BDSDEs with Polynomial Growth Coefficients

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Consider the SPDEs

$$dv(t,x) = [\mathscr{L}v(t,x) + f(x, v(t,x))]dt + g(x, v(t,x))dB_t. \quad (1)$$

Here $\mathscr{L}$ is a second order differential operator given by

$$\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \quad (2)$$

B is a Q-Wiener process on a probability space $(\Omega, \mathcal{F}, P)$ with values in a separable Hilbert space U. Denote by $\{e_i\}_{i=1}^{\infty}$ the countable base of U. Then $Q \in L(U)$ is a symmetric nonnegative trace class operator such that $Qe_i = \lambda_i e_i$ and $\sum_{i=1}^{\infty} \lambda_i < \infty$.

Let $\rho(x) = (1 + |x|)^q$, $q > d + 8p$, is a weight function and p is given later. Denote by $L^k_\rho(\mathbb{R}^d;\mathbb{R}^1)$, $k \geq 2$, the weighted L^k space with the norm $\|l\|_{L^k_\rho(R^d)} = (\int_{\mathbb{R}^d} |l(x)|^k \rho^{-1}(x)dx)^{\frac{1}{k}}$. Denote by $v(t,\omega,v_0) \in L^k_\rho(\mathbb{R}^d;\mathbb{R}^1)$ the solution of Eq. (1) with initial value $v(0) = v_0$, if it exists uniquely (established later in this talk). Then we say $Y : \Omega \to L^k_\rho(\mathbb{R}^d;\mathbb{R}^1)$ is a stationary solution of SPDE (1) if

$$u(t,\omega,Y(\omega)) = Y(\theta_t\omega). \tag{3}$$

Stationary solution a large time limit if the SPDEs which can be regarded as a stochastic dynamical systems. For the deterministic parabolic type PDEs, it is a solution of an elliptic equation. Its distribution is an invariant measure. To find a stationary solution, it is a challenging task. One can think $Y(\omega)$ is initial value of the solution. But it is not an initial value problem, the main task is to find the initial value.

Our observation generally speaking is:
A solution of infinite horizon BDSDEs

$$\begin{aligned} e^{-Ks}Y_s^{t,x} &= \int_s^\infty e^{-Kr} f(X_r^{t,x}, Y_r^{t,x})dr + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr \\ &\quad - \int_s^\infty e^{-Kr} g(X_r^{t,x}, Y_r^{t,x}) d^\dagger \hat{B}_r - \int_s^\infty e^{-Kr} \langle Z_r^{t,x}, dW_r\rangle, \end{aligned} \quad (4)$$

if exists with desired regularity, gives a stationary solution of the corresponding SPDEs.

We consider the following backward SPDE for $0 \le t \le T$:

$$\begin{aligned} u(t,x) &= h(x) + \int_t^T [\mathscr{L}u(s,x) + f(x,u(s,x))]ds \\ &\quad - \int_t^T g(x,u(s,x))d^\dagger \hat{B}_s. \end{aligned} \tag{5}$$

Here $\hat{B}$ is Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ valued in a Hilbert space U. The stochastic integral $\int_t^T g(s,x,u(s,x))d^\dagger \hat{B}_s$ is a backward stochastic integral which will be made clear later. Define $X_s^{t,x}$ to be the solution of the following stochastic differential equations for any given $t \ge 0$ and $x \in \mathbb{R}^d$:

$$\begin{aligned} X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \quad s \ge t, \\ X_s^{t,x} = x, \quad 0 \le s < t, \end{aligned} \tag{6}$$

where W is a Brownian motion on the probability space $(\Omega, \mathscr{F}, P)$ valued in $\mathbb{R}^d$ and is independent of $\hat{B}$.

