Approximation of Invariant Measures for Regime-Switching Diffusions

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Motivations

- A regime-switching diffusion process (RSDP), is a diffusion process in random environments characterized by a Markov chain.
- The state vector of a RSDP is a pair (X_t, Λ_t), where $\{X_t\}_{t\geq 0}$ satisfies a stochastic differential equation (SDE)

$$\mathrm{d}X_t = b(X_t, \Lambda_t)\mathrm{d}t + \sigma(X_t, \Lambda_t)\mathrm{d}W_t, \quad t > 0, \tag{1}$$

with the initial data $X_0 = x \in \mathbb{R}^n, \Lambda_0 = i \in \mathbb{S}$, and $\{\Lambda_t\}_{t\geq 0}$ denotes a continuous-time Markov chain with the state space $\mathbb{S} := \{1, 2 \cdots, N\}$, $1 \leq N \leq \infty$, and the transition rules specified by

$$\mathbb{P}(\Lambda_{t+\triangle} = j | \Lambda_t = i) = \begin{cases} q_{ij}\triangle + o(\triangle), & i \neq j, \\ 1 + q_{ii}\triangle + o(\triangle), & i = j. \end{cases} \tag{2}$$

Motivations (Cont.)

- RSDPs have considerable applications in e.g. control problems, storage modeling, neutral activity, biology and mathematical finance (see e.g. the monographs by Mao-Yuan (2006), and Yin-Zhu (2010)).
- The dynamical behavior of RSDPs may be markedly different from diffusion processes without regime switchings, see e.g. Pinsky -Scheutzow (1992), Mao-Yuan (2006).
- So far, the works on RSDPs have included ergodicity (Cloez-Hairer (2013), Shao (2014)) stability (Mao-Yuan (06), Xi-Yin (2010)), recurrence and transience (Pinsky-Scheutzow (1992), Shao-Xi (2014), Yin-Zhu (2010)), invariant densities (Mattingly et al. (2014)), hypoellipticity (Bakhtin (2012)), and so forth

Motivations (Cont.)

- Since solving RSDPs is still a challenging task, numerical schemes and/or approximation techniques have become one of the viable alternatives (see e.g. Mao-Yuan (2006), Yin-Zhu (2010), Higham et al. (2007)).
- For more details on numerical analysis of diffusion processes without regime switching, please refer to the monograph by Kloeden and Platen (1992).
- Also, approximations of invariant measures for stochastic dynamical sysytems have attracted much attention, see e.g. Mattingly et al. (2010), Talay (1990), Bréhier (2014).

Motivations (Cont.)

- For the counterpart associated with Euler-Maruyama (EM) algorithms, we refer to Yuan-Mao (2005), and Yin-Zhu (2010), where RSDPs therein enjoy finite state space.
- Sufficient conditions imposed in Yuan- Mao (2005), and Yin-Zhu (2010) to guarantee existence of numerical invariant measures are irrelevant to stationary distributions of the continuous-time Markov chains.
- In this talk, we are concerned with the following questions:

(i) Under what conditions, will the discrete-time semigroup generated by EM scheme admit an invariant measure?

(ii) Will the numerical invariant measure, if it exists, converge in some metric to the underlying one?

Invariant Measure

Some notation is listed as below.

- $(\Omega, \mathscr{F}, \mathbb{P})$: probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$;
- $\{W_t\}_{t\geq 0}$: an m-dimensional Brownian motion;
- $\{\Lambda_t\}_{t\geq 0}$: a continuous-time Markov chain with the state space $\mathbb{S} := \{1, 2\cdots, N\}, N < \infty$; The transition rules of $\{\Lambda_t\}_{t\geq 0}$ are specified by (2)
- Q-matrix $Q := (q_{ij})_{N\times N}$ is irreducible and conservative so that $\{\Lambda_t\}_{t\geq 0}$ has a unique stationary distribution $\mu := (\mu_1, \cdots, \mu_N)$.

