Strong Convergence of Wong-Zakai Approximations of Reflected SDEs in A Multidimensional General Domain

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Abstract

In this paper, we obtained the strong convergence of Wong-Zakai approximations of reflected SDEs in a general multidimensional domain giving an affirmative answer to the question posed in [ES] by Evans and Stroock.

Introduction

Let D be a bounded domain in R^d. Consider the reflected stochastic differential equation (SDE):

$$\begin{cases} dX(t) = \sigma(X(t)) \circ dW(t) + b(X(t))dt + dL(t), \\ X(0) = x_0, \quad X(t) \in \bar{D}, t \geq 0, \\ |L|(t) = \int_0^t I_{\partial D}(X(s))d|L|(s), \end{cases} \tag{1}$$

where $W(t), t \geq 0$ is a m-dimensional Brownian motion, $|L|(t)$ stands for the total variation of L on the interval $[0, t]$, $\circ$ indicates a Stratonovich integral. There is a big amount of literature devoted to the study of reflected SDEs. Let us mention a few of them. Reflected SDEs in a convex domain was first studied by H. Tanaka in [T]. Existence and uniqueness of solutions of reflected SDEs in general domains were established by Lions and Sznitman in [LS] and Saisho in [S]. Existence and uniqueness of solutions of reflected SDEs under more general coefficients than the usual Lipschitz conditions were considered in [MR].

Introduction

The purpose of this paper is to study Wong-Zakai type approximations of above reflected SDEs. Let W^n be the $n-$dyadic piecewise linear interpolation of W and X^n the solution of the following reflected random ordinary differential equation:

$$\begin{cases} \dot{X}^n(t) = \sigma(X^n(t))\dot{W}^n(t) + b(X^n(t))dt + \dot{L}^n(t), \\ X^n(0) = x_0, \quad X^n(t) \in \bar{D}, t \geq 0, \\ |L^n|(t) = \int_0^t I_{\partial D}(X^n(s))d|L^n|(s). \end{cases} \tag{2}$$

We are concerned with the strong convergence of X^n to the solution X. Strong convergence of Wong-Zakai approximations to stochastic differential equations is well known, see e.g. [IW]. However, the convergence of Wong-Zakai approximations to stochastic differential equations with reflection (especially in higher dimensions) is not a trivial matter because of the constraints on the solution and the appearance of the boundary local time.

Introduction

There are two main papers related to this question. In [P], Petterson established a Wong-Zakai approximations for SDEs with reflection for convex domains. The convexity is too rigid sometimes for applications. In [ES], Evans and Stroock considered Wong-Zakai approximations for reflected SDEs in general domains (as in [LS]) and proved that X^n converges weakly (in law) to the solution X. In the same paper, the authors also posed the question of whether the strong convergence holds. For some of the interesting applications, we refer the reader to [ES]. The purpose of this paper is to establish the strong convergence (the L^p convergence in $C([0,T],\bar{D})$ of the Wong-Zakai approximations for reflected SDEs in multidimensional general domains, hence giving an affirmative answer to the question in [ES].

The Main result

Let $D \subset R^d$ be a bounded domain with boundary ∂D. For $x \in \partial D$, let $\nu(x) \subset S^{d-1}$ denote a nonempty collection of reflecting directions. Throughout this paper, as in [LS], [ES], we impose the following conditions on the domain.

D.1 $\nu(x) \neq \emptyset$ for every $x \in \partial D$ and there exist a constant $C_0 \geq 0$ such that

$$(x'-x)\cdot\nu + C_0|x-x'|^2 \geq 0 \qquad \text{for all} \quad x' \in D, x \in \partial D \quad \text{and} \quad \nu \in \nu($$

D.2 There exists a function $\phi \in C^2(R^d; R)$ and $\alpha > 0$ such that

$$\nabla\phi(x) \cdot \nu \geq \alpha \qquad \text{for all} \quad x \in \partial D \quad , \nu \in \nu(x).$$

D.3 There exist $n \geq 1$, $\lambda > 0$, $K > 0$, $a_1, a_2, ..., a_n \in S^{d-1}$, and $x_1, x_2, ..., x_n \in \partial D$ such that $\partial D \subset \cup_{i=1}^n B(x_i, K)$ and $x \in \partial D \cap B(x_i, 2K) \Longrightarrow \nu \cdot a_i \geq \lambda$ for all $\nu \in \nu(x)$.