The BDSDE associated with SPDE (5) is

$$
\begin{aligned}
Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x})dr \\
&\quad - \int_s^T g(r, X_r^{t,x}, Y_r^{t,x})d^\dagger \hat{B}_r - \int_s^T \langle Z_r^{t,x}, dW_r\rangle, \quad 0 \leq t \leq s \leq T.
\end{aligned}
\tag{7}
$$

Denote by $\mathcal{N}$ the class of P-null sets of $\mathscr{F}$. For any process $(\eta_s)_{s\geq 0}$, denote $\mathscr{F}_{t,s}^{\eta} = \sigma\{\eta_r - \eta_t; 0 \leq t \leq r \leq s\} \bigvee \mathcal{N}$, $\mathscr{F}_{s,\infty}^{\eta} = \bigvee_{T\geq s} \mathscr{F}_{s,T}^{\eta}$, and let

$$
\begin{aligned}
\mathscr{F}_{s,T} &\triangleq \mathscr{F}_{s,T}^{\hat{B}} \bigvee \mathscr{F}_{t,s}^{W}, \quad \text{for } 0 \leq t \leq s \leq T, \\
\mathscr{F}_{s} &\triangleq \mathscr{F}_{s,\infty}^{\hat{B}} \bigvee \mathscr{F}_{t,s}^{W}, \quad \text{for } 0 \leq t \leq s.
\end{aligned}
$$

Definition 1

Let $\mathbb{S}$ *be a separable Banach space with norm* $\|\cdot\|_{\mathbb{S}}$ *and Borel* σ*-field* $\mathscr{S}$ *and* $q \geq 2$, $K > 0$. *We denote by* $M^{q,-K}([t,\infty);\mathbb{S})$ *the set of* $\mathscr{B}([t,\infty)) \otimes \mathscr{F} / \mathscr{S}$ *measurable random processes* $\{\phi(s)\}_{s \geq t}$ *with values in* $\mathbb{S}$ *satisfying*
(i) $\phi(s) : \Omega \longrightarrow \mathbb{S}$ *is* $\mathscr{F}_s$ *measurable for* $s \geq t$*;*
(ii) $E[\int_t^\infty \mathrm{e}^{-Ks}\|\phi(s)\|_{\mathbb{S}}^q ds] < \infty$.
Also we denote by $S^{q,-K}([t,\infty);\mathbb{S})$ *the set of* $\mathscr{B}([t,\infty)) \otimes \mathscr{F} / \mathscr{S}$ *measurable random processes* $\{\psi(s)\}_{s \geq t}$ *with values in* $\mathbb{S}$ *satisfying*
(i) $\psi(s) : \Omega \longrightarrow \mathbb{S}$ *is* $\mathscr{F}_s$ *measurable for* $s \geq t$ *and* $\psi(\cdot,\omega)$ *is a.s. continuous;*
(ii) $E[\sup_{s \geq t} \mathrm{e}^{-Ks}\|\psi(s)\|_{\mathbb{S}}^q] < \infty$.

Backward stochastic integral.
Let $\{g(s)\}_{s\geq 0}$ be a stochastic process with values in $\mathcal{L}^2_{U_0}(H)$ such that $g(s)$ is $\mathscr{F}_s$ measurable for any $s \geq 0$ and locally square integrable, i.e. for any $0 \leq a \leq b < \infty$, $\int_a^b \|g(s)\|^2_{\mathcal{L}^2_{U_0}(H)} ds < \infty$ a.s.
Since $\mathscr{F}_s$ is a backward filtration with respect to $\hat{B}$. Set

$$B(s) = \hat{B}(T' - s) - \hat{B}(T'). \tag{8}$$

Then $\mathcal{F}^{\hat{B}}_{s,T} = \mathcal{F}^{B}_{T'-T,T'-s}$. Therefore $g(s)$ is measurable with respect to $\mathcal{F}^{B}_{T'-T,T'-s}$ and

$$\int_t^T g(s)d^\dagger \hat{B}_s = -\int_{T'-T}^{T'-t} g(T' - s)dB_s \quad \text{a.s.}$$

