Support Properties of a Class of Λ-Fleming-Viot Processes with Underlying Brownian Motion

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Outline of the Talk

1. Introduction
   - Introduction
   - Coalescents
   - The lookdown construction
2. Compactness and Hausdorff Dimension of the Λ-FV Support
   - Previous results
   - New results
3. Disconnectedness of the FV Support

Fleming-Viot process

- Fleming-Viot process is one of the two fundamental examples of superprocesses. It is a probability-measure-valued Markov process arising as time-space scaling limit of a population genetics model with reproduction and mutation.
- For such a process
  - E denotes the type space. The F-V process X is $M_1(E)$-valued. For $A \subset E$, $X_t(A)$ represents the proportion of individuals in the population with types from A at time t.
  - The mutation is described by a Markov process on E.
  - The reproduction is described by a coalescent process.
- The classical Fleming-Viot process is associated to the Kingman coalescent. The Λ-Fleming-Viot process is associated to the more general Λ-coalescent.
- In this talk we only consider Fleming-Viot processes with Brownian mutation in $\mathbb{R}^d$.

The state space of coalescents

- $[n] = \{1, \ldots, n\}$. $[\infty] = \{1, 2, \ldots\}$.
- A partition $\pi = \{\pi_i, i = 1, 2, \ldots\}$ of $D \subset [\infty]$ is a collection of disjoint blocks such that $\cup_i \pi_i = D$ and $\min \pi_i < \min \pi_j$ for $i < j$.
- $\mathcal{P}_n$ denotes the set of partitions of $[n]$.
- The coalescent process $\{\Pi_n(t), t \geq 0\}$ is a $\mathcal{P}_n$-valued stochastic process such that $\Pi_n(s)$ is a refinement of $\Pi_n(t)$ for every $s < t$.

Kingman's coalescent

- The n-coalescent of binary collisions is a $\mathcal{P}_n$-valued Markov process starting with n blocks such that given there are k blocks, each 2-tuple of blocks merges independently to form a single block at rate 1.
- Kingman (1982) shows that there exists a $\mathcal{P}_\infty$-valued Markov process $\{\Pi(t) : t \geq 0\}$ (called Kingman's coalescent), whose restriction to the first n positive integers is an n-coalescent.

Λ-coalescent

- Pitman (1999) and Sagitov (1999) introduce the Λ-coalescent which allows multiple collisions.
- It is a $\mathcal{P}_\infty$-valued Markov process such that given there are n blocks in the partition, each k-tuple of blocks $(2 \leq k \leq n)$ independently merges to form a single block at rate

$$\lambda_{n,k} = \int_0^1 x^{k-2}(1-x)^{n-k}\Lambda(dx)$$

and Λ is a finite measure on $[0,1]$.

Coming down from infinity

Let $\#\Pi_\infty(t)$ be the number of blocks in the partition $\Pi_\infty(t)$.

- The coalescent comes down from infinity if for all $t > 0$

$$P\left(\#\Pi_\infty(t) < \infty\right) = 1.$$

- The coalescent stays infinite if for all $t > 0$

$$P\left(\#\Pi_\infty(t) = \infty\right) = 1.$$

- Schweinsberg (2000) Suppose that $\Lambda(\{1\}) = 0$. The Λ-coalescent comes down from infinity if and only if
  - either $\Lambda(\{0\}) > 0$;
  - or $\Lambda(\{0\}) = 0$ but

$$\sum_{n=2}^{\infty}\left(\sum_{k=2}^{n}(k-1)\binom{n}{k}\lambda_{n,k}\right)^{-1} < \infty.$$

Examples

- If $\Lambda = \delta_0$, the corresponding coalescent degenerates to Kingman's coalescent and comes down from infinity.
- If
$$\Lambda(dx) = \frac{\Gamma(2)}{\Gamma(2-\beta)\Gamma(\beta)} x^{1-\beta}(1-x)^{\beta-1}\,dx$$
for some $\beta \in (0,2)$, it corresponds to the Beta$(2-\beta,\beta)$-coalescent. The coalescent with $\beta = 1$ is also called U-coalescent.
    - When $\beta \in (0,1]$, it stays infinite;
    - When $\beta \in (1,2)$, it comes down from infinity.
- If $\Lambda = \delta_1$, the corresponding coalescent neither comes down from infinity nor stays infinite.