Convention; Throughout this paper, any function G defined on the positive half line $[0, \infty)$ automatically extends to a function on the whole line by setting $G(s) = G(s \vee 0)$ when necessary.

The main result

Let $W(t) = (W_1(t), W_2(t), ..., W_m(t)), t \geq 0$ be a m-dimensional Brownian motion on a completed filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Suppose $\sigma = (\sigma_{i,j}) \in C^1(\bar{D}; R^d \otimes R^m)$ such that the derivative σ' is Lipschitz continuous and that $b : \bar{D} \to R^d$ is Lipschitz continuous.

For $n \in N$ and $s \in [\frac{k}{2^n}, \frac{k+1}{2^n})$, set $s_n^- = (\frac{k-1}{2^n}) \vee 0$ and $s_n = \frac{k}{2^n}$. Let W^n be the linear interpolation of W defined by

$$W^n(t) = W(\frac{k-1}{2^n}) + 2^n(t - \frac{k}{2^n})(W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})) \quad (3)$$

for $t \in [\frac{k}{2^n}, \frac{k+1}{2^n}), \quad k = 0, 1, 2, ...$ Note that the above convention applies here. Let $\sigma\sigma' : \bar{D} \to R^d$ be defined as

$$(\sigma\sigma'(y))_i = \sum_{j=1}^{m} \sum_{k=1}^{d} \frac{\partial \sigma_{i,j}(y)}{\partial y_k} \sigma_{k,j}(y). \quad (4)$$

The main result

With this notation, equation (1) becomes

$$X(t) = x_0 + \int_0^t \sigma(X(s))dW(s) + \frac{1}{2}\int_0^t \sigma\sigma'(X(s))ds + \int_0^t b(X(s))ds + L(t) \tag{5}$$

Definition

We say that (X, L) is a solution to the reflected SDE (5) if (X, L) is a $\bar{D} \times R^d$-valued, adapted continuous process such that
(i) $L(t), t \geq 0$ is of bounded variation on any finite sub-interval of $[0, \infty)$,
(ii) for $t \geq 0$,

$$X(t) = x_0 + \int_0^t \sigma(X(s))dW(s) + \frac{1}{2}\int_0^t \sigma\sigma'(X(s))ds + \int_0^t b(X(s))ds + L(t)$$

almost surely,

(iii)

$$|L|(t) = \int_0^t I_{\partial D}(X(s))d|L|(s), \qquad L(t) = \int_0^t \nu(X(s))d|L|(s),$$

where $|L|(t)$ stands for the total variation of L on the interval $[0, t]$, the last equality means that $\frac{dL(t)}{d|L|(t)} \in \nu(X(t))$.

The solution (X^n, L^n) to the reflected random ordinary differential equation (2) is defined accordingly.

The main result

Under the above assumptions, the existence and uniqueness of X^n, X are well known now, see, for example, [LS]. Here is the main result.

Theorem[The main result]
Let X^n, X be the solutions to reflected stochastic equations (1) and (2). It holds that for any $p > 0$ and $T > 0$,

$$\lim_{n\to\infty} E[\sup_{0\leq t\leq T} |X^n(t) - X(t)|^p] = 0. \tag{6}$$

The rest of the paper (27 pages) is entirely devoted to the proof of this theorem. Before sketching the proof I like to make a remark.

A remark

Remark. After the submission of this paper to the Annals of Probability, I was made aware of the existence of the following online preprint by the referee:
[1]. S. Aida and K. Sasaki: Wong-Zakai approximation of solutions for reflecting stochastic differential equations on domains in Euclidean spaces.

The work in these two papers was carried out completely independently. The approaches are different. I hope that the method I present to you could also be used for some other reflected stochastic equations.