Polynomial growth assumption on f:
There exists a constant $p \geq 2$ and a function $f_0 : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$ with $\int_0^T \int_{\mathbb{R}^d} |f_0(s,x)|^{8p} \rho^{-1}(x) dx ds < \infty$,

$$
\begin{aligned}
&|f(s,x,y)| \leq L(|f_0(s,x)| + |y|^p)\\
&|\partial_y f(s,x,y)| \leq L(1 + |y|^{p-1}),\\
&|f(s,x_1,y) - f(s,x_2,y)| \leq L(1 + |y|^p)|x_1 - x_2|,\\
&|\partial_y f(s,x_1,y) - \partial_y f(s,x_2,y)| \leq L(1 + |y|^{p-1})|x_1 - x_2|,\\
&|\partial_y f(s,x,y_1) - \partial_y f(s,x,y_2)| \leq L(1 + |y_1|^{p-2} + |y_2|^{p-2})|y_1 - y_2|,\\
&(y_1 - y_2)(f(s,x,y_1) - f(s,x,y_2)) \leq \mu |y_1 - y_2|^2.
\end{aligned}
$$

Typical example is that for an odd integer p

$$f(y) = y - y^p.$$

Theorem 1

The BDSDE (7) has a unique solution $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in S^{8p,0}([t,T]; L_\rho^{8p}(\mathbb{R}^d;\mathbb{R}^1)) \times M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$.

BSDEs

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r)dr - \int_s^T Z_r dW_r.$$

Pardoux and Peng (1991)
Quadratic coefficients in z, Kobylansky (2000)
Linear growth without Lipschitz: Lepeltier and San Martin (1997)
Polynomial growth coefficients (Pardoux (1996), only one paper before us as far as we know) for fixed terminal random variable only, so use the Ascoli-Azela compact argument without involving any ω.

When we consider $\xi = h(X_T^{t,x})$, BSDEs can be concocted with the PDEs:
Smooth coefficients and classical solution of PDEs, Pardoux and Peng (1991);
Viscosiity solutions;
BDSDEs and SPDEs-classical solutions, Pardoux and Peng (1994);
Weak solutions–Lipschitz coefficients, Bally and Matoussi (2001), Zhang and Zhao (2007);
–Linear growth, not Lipschitz coefficients, Zhang and Zhao (2009), Wu and F. Zhang (2012)

Idea of the proof is to use linear growth coefficient approximation and prove the convergence. For each $n \in N$, define

$$f_n(s,x,y) = f(s,x,\Pi_n(y)) + \partial_y f(s,x,\frac{n}{|y|}y)(y - \frac{n}{|y|}y)I_{\{|y|>n\}},$$

where $\Pi_n(y) = \frac{\inf(n,|y|)}{|y|}y$. This function is of linear growth, monotone and C^1. It may not be Lipschitz. In the case of BSDEs, as there is a connection with PDEs. So one can use Rellich-Kondrachov compactness theorem to pass the limit (advantage of BSDEs is that the estimate Z leads to the estimate of ∇u without having to differentiate the equations.) So there was no need to have the second part of f_n. See Zhang and Zhao (2012).
In the case of BSDESs, this does not work as R-K compactness does not work for random fields.

It is not difficult to prove that for any $2 \leq m \leq 8p$

$$\sup_n E[\sup_{s\in[t,T]} \int_{\mathbb{R}^d} |Y_s^{t,x,N,n}|^m \rho^{-1}(x)dx]$$
$$+ \sup_n E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x,N,n}|^m \rho^{-1}(x)dxds]$$
$$+ \sup_n E[(\int_t^T \int_{\mathbb{R}^d} |Z_s^{t,x,N,n}|^2 \rho^{-1}(x)dxds)^{\frac{m}{2}}] < \infty.$$