Λ-FV generator

$$G_f(\mu) = \int f(x_1, \ldots, x_n)\mu^{\otimes n}(d\mathbf{x}).$$

$$LG_f(\mu) = L^{a\delta_0} G_f(\mu) + L^{\Lambda_0} G_f(\mu) + L^B G_f(\mu),$$

where

$$L^{a\delta_0} G_f(\mu) = a \sum_{1\le i<j\le n} \int (f(x_1, \ldots, x_i, \ldots, x_i, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n))\mu^{\otimes n}(d\mathbf{x}),$$

$$L^{\Lambda_0} G_f(\mu) = \int_{[0,1]} \int (G_f((1-\xi)\mu + \xi\delta_x) - G_f(\mu))\mu(dx)\Lambda_0(d\xi)/\xi^2,$$

$$L^B G_f(\mu) = \sum_{i=1}^n \int B_i f(\mathbf{x})\mu^{\otimes n}(d\mathbf{x})$$

where $B_i f$ is B acting on the i-th coordinate of f.

Lookdown construction for Λ-FV with Brownian mutation

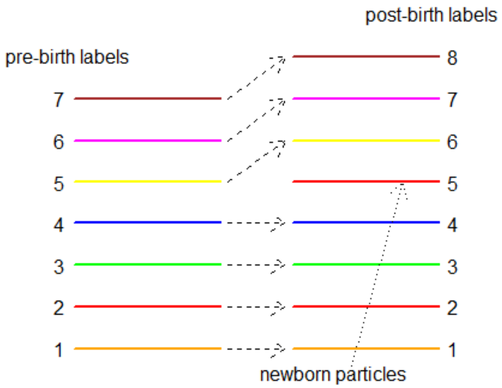
- The (modified) lookdown particle construction of Donnelly and Kurtz is a powerful tool to study the FV processes.
- In this particle system
  - Each particle is attached a "level" from set $\{1, 2, \cdots\}$.
  - The spatial location $X_i(t)$ represents the type of the particle at level i. The motion of X_i represents mutation.
  - Looking forwards in time, the empirical measures of the particles in the lookdown model approximate the FV processes.
  - Looking backwards in time, we can recover the coalescent process describing the genealogy of the lookdown model.
  - The evolution of a particle at level n only depends on the evolution of the particles at lower levels.

- For any finite measure Λ on $[0,1]$, we have $\Lambda = a\delta_0 + \Lambda_0$, where $a\delta_0$ is the restriction of Λ to $\{0\}$ and $\Lambda_0 = \Lambda \mathbf{1}_{(0,1]}$.
- The particle system undergoes reproductions. The particles move according to independent Brownian motions between the reproduction events.
- There are two kinds of reproduction events
  - single-birth events associated to $a\delta_0$;
  - multiple-birth events associated to Λ_0.

Lookdown construction with single-birth

- Let $\{N_{ij}(t) : 1 \leq i < j < \infty\}$ be independent Poisson processes with common rate a.
- At a jump time t of N_{ij},
  - a new particle is born at higher level j and its assumes the spatial location of particle at lower level i,
  - all the other particles with levels above j are "shifted" upwards.

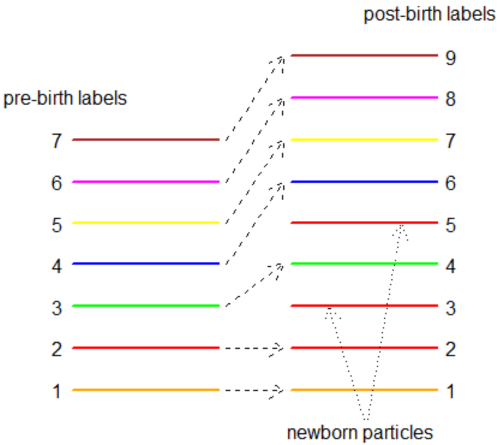
Relabeling of the particles in a Kingman-lookdown event



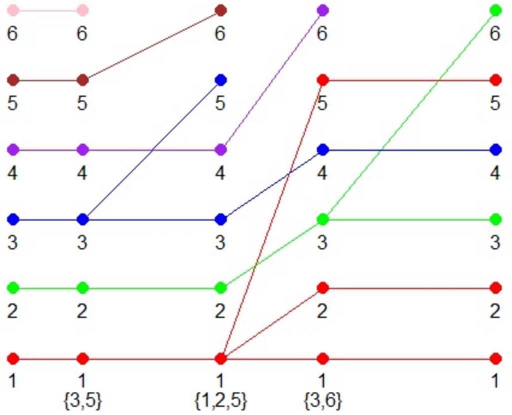
Lookdown construction with multiple-birth

