q_T} \rho_{j_1,\ldots,j_n}(T-t,\boldsymbol{\lambda})
\end{aligned}
$$

- Introduce sequence of truncated solutions corresponding to $\bar{u}$. More precisely, define

$$
\bar{u}_{L,U}(t,\boldsymbol{\lambda}) := \max\left\{L_\xi(t), \min\{\bar{u}(t,\boldsymbol{\lambda}), U_\xi(t)\}\right\}, \quad (t,\boldsymbol{\lambda}) \in Q_T.
$$

The Uniform Boundedness of Approximating Solution II

- Define approximating problem

$$
\begin{aligned}
\frac{\partial \bar{u}^m}{\partial t}(t,\boldsymbol{\lambda}) &= \frac{1}{2}\mathrm{Tr}\left[\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^\top D^2 \bar{u}^m\right](t,\boldsymbol{\lambda}) \\
&+\frac{1}{2}\frac{\left[(\boldsymbol{\nabla}\bar{u}^m)\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^\top(\boldsymbol{\nabla}\underbrace{\bar{u}^m_{L,U}}_{\text{bounded approx}})^\top\right](t,\boldsymbol{\lambda})}{1+\frac{1}{m}\left[(\boldsymbol{\nabla}\bar{u}^m)\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^\top(\boldsymbol{\nabla}\bar{u}^m)^\top\right](t,\boldsymbol{\lambda})} \\
&+\frac{\gamma}{2(1-\gamma)}\frac{\left[(\boldsymbol{\nabla}\bar{u}^m)\bar{\boldsymbol{\sigma}}_{\mathbf{z}}\bar{\boldsymbol{\sigma}}_{\mathbf{z}}^\top(\boldsymbol{\nabla}\underbrace{\bar{u}^m_{L,U}}_{\text{bounded approx}})^\top\right](t,\boldsymbol{\lambda})}{1+\frac{1}{m}\left[(\boldsymbol{\nabla}\bar{u}^m)\bar{\boldsymbol{\sigma}}_{\mathbf{z}}\bar{\boldsymbol{\sigma}}_{\mathbf{z}}^\top(\boldsymbol{\nabla}\bar{u}^m)^\top\right](t,\boldsymbol{\lambda})} \\
&+(\boldsymbol{\nabla}\bar{u}^m)(t,\boldsymbol{\lambda})\bar{\boldsymbol{\theta}}_{j_1,\dots,j_n}(t,\boldsymbol{\lambda}) \\
&+\bar{\xi}_{j_1,\dots,j_n}(t,\boldsymbol{\lambda},\bar{u}^m(t,\boldsymbol{\lambda}))
\end{aligned}
$$

Boundedness property of the Approximating Solutions

- Existence and uniqueness of truncated generalized solution $\bar{u}^m \in L^2([0,T];\mathrm{H}^2(\Delta_{K-1})) \cap \mathrm{H}^1([0,T];L^2(\Delta_{K-1}))$ guaranteed by Schauder's fixed point theorem.
- Uniform boundedness established by

Lemma (Bo and Capponi (2014))

For each $0 \le t \le T$, it holds that Δ_{K-1}-a.s.,

$$L_\xi(t) \le \bar{u}^m(t) \le U_\xi(t), \qquad \forall\ m \in \mathbb{N}$$

Convergence to Generalized Solution

- Solutions of approximation problem are uniformly bounded in Sobolev space.

Lemma (Bo and Capponi (2014))

Let $\bar{u}^m$ be the solution of approximating equation. Then there exists a constant $C > 0$ independent of $m \in \mathbb{N}$ such that $\|\bar{u}^m\|_{L^2([0,T];\mathrm{H}^1(\Delta_{K-1}))} \leq C$.

- Use this result to show that sequence of approximating solutions converges to generalized solution of original HJB PDE.

Theorem (Bo and Capponi (2014))

The HJB-PDE admits a generalized solution $\bar{u} \in L^2([0,T];\mathrm{H}^1(\Delta_{K-1})) \cap L^\infty(Q_T)$. Moreover, the generalized solution is unique.

Verification Theorem

Theorem

Let $(t, \boldsymbol{\lambda}) \in Q_T$, *and* $\mathbf{z} = \mathbf{0}^{j_1,\dots,j_n}$, *for* $n = 1, \dots, N$. *Let* $w(t, \boldsymbol{\lambda}, \mathbf{z}) = \bar{u}(T - t, \boldsymbol{\lambda})$ *with* $\bar{u}$ *being the unique generalized solution to HJB equation. Then,* $w(t, \boldsymbol{\lambda}, \mathbf{z})$ *coincides with the value function, i.e.*

$$w(t, \boldsymbol{\lambda}, \mathbf{z}) := \sup_{\boldsymbol{\pi} \in \tilde{\mathcal{U}}(t,T)} \log\left(G(t, \boldsymbol{\lambda}, \mathbf{z}, \boldsymbol{\pi})\right).$$

where

$$G(t, \boldsymbol{\lambda}, \mathbf{z}, \boldsymbol{\pi}) := \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma \int_t^T \tilde{\eta}(s, \tilde{\mathbf{p}}(s), \boldsymbol{\pi}(s))\mathrm{d}s} \middle| \tilde{\mathbf{p}}(t) = \boldsymbol{\lambda}, \mathbf{H}(t) = \mathbf{z}\right]$$

Moreover, there is a unique admissible optimal Markov feedback strategy $\boldsymbol{\pi}^*_{\mathbf{z}}(t)$ *given by*

$$\boldsymbol{\pi}^*(t) = \frac{(1-\mathbf{z})}{1-\gamma}\left[(\boldsymbol{\Sigma}^\top \boldsymbol{\Sigma})^{-1}\left(\boldsymbol{\Sigma}\boldsymbol{\sigma}(t, \boldsymbol{\lambda})^\top (\nabla w)^\top (t, \boldsymbol{\lambda}, \mathbf{z}) - \boldsymbol{\Gamma}(t, \boldsymbol{\lambda})\right)\right]$$

Reference Papers

- L. Bo and A. Capponi. Optimal Investment in Defaultable Securities Under Information Driven Default Contagion. Preprint available upon request.
- L. Bo and A. Capponi. Bilateral Credit Valuation Adjustment for Large Credit Derivatives Portfolios. *Finance & Stochastics*.
- L. Bo and A. Capponi. Optimal Investment in Credit Derivatives Portfolio under Contagion Risk. *Mathematical Finance*.

Thanks for your attention !