Heat kernels and analyticity of non-symmetric Lévy diffusion semigroups

Xicheng Zhang

Wuhan University

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Problem

Consider the following second order elliptic differential operator in $\mathbb{R}^d$:

$$\mathscr{L}_2^a f(x) = \sum_{i,j=1}^{d} a^{ij}(x)\partial_i\partial_j f(x),$$

where $a^{ij}(x) \in C_b^\infty(\mathbb{R}^d)$ is uniformly positive.

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where $a^{ij}(x) \in C_b^\infty(\mathbb{R}^d)$ is uniformly positive. It is well-known that the C_0-semigroup associated with $\mathscr{L}_2^a$ is analytic in L^p-spaces provided $p \in (1,\infty)$ (cf. Pazy's book). The proof of this fact is based upon the following deep apriori estimate:

$$\|\partial_i\partial_j f\|_p \leqslant C(\|\mathscr{L}_2^a f\|_p + \|f\|_p), \quad f \in \mathbb{W}^{2,p}(\mathbb{R}^d),$$

which is a consequence of singular integral operator theory.

Problem

Now, consider the following non-local Lévy operator: for $\alpha \in (0, 2)$,

$$\mathscr{L}_\alpha^\kappa f(x) := \text{P.V.} \int_{\mathbb{R}^d} (f(x+z) - f(x))\kappa(x,z)|z|^{-d-\alpha}\mathrm{d}z, \tag{1.1}$$

where P.V. stands for the Cauchy principle value, and $\kappa(x, z)$ is a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ and satisfies

$$\kappa(x, z) = \kappa(x, -z), \;\; 0 < \kappa_0 \leqslant \kappa(x, z) \leqslant \kappa_1, \tag{1.2}$$

and for some $\beta \in (0, 1)$

$$|\kappa(x, z) - \kappa(y, z)| \leqslant \kappa_2 |x - y|^\beta. \tag{1.3}$$

Problem

Due to the symmetricity of κ in z, we may write

$$\mathscr{L}_\alpha^\kappa f(x) = \frac{1}{2}\int_{\mathbb{R}^d}(f(x+z)+f(x-z)-2f(x))\kappa(x,z)|z|^{-d-\alpha}\mathrm{d}z.$$

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Let $\Delta^{\frac{\alpha}{2}} := -(-\Delta)^{\frac{\alpha}{2}}$ be the usual fractional Laplacian. By Fourier's transform, it is easy to see that for some constant $c_{d,\alpha} > 0$,

$$\mathscr{L}_\alpha^1 = c_{d,\alpha}\Delta^{\frac{\alpha}{2}}.$$

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Due to the symmetricity of κ in z, we may write

$$\mathscr{L}^{\kappa}_{\alpha} f(x) = \frac{1}{2}\int_{\mathbb{R}^d} (f(x+z) + f(x-z) - 2f(x))\kappa(x,z)|z|^{-d-\alpha}\mathrm{d}z.$$

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$$\mathscr{L}^{1}_{\alpha} = c_{d,\alpha}\Delta^{\frac{\alpha}{2}}.$$

Hence, $\mathscr{L}^{\kappa}_{\alpha}$ can be considered as a generalization of $\Delta^{\frac{\alpha}{2}}$ as $\mathscr{L}^{a}_{2}$ generalises Laplacian Δ.

Problem

- Existence of heat kernel associated with $\mathscr{L}_\alpha^\kappa$, i.e.,

$$\partial_t p(t,x,y) = \mathscr{L}_\alpha^\kappa p(t,\cdot,y)(x), \quad p(0,x,y) = \delta_0(x-y)?$$

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- If yes, heat kernel estimate, gradient and fractional derivative estimate?

- Analyticity of the semigroup associated with $\mathscr{L}_\alpha^\kappa$ in L^p-space?