We assume that, in (1), $b : \mathbb{R}^n \times \mathbb{S} \mapsto \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathbb{S} \mapsto \mathbb{R}^n \otimes \mathbb{R}^m$ satisfy the local Lipschitz condition, i.e., for each $i \in \mathbb{S}$ and $R > 0$, there exists an $L_R > 0$ such that

Invariant Measure (Cont.)

$$|b(x,i)-b(y,i)|+\|\sigma(x,i)-\sigma(y,i)\|\le L_R|x-y|,\quad x,y\in B_R(0).\tag{3}$$

Additionally, we assume that

(H) For each $i\in\mathbb{S}$ and $x,y\in\mathbb{R}^n$, there exist $c_0>0$ and $\beta_i\in\mathbb{R}$ such that

$$2\langle x,b(x,i)\rangle+\|\sigma(x,i)\|^2\le c_0+\beta_i|x|^2,\tag{4}$$

and

$$2\langle x-y,b(x,i)-b(y,i)\rangle+\|\sigma(x,i)-\sigma(y,i)\|^2\le\beta_i|x-y|^2.\tag{5}$$

For any $p>0$, let

$$\text{diag}(\beta):=\text{diag}(\beta_1,\cdots,\beta_N),\quad Q_p:=Q+\frac{p}{2}\text{diag}(\beta),\ \eta_p:=-\max_{\gamma\in\text{spec}(Q_p)}\tag{6}$$

Invariant Measure (Cont.)

where Q is the Q-matrix of $\{\Lambda_t\}_{t\geq 0}$, and spec(Q_p) denotes the spectrum of Q_p.

(Proposition 4.2, Bardet et al. (2010)) Assume that

$$\sum_{i=1}^{N} \mu_i \beta_i < 0. \tag{7}$$

Then, $\eta_p > 0$ for $p < k$, where $k \in (0, \min_{i\in\mathbb{S},\beta_i>0}\{-2q_{ii}/\beta_i\})$ for $\max_{i\in\mathbb{S}} \beta_i > 0$. Remark:

- The RSDP (1) is said to be attractive "in average" if (7). In what follows, we call (7) an "averaging condition".
- $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ is called an invariant measure of $(X_t^{x,i}, \Lambda_t^i)$ if

$$\pi(\Gamma \times \{i\}) = \sum^{N} \int P_t(x, j; \Gamma \times \{i\})\pi(\mathrm{d}x \times \{j\}), \quad t \geq 0.$$

Invariant Measure (Cont.)

(**Theorem 1**) Let $N < \infty$ and assume that (3), (**H**) and (7) hold. Then $(X_t^{x,i}, \Lambda_t^i)$ admits a unique invariant measure $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$.

Proof: The Perron-Frobenius Theorem + Proposition 4.2 (Bardet et al. (2010)).

Example I Let $\{\Lambda_t\}_{t\geq 0}$ be a right-continuous Markov chain taking values in $\mathbb{S} := \{0, 1\}$ with the generator

$$Q = \begin{pmatrix} -4 & 4 \\ \gamma & -\gamma \end{pmatrix}$$

with some $\gamma > 0$. Consider the scalar Ornstein-Uhlenback (O-U) process with regime switching

$$\mathrm{d}X_t = \alpha_{\Lambda_t} X_t \mathrm{d}t + \sigma_{\Lambda_t} \mathrm{d}W_t, \quad t > 0, \quad X_0 = x, \Lambda_0 = i_0 \in \mathbb{S}, \qquad (8)$$

Example I (Cont.)

where $\alpha_\cdot : \mathbb{S} \mapsto \mathbb{R}$ such that $\alpha_0 = 1$, and $\alpha_1 = -1/2$, and $\sigma \in \mathbb{R}$ is a constant.