- Let **N** be a Poisson point process on $\mathbb{R}_+ \times (0,1]$ with intensity measure $dt \otimes x^{-2}\Lambda_0(dx)$.
- "Points" $\{(t_i, x_i)\}$ of **N** correspond to multiple-birth events.
- Intuitively, at each time t_i,
  - a new particle is independently born at each level j with probability x_i;
  - the new born particles assume the (pre-bitrh) spatial location of the particle at the lowest involved level.
  - particles at other levels, keeping their original order, are shifted upwards accordingly.

Relabeling of the particles in a Λ-lookdown event



Sequential lookdowns



Limit of the empirical measures

- Suppose that $(X_i(0))$ is exchangeable.
- Then for each $t > 0$, $(X_i(t))$ is exchangeable, so that

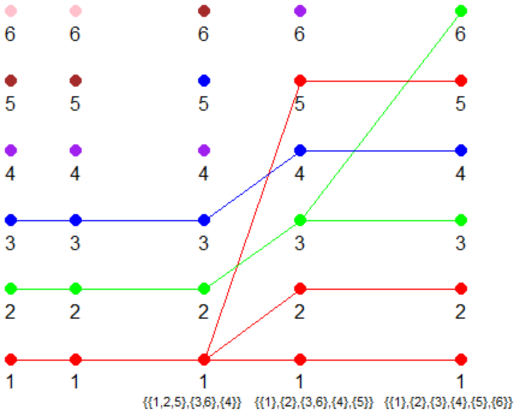
$$X(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)}$$

exists almost surely by de Finetti's theorem, and is the Λ-Fleming-Viot process with Brownian mutation.

Λ-coalescent in the lookdown construction

- Looking backwards from a fixed time $T > 0$, we can recover from the lookdown construction a coalescent describing the genealogy of the particles at time T.
- For fixed $T > 0$, denote $\{\Pi(t) : 0 \leq t \leq T\}$ as a $\mathcal{P}_\infty$-valued process such that i and j belong to same block of $\Pi(t)$ if and only if the particles at levels i and j share the same ancestor at time $T - t$.
- The process $\{\Pi(t) : 0 \leq t \leq T\}$ is a Λ-coalescent process

The coalescent process



Theorem (Dawson and Hochberg (1982))

Let $X(t)$ be the classical Fleming-Viot process in $\mathbb{R}^d$ with Brownian mutation. Then for any fixed $t > 0$, $X(t)$ has a compact support and the Hausdorff dimension of the support at most two.

Theorem (Blath 2009)

Let $X(t)$ be a Λ-FV process with Brownian mutation. If the corresponding Λ-coalescent stays infinite, then for each $t > 0$,

$$SuppX(t) = \mathbb{R}^d.$$

An assumption

Let Π be the Λ-coalescent associated to the Λ-Fleming-Viot process.

$$T_m := \inf\{t : \#\Pi(t) \leq m\}.$$

Assumption: we always assume that there exists $\alpha > 0$ such that

$$\limsup_{m\to\infty} m^{\alpha}\mathbb{E}\,T_m < \infty.$$

The Assumption is sufficient for the Λ-coalescent to come down from infinity.
For Kingman coalescent, $\alpha = 1$.
For Beta$(2-\beta, \beta)$-coalescent, $\alpha = \beta - 1$.

Compact support property

Theorem

For any fixed $t > 0$, the Λ-Fleming-Viot process has a compact support at t. In addition,

$$dim(SuppX_t) \leq 2/\alpha.$$

Proposition

$$dim(SuppX_t) \geq 2.$$

Corollary

If $\Lambda(\{0\}) > 0$, then $dim(SuppX_t) = 2.$

Beta$(2-\beta,\beta)$-Fleming-Viot process

Corollary

The Beta$(2-\beta,\beta)$-Fleming-Viot process has the compact support property if and only if $\beta \in (1,2)$.
Further,

$$dim(SuppX_t) \leq 2/\left(\beta-1\right)$$

for $\beta \in (1,2)$.

- We conjecture that the exact Hausdorff dimension for the support is $2/\left(\beta-1\right)$.