Well-known results

When

$$\kappa(x, z-x) = \kappa(z, x-z),$$

the operator $\mathscr{L}_\alpha^\kappa$ is symmetric in the sense that

$$\int_{\mathbb{R}^d} g(x)\mathscr{L}_\alpha^\kappa f(x)\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\mathscr{L}_\alpha^\kappa g(x)\mathrm{d}x, \quad f, g \in C_0^\infty(\mathbb{R}^d).$$

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In this case, without Hölder's assumption, the following two-sided sharp estimates of the heat kernel $p_\alpha^\kappa(t, x, y)$ of $\mathscr{L}_\alpha^\kappa$ was obtained by Chen and Kumagai (SPA 2003) by using the probabilistic approach:

$$c_0 t(t^{\frac{1}{\alpha}} + |x-y|)^{-d-\alpha} \leqslant p_\alpha^\kappa(t, x, y) \leqslant c_0^{-1} t(t^{\frac{1}{\alpha}} + |x-y|)^{-d-\alpha}.$$

Well-known results

- Bogdan-Jakubowski (CMP 2007): $\Delta^{\frac{\alpha}{2}} + b(x) \cdot \nabla$, where $\alpha \in (1, 2)$. (See also Jakubowski-Szczypkowski(JEE 2010), Jakubowski (Studia Math. 2011), Wang-Zhang(Forum Math. to appear)) (Duhamel's method)

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- Kochubei(Math. USSR 1988) proved the existence for $\mathscr{L}^{\kappa}_{\alpha}$ with smooth κ in y and $\alpha \in [1,2)$; Xie-Zhang(2012): $a_t(x)\Delta^{\frac{1}{2}} + b_t(x)\cdot\nabla$ (Levi's method).

Levi's method

Write

$$\mathscr{L}_\alpha^{\kappa(x)} f(x) = \mathscr{L}_\alpha^{\kappa} f(x) = \frac{1}{2}\int_{\mathbb{R}^d} \delta_f(x;z)\kappa(x,z)|z|^{-d-\alpha}\mathrm{d}z,$$

where

$$\delta_f(x;z) := f(x+z) + f(x-z) - 2f(x).$$

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For fixed $y \in \mathbb{R}^d$, let $\mathscr{L}_\alpha^{\kappa(y)}$ be the freezing operator

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Let $p_y(t,x) := p_\alpha^{\kappa(y)}(t,x)$ be the heat kernel of operator $\mathscr{L}_\alpha^{\kappa(y)}$, i.e.,

$$\partial_t p_y(t,x) = \mathscr{L}_\alpha^{\kappa(y)} p_y(t,x), \quad \lim_{t\downarrow 0} p_y(t,x) = \delta_0(x). \tag{1.4}$$

Levi's method

Now, we want to seek the heat kernel $p^\kappa_\alpha(t,x,y)$ of $\mathscr{L}^\kappa_\alpha$ with the following form:

$$p^\kappa_\alpha(t,x,y) = p_y(t,x-y) + \int_0^t\int_{\mathbb{R}^d} p_z(t-s,x-z)q(s,z,y)\mathrm{d}z\mathrm{d}s. \quad (1.5)$$

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The classical Levi's continuity argument suggests that $q(t,x,y)$ solves the following integral equation:

$$q(t,x,y) = q_0(t,x,y) + \int_0^t\int_{\mathbb{R}^d} q_0(t-s,x,z)q(s,z,y)\mathrm{d}z\mathrm{d}s, \quad (1.6)$$

where

$$\begin{aligned} q_0(t,x,y) &:= (\mathscr{L}^{\kappa(x)}_\alpha - \mathscr{L}^{\kappa(y)}_\alpha)p_y(t,x-y) \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \delta_{p_y}(t,x-y;z)(\kappa(x,z)-\kappa(y,z))|z|^{-d-\alpha}\mathrm{d}z. \end{aligned}$$