Then by Alaoglu Theorem, there is a subsequence weakly converging to $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ in $L_\rho^2(\Omega \times [t,T] \times \mathbb{R}^d; \mathbb{R}^1 \times \mathbb{R}^d)$. The key is to prove

$$f_n(\cdot, X_\cdot^{t,\cdot}, X_\cdot^{t,\cdot,n}) \rightarrow f(\cdot, X_\cdot^{t,\cdot}, Y_\cdot^{t,\cdot});$$
$$\int_t^T g(s, X_s^{t,\cdot}, Y_s^{t,\cdot,n})d^\dagger \hat{B}_s \rightarrow \int_t^T g(s, X_s^{t,\cdot}, Y_s^{t,\cdot})d^\dagger \hat{B}_s;$$

Theorem 2

(Theorem 2, Bally and Saussereau (2004)) Let $(u_n)_{n\in\mathbb{N}}$ *be a sequence of* $L^2([0,T]\times\Omega; H^1(\mathcal{O}))$. *Suppose that*

(1) $\sup_n E[\int_0^T \|u_n(s,\cdot)\|^2_{H^1(\mathcal{O})} ds] < \infty$.

(2) For all $\varphi \in C_c^k(\mathcal{O})$ *and* $t \in [0,T]$, $u_n^\varphi(s) \in \mathbb{D}^{1,2}$ *and* $\sup_n \int_0^T \|u_n^\varphi(s)\|^2_{\mathbb{D}^{1,2}} ds < \infty$.

(3) For all $\varphi \in C_c^k(\mathcal{O})$, *the sequence* $(E[u_n^\varphi])_{n\in\mathbb{N}}$ *of* $L^2([0,T])$ *satisfies*

(3i) For any $\varepsilon > 0$, *there exists* $0 < \alpha < \beta < T$ *s.t.*

$$\sup_n \int_{[0,T]\setminus(\alpha,\beta)} |E[u_n^\varphi(s)]|^2 ds < \varepsilon.$$

(3ii) For any $0 < \alpha < \beta < T$ *and* $h \in \mathbb{R}^1$ *s.t.* $|h| < min(\alpha, T-\beta)$, *it holds*

$$\sup_n \int_\alpha^\beta |E[u_n^\varphi(s+h)] - E[u_n^\varphi(s)]|^2 ds < C_p|h|.$$

(4) For all $\varphi \in C_c^k(\mathcal{O})$, the following conditions are satisfied:
(4i) For any $\varepsilon > 0$, there exists $0 < \alpha < \beta < T$ and $0 < \alpha' < \beta' < T$ s.t.

$$\sup_n E[\int_{[0,T]^2 \setminus (\alpha,\beta)\times(\alpha',\beta')} |D_\theta u_n^\varphi(s)|^2 d\theta ds] < \varepsilon.$$

(4ii) For any $0 < \alpha < \beta < T$, $0 < \alpha' < \beta' < T$ and $h, h' \in \mathbb{R}^1$ s.t. $max(|h|, |h'|) < min(\alpha, \alpha', T-\beta, T-\beta')$, it holds that

$$\sup_n E[\int_\alpha^\beta \int_{\alpha'}^{\beta'} |D_{\theta+h} u_n^\varphi(s+h') - D_\theta u_n^\varphi(s)|^2 d\theta ds] < C_p(|h| + |h'|).$$

Then $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(\Omega \times [0,T] \times \mathcal{O}; \mathbb{R}^1)$.
Malliavin, Nualart and Da Prato (1992), Peszat (1993), $L^2(\Omega)$; Bally and Saussereau (2004), $L^2(\Omega \times [0,T] \times \mathcal{O})$; Feng and Zhao (2012), $C^0([0,T], L^2(\Omega \times \mathcal{O}))$.