Remark:

- By an M-Matrix approach, (8) has a unique invariant measure for $\gamma \in (0, 1)$, see e.g. Example 5.1 (Yuan-Mao, 2003).
- By **Theorem 1**, $(X_t^{x,i}, \Lambda_t^i)$ admits a unique invariant measure $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ for $\gamma \in (0, 2)$.
- **Theorem I** can apply more interesting examples than the existing literature.
- For a scalar RSDP, existence and uniqueness of invariant measure can be determined only by the drift coefficient in some cases.

Numerical Invariant Measure

For a given stepsize $\delta \in (0,1)$, define the discrete-time EM scheme associated with (1) as follows

$$\overline{Y}^{x,i}_{(k+1)\delta} := \overline{Y}^{x,i}_{k\delta} + b(\overline{Y}^{x,i}_{k\delta}, \Lambda^i_{k\delta})\delta + \sigma(\overline{Y}^{x,i}_{k\delta}, \Lambda^i_{k\delta})\triangle W_k, \ \ k \geq 0, \tag{9}$$

with $\overline{Y}^{x,i}_0 = x, \Lambda^i_0 = i \in \mathbb{S}$, where $\triangle W_k := W_{(k+1)\delta} - W_{k\delta}$ stands for the Brownian motion increment. Remark:

- $(\overline{Y}^{x,i}_{k\delta}, \Lambda^i_{k\delta})$ is a time homogeneous Markov chain.
- If $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ such that

$$\pi^\delta(\Gamma \times \{i\}) = \sum_{j=1}^N \int_{\mathbb{R}^n} P^\delta_{k\delta}(x, j; \Gamma \times \{i\})\pi^\delta(\mathrm{d}x \times \{j\}), \ \ \Gamma \in \mathscr{B}(\mathbb{R}^n),$$

  then we call $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ an invariant measure of $(\overline{Y}^{x,i}_{k\delta}, \Lambda^i_{k\delta})$.

Numerical Invariant Measure: Additive Noise

we further assume that, for each $i \in \mathbb{S}$ and $x, y \in \mathbb{R}^n$, there exists an $L > 0$ such that

$$|b(x,i) - b(y,i)| + \|\sigma(x,i) - \sigma(y,i)\| \le L|x-y|. \tag{10}$$

(**Theorem 2**) Let $N < \infty$, and assume further that (**H**), (7), and (10) hold with $\sigma(\cdot,\cdot) \equiv \sigma(\cdot)$. Then, $(\overline{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$ admits a unique invariant measure $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ whenever the stepsize is sufficiently small.

Remark:

- Example II: The EM scheme associated with (8) has a unique invariant measure whenever the stepsize $\delta \in (0,1)$ is sufficiently small.
- Under the averaging condition (7), existence and uniqueness of numerical invariant measure for (1) with multiplicative noise is still open.

Proof of **Theorem 2**

(i) Existence of an Invariant Measure. For each integer $q \geq 1$, define the measure

$$\mu_q(B_R \times \mathbb{S}) := \frac{1}{q}\sum_{k=0}^{q}\mathbb{P}((\overline{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^{i}) \in B_R \times \mathbb{S}),$$

where $B_R := \{x \in \mathbb{R}^n : |x| \leq R\}$, a compact subset of $\mathbb{R}^n$, for some $R > 0$. To show existence of an invariant measure, it suffices to show that, for any $x \in \mathbb{R}^n$,

$$\sup_{k\geq 0}\mathbb{E}|\overline{Y}_{k\delta}^{x,i}|^p < \infty. \tag{11}$$

Indeed, if so, the Chebyshev inequality yields that the measure sequence $\{\mu_q(\cdot)\}_{q\geq 1}$ is tight. Then, one can extract a subsequence which converges weakly to an invariant measure (see e.g. Meyn-Tweedie (1992)).

Proof of **Theorem 2** (Cont.)