Levi's method

In fact, we formally have

$$
\begin{aligned}
\partial_t p_\alpha^\kappa(t,x,y) &= \mathscr{L}_\alpha^{\kappa(y)} p_y(t,x-y) + q(t,x,y) \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \partial_t p_z(t-s,x-z) q(s,z,y) \mathrm{d}z \mathrm{d}s \\
&= \mathscr{L}_\alpha^{\kappa(x)} p_y(t,x-y) \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \mathscr{L}_\alpha^{\kappa(x)} p_z(t-s,x-z) q(s,z,y) \mathrm{d}z \mathrm{d}s \\
&= \mathscr{L}_\alpha^{\kappa(x)} p_\alpha^\kappa(t,x,y).
\end{aligned}
$$

Case: $\kappa(x, z) = \kappa(z)$

Let p_α^κ be the heat kernel of $\mathscr{L}_\alpha^\kappa$. Write

$$\delta_{p_\alpha^\kappa}(t, x; z) := p_\alpha^\kappa(t, x + z) + p_\alpha^\kappa(t, x - z) - 2p_\alpha^\kappa(t, x)$$

and

$$\varrho_\gamma^\beta(t, x) := t^{\frac{\gamma}{\alpha}}(|x|^\beta \wedge 1)(t^{\frac{1}{\alpha}} + |x|)^{-d-\alpha}.$$

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Let p^κ_α be the heat kernel of $\mathscr{L}^\kappa_\alpha$. Write

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We have

Lemma 1 (Fractional derivative estimate)

- $$\int_{\mathbb{R}^d} |\delta_{p^\kappa_\alpha}(t,x;z)| \cdot |z|^{-d-\alpha} \mathrm{d}z \leqslant C\varrho^0_0(t,x).$$

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- $$\int_{\mathbb{R}^d} |\delta_{p^\kappa_\alpha}(t,x;z) - \delta_{p^\kappa_\alpha}(t,x';z)| \cdot |z|^{-d-\alpha} \mathrm{d}z$$
$$\leqslant C((t^{-\frac{1}{\alpha}}|x-x'|) \wedge 1)\varrho^0_0(t,x).$$

Case: $\kappa(x, z) = \kappa(z)$

Lemma 2 (Continuous dependence of heat kernel)

Let κ *and* $\hat{\kappa}$ *be two kernel functions. For any* $\gamma \in (0, \alpha \wedge 1)$*, there exists a constant* $C = C(d, \alpha, \kappa_0, \kappa_1, \gamma) > 0$ *such that*

- $$|p_\alpha^\kappa(t, x) - p_\alpha^{\hat{\kappa}}(t, x)| \leqslant C\|\kappa - \hat{\kappa}\|_\infty(\varrho_\alpha^0 + \varrho_{\alpha-\gamma}^\gamma)(t, x).$$

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- $$|p^{\kappa}_{\alpha}(t,x)-p^{\hat\kappa}_{\alpha}(t,x)|\leqslant C\|\kappa-\hat\kappa\|_\infty(\varrho^0_\alpha+\varrho^\gamma_{\alpha-\gamma})(t,x).$$

- $$|\nabla p^{\kappa}_{\alpha}(t,x)-\nabla p^{\hat\kappa}_{\alpha}(t,x)|\leqslant C\|\kappa-\hat\kappa\|_\infty t^{-\frac{1}{\alpha}}(\varrho^0_\alpha+\varrho^\gamma_{\alpha-\gamma})(t,x).$$

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- $$\int_{\mathbb{R}^d}\frac{|\delta_{p^\kappa_\alpha}(t,x;z)-\delta_{p^{\hat\kappa}_\alpha}(t,x;z)|\mathrm{d}z}{|z|^{d+\alpha}}\leqslant C\|\kappa-\hat\kappa\|_\infty(\varrho^0_0+\varrho^\gamma_{-\gamma})(t,x).$$

Idea of the proof

If we set

$$\hat{\kappa}(z) := \kappa(z) - \frac{\kappa_0}{2},$$

then by convolution technique, one can write

$$p^{\kappa}_{\alpha}(t,x) = \int_{\mathbb{R}^d} p^{\kappa_0/2}_{\alpha}(t, x-y) p^{\hat{\kappa}}_{\alpha}(t,y) \mathrm{d}y.$$