Sketch of the proof

First of all we recall the following estimates from [ES].

lemma[1] Let $p \geq 2,\ T > 0$. Then there exists a constant $C_1(T,p)$ independent of n such that

$$E[|X^n(t) - X^n(s)|^p] \leq C_1(T,p)|t-s|^{\frac{p}{2}}, \tag{7}$$

for $0 \leq s, t \leq T$.

lemma[2] Let $p \geq 2,\ T > 0$. Then there exists a constant $C_2(T,p)$ such that

$$E[|X(t) - X(s)|^p] \leq C_2(T,p)|t-s|^{\frac{p}{2}}, \tag{8}$$

for $0 \leq s, t \leq T$.

Sketch of the proof

Due to (7), (8) above, to prove the main result, it can be shown that one only needs to prove that for any fixed $t > 0$

$$\lim_{n\to\infty} E[|X^n(t) - X(t)|^2] = 0. \tag{9}$$

To prove (9). again because of (7), (8) we may assume that t is a dyadic number, i.e., $t = \frac{k_0}{2^{n_0}}$ for some positive integers k_0, n_0 and we may also assume $n \geq n_0$.

Sketch of the proof

Let $f(y_1, y_2, y_3) = exp(r(y_1 + y_2))y_3$. Recall ϕ is the function specified in (D.2). To simplify the exposure, we introduce the following notation:

$$y_1(t) := \phi(X(t)), y_2^n(t) := \phi(X^n(t)), y_3^n(t) := |X^n(t) - X(t)|^2.$$

$$f_n(t) := f(y_1(t), y_2^n(t), y_3^n(t)), g_n(t) := exp(ry_1(t) + ry_2^n(t)).$$

Since X^n, X take values in the bounded domain $\bar{D}$, we have

$$c_1|X^n(t) - X(t)|^2 \leq f_n(t) \leq c_2|X^n(t) - X(t)|^2, \tag{10}$$

where c_1, c_2 are positive constants independent of n. Thus the proof of (9) reduces to show

$$\lim_{n\to\infty} E[f_n(t)] = 0. \tag{11}$$

Sketch of the proof

By Ito's formula, we have

$$
\begin{aligned}
& f_n(t) \\
= \; & r \int_0^t f_n(s) < \nabla\phi(X(s)), \sigma(X(s)) dW(s) > \\
& + r \int_0^t f_n(s) < \nabla\phi(X(s)), b(X(s) > ds \\
+ \; & \frac{1}{2} r \int_0^t f_n(s) tr(\phi''(\sigma\sigma^*)(X(s))) ds \\
& + \frac{1}{2} r \int_0^t f_n(s) < \nabla\phi(X(s), \sigma\sigma'(X(s)) > ds \\
+ \; & r \int_0^t f_n(s) < \nabla\phi(X(s)), \nu(X(s)) > d|L|(s) \\
& + r \int_0^t f_n(s) < \nabla\phi(X^n(s)), \sigma(X^n(s)) dW^n(s) >
\end{aligned}
$$

Sketch of the proof

$$
\begin{aligned}
&+ \quad r\int_0^t f_n(s) < \nabla\phi(X^n(s)), b(X^n(s)) > ds \\
&\quad +r\int_0^t f_n(s) < \nabla\phi(X^n(s)), \nu(X^n(s)) > d|L^n|(s) \\
&+ \quad 2\int_0^t g_n(s) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
&- \quad 2\int_0^t g_n(s) < X^n(s) - X(s), \sigma(X(s))dW(s) > \\
&+ \quad 2\int_0^t g_n(s) < X^n(s) - X(s), b(X^n(s)) - b(X(s)) > ds
\end{aligned}
$$

Sketch of the proof

$$
\begin{aligned}
&- \int_0^t g_n(s) < X^n(s) - X(s), \sigma\sigma'(X(s)) > ds \\
&+ 2\int_0^t g_n(s) < X^n(s) - X(s), \nu(X^n(s))d|L^n|(s) - \nu(X(s))d|L|(s) > \\
&+ \int_0^t g_n(s) tr(\sigma\sigma^*(X(s)))ds \\
&\quad + \frac{1}{2}r^2 \int_0^t f_n(s)|\sigma^*\nabla\phi|^2(X(s))ds \\
&- 2r\int_0^t g_n(s) < \sigma^*(X(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X(s)) > ds. \qquad (12)
\end{aligned}
$$