(ii) Uniqueness of Invariant Measure. It is sufficient to claim

$$\mathbb{E}(Y_{k\delta}^{x,i} - Y_{k\delta}^{x,i}|^p) \le c e^{-\rho k\delta}|x-y|^p. \tag{12}$$

Remark:

- (11) and (12) can be obtained by using the Perron-Frobenius Theorem plus Proposition 4.2 (Bardet et al. (2010)).
- Actually, a upper bound of $\delta \in (0,1)$ such that **Theorem 2** holds can be given as follows

$$\delta < (1/(16L^2)) \wedge (\eta_p/\alpha)^{2/p}.$$

- $(\overline{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$, associated with Example I, admits a unique invariant measure $\pi^\delta \in \mathcal{P}(\mathbb{R}\times\mathbb{S})$ whenever the stepsize is sufficiently small.

Numerical Invariant Measure: Multiplicative Noise

(**Theorem 3**) Under conditions of **Theorem 2**, for sufficiently small $\delta \in (0,1)$,

$$W_p(\mu, \mu^\delta) \le c\delta^{p/2}, \qquad p \in (0, 1 \wedge p_0),$$

where

$$W_p(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{S}} \int_{\mathbb{R}^n \times \mathbb{S}} d(x,y)^p \pi(\mathrm{d}x, \mathrm{d}y), \quad p \in (0,1],$$

(**Theorem 4**) Let $N < \infty$, (10), (**H**), and (7) hold. Assume further that

$$\min_{i \in \mathbb{S}} \{-q_{ii}/\beta_i, \beta_i > 0\} > 1. \tag{13}$$

Then $(\overline{Y}^{x,i}_{k\delta}, \Lambda^i_{k\delta})$ has a unique invariant measure $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ whenever the stepsize $\delta \in (0,1)$ is sufficiently small.

Example II

Let $\{\Lambda_t\}_{t\geq 0}$ be a right-continuous Markov chain taking values in $\mathbb{S} := \{0,1,2\}$ with the generator

$$Q = \begin{pmatrix} -(3+\nu) & \nu & 3 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{pmatrix}$$

for some $\nu \geq 0$. Consider a scalar linear SDE with regime switching

$$\mathrm{d}X_t = \alpha_{\Lambda_t} X_t \mathrm{d}t + \sigma_{\Lambda_t} X_t \mathrm{d}W_t, \quad t \geq 0, \ \ X_0 = x, \ \Lambda_0 = i_0, \tag{14}$$

where $\alpha_{\cdot}, \sigma_{\cdot} : \mathbb{S} \mapsto \mathbb{R}$ such that

$$\alpha_0 = \frac{1}{2}, \alpha_1 = -2, \alpha_2 = -3, \qquad \sigma_0 = \frac{1}{3}, \sigma_1 = 2, \sigma_2 = 1.$$

Example II (Cont.)

Observe that (10) holds with $L = 4$, and (**H**) holds for $\beta_0 = \frac{10}{9}, \beta_1 = 0$, and $\beta_2 = -5$. Since the Markov chain possesses the stationary distribution

$$\mu = (\mu_0, \mu_1, \mu_2) = \Big(\frac{5}{20+5\nu}, \frac{6+3\nu}{20+5\nu}, \frac{9+2\nu}{20+5\nu}\Big),$$

it is easy to see that (7) and (13) are satisfied respectively for any $\nu \geq 0$. Then, $(\overline{Y}^{x,i}_{k\delta}, \Lambda^i_{k\delta})$ has a unique invariant measure for sufficiently small $\delta \in (0, 1)$.