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$$p_\alpha^\kappa(t,x) = \int_{\mathbb{R}^d} p_\alpha^{\kappa_0/2}(t,x-y)p_\alpha^{\hat{\kappa}}(t,y)\mathrm{d}y.$$

On the other hand, by Duhamel's formula,

$$\begin{aligned} p_\alpha^\kappa(t,x) - p_\alpha^{\hat{\kappa}}(t,x) &= \int_0^t\int_{\mathbb{R}^d} p_\alpha^\kappa(t-s,x-y)(\mathscr{L}_\alpha^\kappa - \mathscr{L}_\alpha^{\hat{\kappa}})p_\alpha^{\hat{\kappa}}(s,y)\mathrm{d}y\mathrm{d}s \\ &= \int_0^t\int_{\mathbb{R}^d} (\mathscr{L}_\alpha^\kappa - \mathscr{L}_\alpha^{\hat{\kappa}})p_\alpha^\kappa(t-s,x-y)p_\alpha^{\hat{\kappa}}(s,y)\mathrm{d}y\mathrm{d}s. \end{aligned}$$

Statement of Main Theorem

Under (1.2) and (1.3), there exists a unique nonnegative continuous function $p^\kappa_\alpha(t,x,y)$ on $(0,1)\times\mathbb{R}^d\times\mathbb{R}^d$ solving

$$\partial_t p^\kappa_\alpha(t,x,y) = \mathscr{L}^\kappa_\alpha p^\kappa_\alpha(t,\cdot,y)(x), \quad t>0,$$

and satisfying that

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and satisfying that

- (Upper bound) For all $t\in(0,1)$ and $x,y\in\mathbb{R}^d$,

$$p^\kappa_\alpha(t,x,y) \leqslant c_1 t \varrho^0_0(t,x-y).$$

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- (Hölder's estimate) For all $\gamma\in(0,\alpha\wedge1)$, $t\in(0,1)$ and $x,x',y\in\mathbb{R}^d$,

$$\begin{aligned}&|p^\kappa_\alpha(t,x,y)-p^\kappa_\alpha(t,x',y)|\\ &\quad\leqslant c_2|x-x'|^\gamma t^{1-\frac{\gamma}{\alpha}}\Big\{\varrho^0_0(t,x-y)+\varrho^0_0(t,x'-y)\Big\}.\end{aligned}$$

Statement of Main Theorem

- (Fractional derivative estimate) For all $x, y \in \mathbb{R}^d$, the mapping $t \mapsto \mathscr{L}^\kappa_\alpha p^\kappa_\alpha(t, \cdot, y)(x)$ is continuous on $(0, 1)$, and

$$|\mathscr{L}^\kappa_\alpha p^\kappa_\alpha(t, \cdot, y)(x)| \leqslant c_3 \varrho^0_0(t, x - y).$$

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$$|\mathscr{L}_\alpha^\kappa p_\alpha^\kappa(t,\cdot,y)(x)| \leqslant c_3 \varrho_0^0(t, x-y).$$

- (Continuity) For any bounded and uniformly continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_\alpha^\kappa(t,x,y) f(y) \mathrm{d}y - f(x) \right| = 0.$$

Statement of Main Theorem

- (Conservativity) For all $(t, x) \in (0, 1) \times \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p_\alpha^\kappa(t, x, y)\mathrm{d}y = 1.$$

Statement of Main Theorem

- (Conservativity) For all $(t, x) \in (0, 1) \times \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^{\kappa}_{\alpha}(t, x, y)\mathrm{d}y = 1.$$

