COMPARISON THEOREMS OF SPECTRAL GAPS OF SCHRÖDINGER OPERATORS AND DIFFUSION OPERATORS ON ABSTRACT WIENER SPACES

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Key words: Spectral gap, Schrödinger operator, abstract Wiener space, min-max principle, Malliavin calculus.

Mathematical Subject Classification: primary 35P15, 35J10; secondary 60H10.

Abstract: How to use the information of coefficients in partial differential operators to get the information of spectrum of the operators? There exists a long literature of studying this problem from theory of diffusion processes and partial differential equations, and there are a lot of interesting problems need to answer. Among them there is a *fundamental gap conjecture* observed by Michiel van den Berg [J. Statist. Phys. 31(1983),no.3,623-637] and was independently suggested by Ashbaugh and Benguria [Proc. Amer. Math. Soc. 105(1989),no.2,419-424] and Yau [*Nonlinear analysis in geometry*, Monographies de L'Enseignement Mathématique,Vol.33,L'Enseignement Mathématique,Geneva, 1986. Série des Conférences de l'Union Mathématique Internationale,8], which gave an optimal lower bound of $\lambda_1 - \lambda_0$, the distance between the first two Dirichlet eigenvalues of a Schrödinger operator $-\Delta + V$ on a bounded uniformly convex domain Ω with a weakly convex potential V. By introducing the notion of *modulus of convexity* for functions, and studying the relationship between the modulus of convexity for V and the modulus of log-concavity for the first eigenfuncition (i.e. ground state) of Schrödinger operator $-\Delta + V$ through that of the one dimensional corresponding problems, Andrews and Clutterbuck [J. Amer. Math. Soc. 24 (2011), no. 3, 899–916] recently solved the fundamental gap conjecture. More interestingly, they proved a fundamental gap comparison theorem, that compare the fundamental gap of the Schrödinger operator $-\Delta + V$ with that of the one dimensional corresponding operator.

Note that, for the spectral gap of Schrödinger Operators and Diffusion Operators there are some sharp results for exponential integrability conditions of potential functions and diffusion coefficients such as Theorem 4.5 in [Simon and Hoegh-Krohn, JFA 1972] and Corollary 7.2 in [Fuzhou Gong and Liming Wu, J. Math. Pures Appl. 2006] in the literature. However, there was no nice estimates on the spectral gap or ground state. Roughly speaking, we have to make some control on the "derivative" of potential functions or diffusion coefficients, otherwise a high-frequency vibration on them will impact heavily on the spectral gap or ground state, but make no difference to the integrability.

In this talk we extend the fundamental gap comparison theorem of Andrews and Clutterbuck to the infinite dimensional setting. More precisely, we proved that the fundamental gap of the Schrödinger operator $-\mathcal{L}_* + V$ ($\mathcal{L}_*$ is the Ornstein–Uhlenbeck operator) on the abstract Wiener space is greater than that of the one dimensional operator $-\frac{d^2}{ds^2} + s\frac{d}{ds} + \tilde{V}(s)$, provided that $\tilde{V}$ is a modulus of convexity for V. Similar result is established for the diffusion operator $-\mathcal{L}_* + \nabla F \cdot \nabla$. The main results are as follows.

Let (W, H, μ) be an abstract Wiener space and $\mathcal{L}_*$ the Ornstein–Uhlenbeck operator on W associated to the symmetric Dirichlet form $\mathcal{E}_*(f,f) = (f, -\mathcal{L}_* f)$ with domain $\mathcal{D}[\mathcal{E}_*] = D_1^2(W,\mu)$ (i.e. $f \in L^2(W,\mu)$ with its Malliavin derivative $\nabla f \in L^2(W,H)$). Let $V \in D_1^p(W,\mu)$ for some $p > 1$ be a potential satisfying the *KLMN condition*, then one can define $-\mathcal{L} = -\mathcal{L}_* + V$ to be a self-adjoint Schrödinger operator bounded from below.