Sketch of the proof

$$
\begin{aligned}
& g_n(t) \\
= \ & exp(2r\phi(x_0)) + r\int_0^t g_n(s) < \nabla\phi(X(s)), \sigma(X(s))dW(s) > \\
+ \ & r\int_0^t g_n(s) < \nabla\phi(X(s)), b(X(s)) > ds \\
+ \ & \frac{1}{2} r\int_0^t g_n(s) tr(\phi''(\sigma\sigma^*)(X(s)))ds \\
& + \frac{1}{2} r\int_0^t g_n(s) < \nabla\phi(X(s), \sigma\sigma'(X(s)) > ds
\end{aligned}
$$

Sketch of the proof

$$
\begin{aligned}
&+ \quad r\int_0^t g_n(s) < \nabla\phi(X(s)), \nu(X(s)) > d|L|(s) \\
&+ \quad r\int_0^t g_n(s) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
&+ \quad r\int_0^t g_n(s) < \nabla\phi(X^n(s)), b(X^n(s)) > ds \\
&\quad\quad +r\int_0^t g_n(s) < \nabla\phi(X^n(s)), \nu(X^n(s)) > d|L^n|(s) \\
&+ \quad \frac{1}{2}r^2\int_0^t g_n(s)|\sigma^*\nabla\phi|^2(X(s))ds \qquad (13)
\end{aligned}
$$

Sketch of the proof

To bound $E[f_n(t)]$, the crucial step is to get proper estimates for the terms

$$rE[\int_0^t f_n(s) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >],$$

and

$$rE[\int_0^t g_n(s) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) >].$$

This will be done in the following two lemmas.

Sketch of proof

lemma[3]
It holds that

$$
\begin{aligned}
& rE[\int_0^t f_n(s) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >] \\
\leq \quad & C(\frac{1}{2^n})^{\frac{1}{2}} + r^2 E[\int_0^t f_n(s) < \sigma^*\nabla\phi(X(s)), \sigma^*\nabla\phi(X^n(s)) > ds] \\
& + \frac{1}{2} r^2 E[\int_0^t f_n(s) |\sigma^*\nabla\phi|^2 (X^n(s)) ds] \\
& + r \int_0^t < g_n(s)\sigma^*(X^n(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds \\
& + \frac{1}{2} r \int_0^t f_n(s) \sum_{i=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_i (X^n(s)) ds \\
& - 2r \int_0^t < g_n(s)\sigma^*(X(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds \qquad (14)
\end{aligned}
$$

Main steps of the proof of Lemma 3

Set

$$A = r\int_0^t f_n(s) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > .$$

Write

$$\begin{aligned} A &= r\int_0^t f_n(s_n^-) < \nabla\phi(X^n(s_n^-)), \sigma(X^n(s_n^-))dW^n(s) > \\ &+ r\int_0^t (f_n(s) - f_n(s_n^-)) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\ &+ r\int_0^t f_n(s_n^-) < \sigma^*\nabla\phi(X^n(s)) - \sigma^*\nabla\phi(X^n(s_n^-)), dW^n(s) > \\ &:= A_1 + A_2 + A_3. \end{aligned} \tag{15}$$

As a stochastic integral, it is easy to see that $E[A_1] = 0$.