- (C-K equation) For all $s, t \in (0, 1)$ and $x, y \in \mathbb{R}^d$, the following Chapman-Kolmogorov's equation holds:

$$\int_{\mathbb{R}^d} p^{\kappa}_{\alpha}(t, x, z) p^{\kappa}_{\alpha}(s, z, y)\mathrm{d}z = p^{\kappa}_{\alpha}(t + s, x, y).$$

Statement of Main Theorem

Moreover, if $\alpha \in [1,2)$, then

- (Gradient estimate) for all $x, y \in \mathbb{R}^d$ and $t \in (0,1)$,

$$|\nabla p^{\kappa}_{\alpha}(t,\cdot,y)(x)| \leqslant c_4 t^{1-\frac{1}{\alpha}} \varrho^0_0(t, x-y);$$

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and if $\nabla_x \kappa(x,z)$ and $\nabla^2_x \kappa(x,z)$ are bounded, then we also have the following conclusions:

- (Generator) For all $f, g \in C^\infty_0(\mathbb{R}^d)$,

$$\lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} g(x)\Big(\mathscr{P}^\kappa_t f(x) - f(x)\Big)\mathrm{d}x = \int_{\mathbb{R}^d} g(x)\mathscr{L}^\kappa_\alpha f(x)\mathrm{d}x,$$

  where

$$\mathscr{P}^\kappa_t f(x) := \int_{\mathbb{R}^d} p^\kappa_\alpha(t,x,y) f(y)\mathrm{d}y.$$

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Moreover, if $\alpha \in [1, 2)$, then

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and if $\nabla_x \kappa(x, z)$ and $\nabla_x^2 \kappa(x, z)$ are bounded, then we also have the following conclusions:

- (Generator) For all $f, g \in C_0^\infty(\mathbb{R}^d)$,

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} g(x)\Big(\mathscr{P}_t^\kappa f(x) - f(x)\Big)\mathrm{d}x = \int_{\mathbb{R}^d} g(x)\mathscr{L}_\alpha^\kappa f(x)\mathrm{d}x,$$

  where

$$\mathscr{P}_t^\kappa f(x) := \int_{\mathbb{R}^d} p_\alpha^\kappa(t, x, y) f(y)\mathrm{d}y.$$

- (Analyticity) The C_0-semigroup $(\mathscr{P}_t^\kappa)_{t \geqslant 0}$ is analytic in $L^p(\mathbb{R}^d)$ provided $p \in [1, \infty)$.

A key lemma (Kochubei)

If $\beta \in [0, \alpha)$, then

$$\int_{\mathbb{R}^d} \varrho_\gamma^\beta(t, x)\mathrm{d}x \preceq t^{\frac{\gamma+\beta-\alpha}{\alpha}}.$$

A key lemma (Kochubei)

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If $\beta_1, \beta_2 \in [0, \alpha)$ and $\gamma_1 + \beta_1 > 0$, $\gamma_2 + \beta_2 > 0$, then

$$\int_0^t \int_{\mathbb{R}^d} \varrho_{\gamma_1}^{\beta_1}(t-s, x-z)\varrho_{\gamma_2}^{\beta_2}(s, z)\mathrm{d}z\mathrm{d}s$$
$$\preceq \mathcal{B}\left(\tfrac{\gamma_1+\beta_1}{\alpha}, \tfrac{\gamma_2+\beta_2}{\alpha}\right)\Big\{\varrho^0_{\gamma_1+\gamma_2+\beta_1+\beta_2} + \varrho^{\beta_1}_{\gamma_1+\gamma_2+\beta_2} + \varrho^{\beta_2}_{\gamma_1+\gamma_2+\beta_1}\Big\}(t, x),$$

where $\mathcal{B}(\gamma, \beta)$ is the usual Beta function defined by

$$\mathcal{B}(\gamma, \beta) := \int_0^1 (1-s)^{\gamma-1}s^{\beta-1}\mathrm{d}s, \quad \gamma, \beta > 0.$$