Numerical Invariant Measure: Reversible Case

In this section, we assume that the Markov chain $\{\Lambda_t\}_{t\geq 0}$ is reversible, i.e., $\pi_i q_{ij} = \pi_j q_{ji}, i, j \in \mathbb{S}$, for some probability measure $\pi := (\pi_1, \cdots, \pi_N)$. To begin with, we need to introduce some notation. Let

$$L^2(\pi) := \Big\{ f \in \mathscr{B}(\mathbb{S}) : \sum_{i=1}^{N} \pi_i f_i^2 < \infty \Big\}.$$

Then $(L^2(\pi), \langle \cdot, \cdot \rangle_0, \| \cdot \|_0)$ is a Hilbert space. Define the bilinear form $(D(f), \mathscr{D}(D))$ as

$$D(f) := \frac{1}{2} \sum_{i,j=1}^{N} \pi_i q_{ij} (f_j - f_i)^2 - \sum_{i=1}^{N} \pi_i \beta_i f_i^2, \quad f \in L^2(\pi),$$

where $\beta_i \in \mathbb{R}, i \in \mathbb{S}$, is given in (**H**), and the domain

$$\mathscr{D}(D) := \{ f \in L^2(\pi) : D(f) < \infty \}.$$

Numerical Invariant Measure: Reversible Case (Cont.)

The principal eigenvalue λ_0 of $D(f)$ is defined by

$$\lambda_0 := \inf\{D(f) : f \in \mathscr{D}(D), \|f\|_0 = 1\}.$$

For more details on the first eigenvalue, refer to Chen (2000, 2005) . Due to the fact that the state space of $\{\Lambda_t\}_{t\geq 0}$ is finite, there exists $\xi = (\xi_1, \cdots, \xi_N) \in \mathscr{D}(D)$ such that

$$D(\xi) = \lambda_0 \|\xi\|_0^2. \tag{15}$$

Define the operator

$$\Omega := Q + \text{diag}(\beta_1, \cdots, \beta_N),$$

where Q is the Q-matrix of $\{\Lambda_t\}_{t\geq 0}$, and $\beta_i \in \mathbb{R}$ such that **(H)**.

Numerical Invariant Measure: Reversible Case (Cont.)

(**Theorem 5**) Let $N < \infty$, (10) and (**H**) hold, and assume further $\lambda_0 > 0$. Then, $(\overline{Y}^{x,i}_{k\delta}, \Lambda^i_{k\delta})$ admits a unique measure $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ whenever the stepsize $\delta \in (0,1)$ is sufficiently small.

Proof: Recalling (15) and checking the argument of Theorem 3.2] (Shao-Xi, 2014), one has

$$\xi \gg \mathbf{0} \quad \text{and} \quad (Q\xi)(i) + \beta_i \xi_i = -\lambda_0 \xi_i, \ i \in \mathbb{S}.$$

Then, the desired assertion follows by following an argument of Theorem 2.

Example IV

Let $\{\Lambda_t\}_{t\geq 0}$ be a right-continuous Markov chain taking values in $\mathbb{S} := \{0,1,2\}$ with the generator

$$Q = \begin{pmatrix} -b & b & 0 \\ 2a & -2(a+b) & 2b \\ 0 & 3a & -3a \end{pmatrix}$$

for some $a, b > 0$. Consider a scalar SDE with regime switching

$$\mathrm{d}X_t = \alpha_{\Lambda_t} X_t \mathrm{d}t + \sigma_{\Lambda_t} X_t \mathrm{d}W_t, \quad t \geq 0, \ \ X_0 = x, \tag{16}$$

where $\alpha_\cdot, \sigma_\cdot : \mathbb{S} \mapsto \mathbb{R}$ such that

$$c_0 = 2\alpha_0 + \sigma_0^2 < 0, \ \ c_1 = 2\alpha_1 + \sigma_1^2, \ \ c_2 = 2\alpha_2 + \sigma_2^2.$$

Example IV (Cont.)