Main steps of the proof of Lemma 3

In view of (12), we further write A_2 as

$$
\begin{aligned}
& A_2 \\
= \;& r^2 \int_0^t \Big(\int_{s_n^-}^s f_n(u) \\
& \times < \nabla\phi(X(u)), \sigma(X(u)) dW(u) > \Big) < \nabla\phi(X^n(s)), \sigma(X^n(s)) dW^n(s \\
+ \;& r^2 \int_0^t \Big(\int_{s_n^-}^s f_n(u) < \nabla\phi(X(u)), b(X(u)) du > \Big) \\
& \quad \times < \nabla\phi(X^n(s)), \sigma(X^n(s)) dW^n(s) > \\
+ \;& \frac{1}{2} r^2 \int_0^t \Big(\int_{s_n^-}^s f_n(u) tr(\phi''(\sigma\sigma^*)(X(u)) du \Big) \\
& \quad \times < \nabla\phi(X^n(s)), \sigma(X^n(s)) dW^n(s) >
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{2} r^2 \int_0^t \Big(\int_{s_n^-}^{s} f_n(u) < \nabla\phi(X(u)), (\sigma\sigma')(X(u)) > du \Big) \\
&\qquad \times < \nabla\phi(X^n(s)), \sigma(X^n(s)) dW^n(s) > \\
&+ \quad r^2 \int_0^t \Big(\int_{s_n^-}^{s} f_n(u) < \nabla\phi(X(u)), \nu(X(u)) > d|L|(u) \Big) \\
&\qquad \times < \nabla\phi(X^n(s)), \sigma(X^n(s)) dW^n(s) > \\
&+ \quad \cdots \\
&:= \quad \sum_{i=1}^{17} A_{2i} \qquad (16)
\end{aligned}
$$

We will bound each of the terms. Since $\nabla\phi$, b, σ are bounded on $\bar{D}$, we have

$$\begin{aligned} E[|A_{22}|] &\leq C\int_0^t (s-s_n^-)E[|\dot{W}^n(s)|]ds \\ &\leq C\frac{1}{2^n}\int_0^t (2^n)^{\frac{1}{2}}ds \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \end{aligned} \tag{17}$$

Similarly, it holds that

$$E[|A_{2i}|] \leq C(\frac{1}{2^n})^{\frac{1}{2}}, \quad i=3,4,7,11,12,15,16,17. \tag{18}$$

To bound A_{21}, we write it as

$$\begin{aligned}
& A_{21} \\
= \; & r^2 \int_0^t \Big(\int_{s_n^-}^s [f_n(u) < \nabla\phi(X(u)), \sigma(X(u)dW(u) > -f_n(s_n^-) \times \\
< \; & \nabla\phi(X(s_n^-)), \sigma(X(s_n^-))dW(u) >] \Big) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) \\
+ \; & r^2 \int_0^t \Big(\int_{s_n^-}^s f_n(s_n^-) < \nabla\phi(X(s_n^-)), \sigma(X(s_n^-))dW(u) > \Big) \\
\times \; & [< \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
& \qquad - < \nabla\phi(X^n(s_n^-)), \sigma(X^n(s_n^-))dW^n(s) >] \\
+ \; & r^2 \int_0^t f_n(s_n^-) < \nabla\phi(X(s_n^-)), \sigma(X(s_n^-))(W(s) - W(s_n^-)) > \\
& \qquad \times < \nabla\phi(X^n(s_n^-)), \sigma(X^n(s_n^-))dW^n(s) > \\
:= \; & A_{21,1} + A_{21,2} + A_{21,3}. \qquad (
\end{aligned}$$

Main steps of the proof of Lemma 3

By Ito isometry and Hölder's inequality,

$$
\begin{aligned}
& E[A_{21,1}] \\
\leq\ & C\int_0^t (E[\int_{s_n^-}^s |f_n(u)\sigma^*\nabla\phi(X(u)) - f_n(s_n^-)\sigma^*\nabla\phi(X(s_n^-))|^2 \\
& \times du])^{\frac{1}{2}}(E[|\dot{W}^n|^2(s)])^{\frac{1}{2}} ds \\
\leq\ & C\int_0^t (2^n)^{\frac{1}{2}}(E[\int_{s_n^-}^s |f_n(u)\sigma^*\nabla\phi(X(u)) - f_n(s_n^-) \\
& \times\sigma^*\nabla\phi(X(s_n^-))|^2 du])^{\frac{1}{2}} ds \\
\leq\ & C\int_0^t (2^n)^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}} ds \leq C(\frac{1}{2^n})^{\frac{1}{2}}, \qquad (20)
\end{aligned}
$$

where (7), (8) have been used.

Main steps of the proof of Lemma 3

For the term $A_{21,2}$, we have

$$
\begin{aligned}
& E[A_{21,2}] \\
\leq \ & C\int_0^t (E[|W(s)-W(s_n^-)|^3])^{\frac{1}{3}} \\
& \times (E[|\sigma^*\nabla\phi(X^n(s))-\sigma^*\nabla\phi(X^n(s_n^-))|^3])^{\frac{1}{3}}(E[|\dot{W}^n|^3(s)])^{\frac{1}{3}}\,ds \\
\leq \ & C\int_0^t (2^n)^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}}\,ds \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \qquad (21)
\end{aligned}
$$

where (7) has been used.