Correspondingly, let $\tilde{\mathcal{L}}_* = \frac{d^2}{ds^2} - s\frac{d}{ds}$ be the one-dimensional Ornstein–Uhlenbeck operator on R^1 with respect to the Gaussian measure $d\gamma_1 = (4\pi)^{-\frac{1}{2}} \exp(-\frac{s^2}{4})\, ds$. Let $\tilde{V} \in C^1(R^1) \cap L^1(R^1, \gamma_1)$ be a symmetric potential satisfying the KLMN condition too. Then $-\tilde{\mathcal{L}} = -\tilde{\mathcal{L}}_* + \tilde{V}$ is bounded from below. For convenience, a tilde will be added to all notations relative to $\tilde{\mathcal{L}}_*$ and $\tilde{V}$.

Let $< .,. >_H$ denote the inner product in the Cameron–Martin space H, and $|.|_H$ the norm.

Theorem A: Suppose for almost all $w \in W$ and every $h \in H$ with $h \neq 0$,

$$\left\langle \nabla V(w+h) - \nabla V(w), \frac{h}{|h|_H} \right\rangle_H \geq 2\tilde{V}'\left(\frac{|h|_H}{2}\right).$$

Then there exists a comparison

$$\lambda_1 - \lambda_0 \geq \tilde{\lambda}_1 - \tilde{\lambda}_0.$$

Hence, the existence of the spectral gap of $-\mathcal{L}$ on Wiener space can sometimes be reduced to one dimensional case. According to Andrews and Clutterbuck's notion, $\tilde{V}$ is a modulus of convexity for V. However, V doesn't need to be convex at all.

There are examples to show that, the above result is sharp, and the sharp exponential integrability of potential functions such as Theorem 4.5 in [Simon and Hoegh-Krohn, JFA 1972] can not be used but the above result can.

The next result gives the modulus of log-concavity for the ground state ϕ_0 of $-\mathcal{L}$.

Theorem B: Assume the same condition as in Theorem A and the gap $\tilde{\lambda}_1 - \tilde{\lambda}_0 > 0$. Then $-\mathcal{L}$ and $-\tilde{\mathcal{L}}$ have a unique ground state ϕ_0 and $\tilde{\phi}_0$ respectively. Moreover, for almost all $w \in W$ and every $h \in H$ with $h \neq 0$,

$$\left\langle \nabla \log \phi_0(w+h) - \nabla \log \phi_0(w), \frac{h}{|h|_H} \right\rangle_H \leq 2(\log \tilde{\phi}_0)' \left(\frac{|h|_H}{2} \right).$$

We also consider the diffusion operator $-\mathcal{L} = -\mathcal{L}_* + \nabla F \cdot \nabla$ on the Wiener space and we want to compare its spectral gap with the one dimensional operator $-\tilde{\mathcal{L}} = -\frac{d^2}{ds^2} + (s + \omega'(s))\frac{d}{ds}$. Although this kind of diffusion operator can be transformed to the Schrödinger type operator and their spectrum coincide with each other, the expression for the potential function V is a little complicated, hence it seems inappropriate to derive the gap comparison of diffusion operators from that of the transformed Schrödinger type operators. We shall directly establish the comparison theorem for spectral gaps of diffusion operators, and the main result is as follows.

Theorem C: Assume that $F \in D_1^p(W, R^1)$ satisfies $\int_W e^{-F}\, d\mu = 1$ and two functions F and ω are related by the following inequality: for all $h \in H$ and μ-a.e. $w \in W$,

$$\left\langle \nabla F(w+h) - \nabla F(w), \frac{h}{|h|_H} \right\rangle_H \geq 2\omega' \left(\frac{|h|_H}{2} \right).$$

Suppose also that $\omega \in C^1(R^1)$ is even, satisfying $\int_{R^1} e^{-\omega}\, d\gamma_1 = 1$ and $\lim_{t\to\infty} (\omega'(t) + t) = +\infty$. Then we have

$$\lambda_1 \geq \tilde{\lambda}_1.$$

There are also examples to show that, the sharp exponential integrability for diffusion coefficients ∇F such as Corollary 7.2 in [Fuzhou Gong and Liming Wu, J. Math. Pures Appl. 2006] and can not be used but the above result can.

The paper about the above all results has been published in [JFA,266(2014),5639-5675].

Furthermore, we give the probabilistic proofs of fundamental gap conjecture and spectral gap comparison theorem of Andrews and Clutterbuck in finite dimensional case via the coupling by reflection of the diffusion processes.