Lemma 1

(generalized equivalence of norm principle) If $s \in [t, T]$, $\varphi : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$ *is independent of the* σ*-field* $\mathscr{F}^W_{t,s}$ *and* $\varphi\rho^{-1} \in L^1(\Omega \times \mathbb{R}^d)$, *then there exist two constants* $c > 0$ *and* $C > 0$ *s.t.*

$$\begin{aligned} cE[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)dx] &\leq E[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})|\rho^{-1}(x)dx] \\ &\leq CE[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)dx]. \end{aligned}$$

Let $u(t,x) = Y_t^{t,x}$. Once Theorem 1 is proved, then

$$\begin{aligned} E\int_{R^d} |u(s,x)|^{2p}\rho^{-1}(x)dx &\leq \frac{1}{c}E\int_{R^d} |u(s,X_s^{t,x})|^{2p}\rho^{-1}(x)dx \\ &= \frac{1}{c}E\int_{R^d} |Y_s^{t,x}|^{2p}\rho^{-1}(x)dx. \end{aligned}$$

Theorem 3

Define $u(t,x) = Y_t^{t,x}$*, then* $u(t,x)$ *is the unique weak solution of SPDE (5). Moreover,*

$$u(s, X_s^{t,x}) = Y_s^{t,x}, \ (\sigma^* \nabla u)(s, X_s^{t,x}) = Z_s^{t,x} \text{ for a.e. } s \in [t,T], \ x \in \mathbb{R}^d \text{ a.s.}$$

First to use the standard limit procedure taking $T \to \infty$, we can prove

Theorem 4

BDSDE (4) has a unique solution $(Y_{\cdot}^{t,\cdot}, Z_{\cdot}^{t,\cdot}) \in S^{2p,-K} \bigcap M^{2p,-K}([t,\infty]; L_{\rho}^{2p}(\mathbb{R}^d;\mathbb{R}^1)) \times M^{2,-K}([t,\infty]; L_{\rho}^{2}(\mathbb{R}^d;\mathbb{R}^d))$.

Let $\tilde{\theta}_t = \hat{\theta}_t \circ \check{\theta}_t$, $t \geq 0$, where $\hat{\theta}_t, \check{\theta}_t : \Omega \longrightarrow \Omega$ are measurable mappings on $(\Omega, \mathscr{F}, P)$ defined by

$$\hat{\theta}_t \begin{pmatrix} \hat{B} \\ W \end{pmatrix}(s) = \begin{pmatrix} \hat{B}_{s+t} - \hat{B}_t \\ W_s \end{pmatrix}, \quad \check{\theta}_t \begin{pmatrix} \hat{B} \\ W \end{pmatrix}(s) = \begin{pmatrix} \hat{B}_s \\ W_{s+t} - W_t \end{pmatrix}.$$

Set

$$\tilde{\theta}_t \circ \phi(\omega) = \phi(\tilde{\theta}_t(\omega)).$$

Recall standard existing result

$$\tilde{\theta}_r \circ X_s^{t,\cdot} = \check{\theta}_r \circ X_s^{t,\cdot} = X_{s+r}^{t+r,\cdot} \text{ for all } r,s,t \geq 0 \text{ a.s.}$$

Theorem 5

$$\tilde{\theta}_r \circ Y_s^{t,\cdot} = Y_{s+r}^{t+r,\cdot}, \quad \tilde{\theta}_r \circ Z_s^{t,\cdot} = Z_{s+r}^{t+r,\cdot} \quad \text{for all } r \geq 0,\ s \geq t \text{ a.s.}$$

In particular, for any $t \geq 0$,

$$\hat{\theta}_r \circ Y_t^{t,\cdot} = Y_{t+r}^{t+r,\cdot} \quad \text{for all } r \geq 0 \text{ a.s.} \tag{9}$$

Another observation is that that $t \to Y_t^{t,\cdot}$ is continuous.
Consider the forward SPDE (1) with Brownian motion B. Set $\hat{B}$ as the time reversal BM at T', $\theta : (\theta_t B)(s) = B_{t+s} - B_t$. Then $\hat{\theta}_t = \theta_t^{-1} = \theta_{-t}$ is the shift operator of $\hat{B}$.