A nonlocal maximal principle

Let $u(t,x) \in C_b([0,1] \times \mathbb{R}^d)$ with

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} |u(t,x) - u(0,x)| = 0.$$

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Suppose that for each $x \in \mathbb{R}^d$,

$$t \mapsto \mathscr{L}_\alpha^\kappa u(t,x) \text{ is continuous on } (0,1],$$

and for any $\varepsilon \in (0,1)$ and some $\gamma_\varepsilon \in ((\alpha - 1) \vee 0, 1)$,

$$\sup_{t \in (\varepsilon,1)} |u(t,x) - u(t,x')| \leqslant C_\varepsilon |x - x'|^{\gamma_\varepsilon}.$$

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and for any $\varepsilon \in (0,1)$ and some $\gamma_\varepsilon \in ((\alpha - 1) \vee 0, 1)$,

$$\sup_{t \in (\varepsilon, 1)} |u(t,x) - u(t,x')| \leqslant C_\varepsilon |x - x'|^{\gamma_\varepsilon}.$$

If $u(t,x)$ satisfies the following equation: for all $(t,x) \in (0,1) \times \mathbb{R}^d$,

$$\partial_t u(t,x) = \mathscr{L}^\kappa_\alpha u(t,x),$$

A nonlocal maximal principle

Let $u(t,x) \in C_b([0,1]\times\mathbb{R}^d)$ with

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}|u(t,x)-u(0,x)|=0.$$

Suppose that for each $x\in\mathbb{R}^d$,

$$t\mapsto \mathscr{L}^\kappa_\alpha u(t,x) \text{ is continuous on } (0,1],$$

and for any $\varepsilon\in(0,1)$ and some $\gamma_\varepsilon\in((\alpha-1)\vee 0,1)$,

$$\sup_{t\in(\varepsilon,1)}|u(t,x)-u(t,x')|\leqslant C_\varepsilon|x-x'|^{\gamma_\varepsilon}.$$

If $u(t,x)$ satisfies the following equation: for all $(t,x)\in(0,1)\times\mathbb{R}^d$,

$$\partial_t u(t,x)=\mathscr{L}^\kappa_\alpha u(t,x),$$

then for all $t\in(0,1)$,

$$\sup_{x\in\mathbb{R}^d}u(t,x)\leqslant\sup_{x\in\mathbb{R}^d}u(0,x).$$

Proof of analyticity

Below, we write

$$\mathscr{P}_t^\kappa f(x) := \int_{\mathbb{R}^d} p_\alpha^\kappa(t,x,y) f(y) \mathrm{d}y.$$

Lemma 3

For any $p \in [1,\infty)$ and $f \in L^p(\mathbb{R}^d)$, $(0,1) \ni t \mapsto \mathscr{L}_\alpha^\kappa \mathscr{P}_t^\kappa f \in L^p(\mathbb{R}^d)$ is continuous. In the case of $p = \infty$, i.e., if f is a bounded measurable function on $\mathbb{R}^d$, then for each $x \in \mathbb{R}^d$, $t \mapsto \mathscr{L}_\alpha^\kappa \mathscr{P}_t^\kappa f(x)$ is a continuous function on $(0,1)$. Moreover, for any $p \in [1,\infty]$, there exists a constant $C = C(p,d,\alpha,\kappa_0,\kappa_1) > 0$ such that for all $f \in L^p(\mathbb{R}^d)$ and $t \in (0,1)$,

$$\boxed{\|\mathscr{L}_\alpha^\kappa \mathscr{P}_t^\kappa f\|_p \leqslant C t^{-1} \|f\|_p}.$$

Open questions

- Lower bound estimate of heat kernel of $\mathscr{L}_\alpha^\kappa$?

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- Existence of Markov process associated with $\mathscr{L}_\alpha^\kappa$?
- Can we do the estimates for more general operators like

$$\mathscr{L}_\alpha^\kappa f(x) := \int_{\mathbb{R}^d} \delta_f(x; y)\kappa(x, y)\nu(\mathrm{d}y)?$$

Thank you very much for your kind attention!