Modulus of continuity for the support

For any set A, write $\mathbb{B}(A, r)$ for the closed r-neighborhood of A.

Theorem

For any fixed $t \geq 0$, there exist positive random variable $\theta \equiv \theta(t, d, \alpha) < 1$ and constant $C \equiv C(d, \alpha)$ such that for any Δt with $0 < \Delta t \leq \theta$ we have $\mathbb{P}$-a.s.

$$supp\, X(t + \Delta t) \subseteq \mathbb{B}\left(supp\, X(t), C\sqrt{\Delta t \log(1/\Delta t)}\right).$$

More compactness

- $\forall\ I \subseteq [0, \infty)$, write $\mathcal{R}(I) \equiv \overline{\cup_{s \in I} \mathrm{supp}\, X(s)}$ for the range of the support for X over I.

Applying the modulus of continuity, we have the following result.

Theorem

$\mathrm{supp}\, X(t)$ is compact for all $t > 0$ $\mathbb{P}$-a.s.
If $\mathrm{supp}\, X(0)$ is compact, $\mathcal{R}([0, t))$ is compact for all $t > 0$ $\mathbb{P}$-a.s.

More Hausdorff dimensions

- $\lambda_n \equiv \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k}$: the total coalescence rate.
- Condition A: There exists a constant $\alpha > 0$ s.t. $\limsup_{m\to\infty} m^\alpha \sum_{n=m+1}^{\infty} \lambda_n{}^{-1} < \infty$.
- Kingman's coalescent: $\lambda_n = O\left(n^2\right)$.
- Beta$(2-\beta, \beta)$-coalescent: $\lambda_n = O\left(n^\beta\right)$ for $\beta \in (1,2)$.

Theorem

Suppose that Condition A holds. Then for all $t > 0$, $\dim \operatorname{supp} X(t) \leq 2/\alpha$ $\mathbb{P}$-a.s.

Theorem

Suppose that Condition A holds. Then for all $0 < \delta < T$, $\dim \mathcal{R}([\delta, T)) \leq 2 + 2/\alpha \quad \mathbb{P}$-a.s.

Previous results on superBrownian motions

- The disconnectedness of super-Brownian support is a question first asked by Dawson.
- For super-Brownian motion with Brownian branching, the question was partially answered by Perkins.
- For super-Brownian motion with stable branching, the question was partially answered by Delmas.

New results on Fleming-Viot processes

Recall the assumption

$$\limsup_{m\to\infty} m^\alpha \mathbb{E}\, T_m < \infty.$$

Theorem

Given $d > 4/\alpha$, for any $T > 0$, $suppX_T$ is totally disconnected.

Theorem

Given $d > 2 + 4/\alpha$, $suppX_t$ is totally disconnected for all $t > 0$.

Outline of the proof

- For fixed $T > 0$ we consider the ancestors of those particles at times $T - n^{-1}$ and $T - n^{-\epsilon}$ with $0 < \epsilon < 1$, respectively.
- We group the ancestors at time $T - n^{-1}$ together according to their respective ancestors at time $T - n^{-\epsilon}$.
- On one hand, the typical distances of ancestors from different groups are of the order $n^{-\epsilon/2}$ since their positions are determined by independent Brownian motions.
- On the other hand, the typical distance of a particle at time T from its respective ancestor at time $T - n^{-1}$ is of the order $n^{-1/2}$, and from its ancestor at time $T - n^{-\epsilon}$ is of the order $n^{-\epsilon/2}$ by the modulus of continuity for the ancestry process.
- For large n ant at time T each particle typically stays away from particles with different ancestors at time $T - n^{-\epsilon}$, and the maximal distance among particles with the same ancestors at time $T - n^{-\epsilon}$ are typically of the order $n^{-\epsilon/2}$.

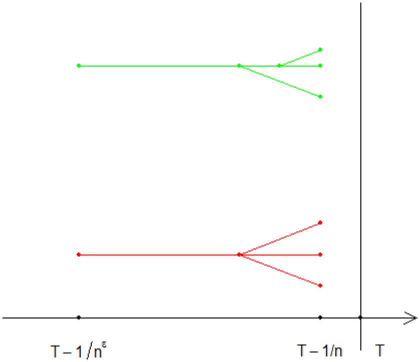


Figure : Disconnectedness of support

Look forward to the next workshop