Note $(\hat{\theta}_{T'-t}\hat{B})(s) = B_{t-s} - B_s$ is independent of T'. Since $v(t,\cdot) = u(T'-t,\cdot) = Y_{T'-t}^{T'-t,\cdot}$ a.s., so

$$\begin{aligned} \theta_r v(t,\cdot,\omega) &= \hat{\theta}_{-r} u(T'-t,\cdot,\hat{\omega}) = \hat{\theta}_{-r}\hat{\theta}_r u(T'-t-r,\cdot,\hat{\omega}) \\ &= u(T'-t-r,\cdot,\hat{\omega}) = v(t+r,\cdot,\omega), \end{aligned}$$

for all $r \geq 0$ and $T' \geq t+r$ a.s. In particular, let $Y(\cdot,\omega) = v(0,\cdot,\omega) = Y_{T'}^{T',\cdot}(\hat{\omega})$, then the above formula implies:

$$\begin{aligned} \theta_t Y(\cdot,\omega) &= Y(\cdot,\theta_t\omega) = v(t,\cdot,\omega) \\ &= v(t,\cdot,\omega,v(0,\cdot,\omega)) = v(t,\cdot,\omega,Y(\cdot,\omega)) \text{ for all } t \geq 0 \text{ a.s.} \end{aligned}$$

It turns out that $v(t,\cdot,\omega) = Y(\cdot,\theta_t\omega) = Y_{T'-t}^{T'-t,\cdot}(\hat{\omega})$ *is a stationary solution of SPDE (1) w.r.t.* θ.

We can also prove that

$$Y_{\cdot}^{t,\cdot,T}(h) \longrightarrow Y_{\cdot}^{t,\cdot} \; as \; T \rightarrow \infty$$

in $S^{2p,-K} \bigcap M^{2p,-K}([t,\infty]; L_{\rho}^{2p}(\mathbb{R}^d;\mathbb{R}^1))$. This implies that

$$Y_t^{t,\cdot,T}(h) \longrightarrow Y_t^{t,\cdot} \; as \; T \rightarrow \infty.$$

In particular,

$$Y_{T'-t}^{T'-t,\cdot,T}(h) \longrightarrow Y_{T'-t}^{T'-t,\cdot} = v(t,\cdot) = Y(\theta_t\cdot) \text{ and } Y_{T'}^{T',\cdot,T}(h) \longrightarrow Y_{T'}^{T',\cdot} = Y(\cdot).$$

Thus

$$\begin{aligned} v(T-T',h,\theta_{-(T-T')}\omega) &= u(T-(T-T'),h,\widehat{\theta_{-(T-T')}\omega}^{T}) \\ &= u(T',h,\hat{\omega}) = Y_{T'}^{T',\cdot,T}(h,\hat{\omega}) \end{aligned}$$

since

$$
\begin{aligned}
(\theta_{-(T-T')}\widehat{\omega^T})(s) &= (\theta_{-(T-T')}\omega)(T-s) - (\theta_{-(T-T')}\omega)(T) \\
&= \omega(-(T-T')+T-s) - \omega(-(T-T')+T) \\
&= \omega(T'-s) - \omega(T') = \hat{\omega}(s).
\end{aligned}
$$

That is to say that the time reversal of the Brownian motion B at time T' is the same as the time reversal of the Brownian motion $\theta_{-(T-T')}\omega$ at T. Therefore as $T \to \infty$, $v(T-T', h, \theta_{-(T-T')}\cdot) = Y_{T'}^{T',\cdot,T}(h,\cdot) \longrightarrow Y(\cdot)$ in $L^{2p}(\Omega; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1))$. The result does not depend on the choice of T'. So we have proved

Theorem 6

As $T \to \infty$, $v(T, h, \theta_{-T}\cdot) \longrightarrow Y(\cdot)$ in $L^{2p}(\Omega; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1))$, and $Y(\theta_t\omega)$ is the stationary solution of the SPDE (1).