We further assume that

$$b + c_0 < 0, \quad a - b - c_1 > 0, \quad a - c_2 > 0. \tag{17}$$

Note that (10) holds with $L = \max_{i \in \mathbb{S}}\{|\alpha_i| + |\sigma_i|\}$ and (**H**) holds with

$$\beta_0 = c_0, \quad \beta_1 = c_1, \quad , \beta_2 = c_2.$$

Moreover, by the notion of Ω, for $\xi_i = i + 1$, $i = 0, 1, 2$, we deduce that

$$(\Omega\xi)(0) = -(-b - c_0)\xi_0, \quad (\Omega\xi)(1) = -(a - b - c_1)\xi_1, \quad (\Omega\xi)(2) = -(a - c_2$$

Taking

$$\lambda = \min\{-b - c_0, a - b - c_1, a - c_2\} > 0$$

Example IV (Cont.)

thanks to (17), one finds that

$$(\Omega\xi)(i) \le -\lambda\xi_i, \quad i = 0, 1, 2.$$

Then $\lambda_0 > 0$ due to Theorem 4.4 (Shao-Xi, 2014, see also Chen, 2000). As a result, $(\overline{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$ has a unique invariant measure whenever the stepsize is sufficiently small.

Remark: **Theorem 5** can also be extended into the case of RSDPs with a finite state space (i.e. $N = \infty$) provided that λ_0 is attainable, i.e., there exists $f \in L^2(\pi), f \neq 0$, such that $D(f) = \lambda_0 \|f\|_0^2$.

Numerical Invariant Measure: Countable State Space

We further suppose that

$$K := \sup_{i\in\mathbb{S}} \beta_i < \infty \qquad \text{and} \qquad \sup_{i\in\mathbb{S}} (-q_{ii}) < \infty, \tag{18}$$

where $\beta_i \in \mathbb{R}$ is given in (**H**). Let us insert m points in the interval $(-\infty, K]$ as follows:

$$-\infty =: k_0 < k_1 < \cdots < k_m < k_{m+1} := K.$$

Then, the interval $(-\infty, K]$ is divided into $m+1$ sub-intervals $(k_{i-1}, k_i]$ indexed by i. Let

$$F_i := \{j \in \mathbb{S} : \beta_j \in (k_{i-1}, k_i]\}, \quad i = 1, \cdots, m+1.$$

Without loss of generality, we can and do assume that each F_i is not empty.

Numerical Invariant Measure: Countable State Space (Cont.)

Then

$$F := \{F_1, \cdots, F_{m+1}\}$$

is a finite partition of $\mathbb{S}$. For $i, j = 1, \cdots, m+1$, set

$$q_{ij}^F := \begin{cases} \sup_{r\in F_i} \sum_{k\in F_j} q_{rk}, & j < i, \\ \inf_{r\in F_i} \sum_{k\in F_j} q_{rk}, & j > i, \\ -\sum_{j\neq i} q_{ij}^F, & i = j. \end{cases}$$

So $Q^F := (q_{ij}^F)$ is the Q-matrix for some Markov chain with the state space $\mathbb{S}_0 := \{1, \cdots, m+1\}$.

For $i = 1, \cdots, m+1$, let

$$\beta_i^F := \sup_{j \in F_i} \beta_j, \qquad H_{m+1} := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(m+1)\times(m+1)}.$$

(**Theorem 5**) Let $N = \infty$, (10), (**H**) and (18) hold. Assume further that $\{\Lambda_t\}_{t\geq 0}$ is ergodic and that

$$-(Q^F + \text{diag}(\beta_1^F, \cdots, \beta_{m+1}^F))H_{m+1}$$

is a nonsingular M-matrix. Then $(\overline{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$ admits an invariant measure $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ whenever the stepsize $\delta \in (0,1)$ is sufficiently small.

Summary

In this talk, we are concerned with long-time behavior for EM schemes associated with a range of regime-switching diffusion processes. In particular, existence and uniqueness of numerical invariant measures are addressed

- (i) For regime-switching diffusion processes with finite state spaces by the Perron-Frobenius theorem if the "averaging condition" holds,
- And, with regard to reversible Markov chain, via the principal eigenvalue approach provided that the principal eigenvalue is positive;
- (ii) For regime-switching diffusion processes with countable state spaces by a finite partition method and an M-Matrix theory.
- Also, we reveal that numerical invariant measures converge in the Wasserstein metric to the underlying ones.

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Thanks A Lot !