Main steps of the proof of Lemma 3

Now,

$$\begin{aligned}
& A_{21,3} \\
= \;& r^2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} f_n(\frac{k-1}{2^n}) < \nabla\phi(X(\frac{k-1}{2^n})), \sigma(X(\frac{k-1}{2^n}))(W(s) - \\
& W(\frac{k}{2^n})) >< \nabla\phi(X^n(\frac{k-1}{2^n})), \sigma(X^n(\frac{k-1}{2^n}))(W(\frac{k}{2^n}) - W(\frac{k-1}{2^n}) \\
+ \;& r^2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} f_n(\frac{k-1}{2^n}) < \sigma^*\nabla\phi(X(\frac{k-1}{2^n})), W(\frac{k}{2^n}) - W(\frac{k-1}{2^n}) \\
& > \times < \sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})), W(\frac{k}{2^n}) - W(\frac{k-1}{2^n}) > ds \\
:= \;& A_{21,31} + A_{21,32}.
\end{aligned}$$

Conditioning on $\mathcal{F}_{\frac{k}{2^n}}$, it is easy to see that $E[A_{21,31}] = 0$.

Moreover,

$$
\begin{aligned}
& A_{21,32} \\
= \;& r^2 \sum_k f_n(\frac{k-1}{2^n}) \sum_{i=1}^m (\sigma^* \nabla \phi)_i (X(\frac{k-1}{2^n}))(\sigma^* \nabla \phi)_i (X^n(\frac{k-1}{2^n})) \\
& \times (|W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n})|^2 - \frac{1}{2^n}) \\
+ \;& r^2 \sum_k f_n(\frac{k-1}{2^n}) \sum_{i \neq j} (\sigma^* \nabla \phi)_i (X(\frac{k-1}{2^n}))(\sigma^* \nabla \phi)_j (X^n(\frac{k-1}{2^n})) \\
& \times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n})) \\
+ \;& r^2 \sum_k f_n(\frac{k-1}{2^n}) \sum_{i=1}^m (\sigma^* \nabla \phi)_i (X(\frac{k-1}{2^n}))(\sigma^* \nabla \phi)_i (X^n(\frac{k-1}{2^n}))(\frac{1}{2^n}) \\
:= \;& A_{21,321} + A_{21,322} + A_{21,323}. \qquad (23)
\end{aligned}
$$

The main steps of proof of Lemma 3

Conditioning on $\mathcal{F}_{\frac{k-1}{2^n}}$ and using the independence of W_i, W_j for $i \neq j$, we find that $E[A_{21,321}] = 0$ and $E[A_{21,322}] = 0$.

The main steps of proof of Lemma 3

On the other hand,

$$
\begin{aligned}
& E[A_{21,323}] \\
= \;& r^2 E[\int_0^t f_n(s) < \sigma^* \nabla\phi(X(s)), \sigma^* \nabla\phi(X^n(s)) > ds] \\
+ \;& r^2 E[\int_0^t \{f_n(s_n^-) < \sigma^* \nabla\phi(X(s_n^-)), \sigma^* \nabla\phi(X^n(s_n^-)) > \\
& -f_n(s) < \sigma^* \nabla\phi(X(s)), \sigma^* \nabla\phi(X^n(s)) >\} ds] \\
\leq \;& r^2 E[\int_0^t f_n(s) < \sigma^* \nabla\phi(X(s)), \sigma^* \nabla\phi(X^n(s)) > ds] \\
& +C(\frac{1}{2^n})^{\frac{1}{2}}, \qquad (24)
\end{aligned}
$$

where (7), (8) again have been used. Putting together (19)—(24) we arrive at

$$E[A_{21}] \leq CE[\int_0^t f_n(s) ds] + C(\frac{1}{2^n})^{\frac{1}{2}}. \qquad (25)$$

The main steps of proof of Lemma 3

The term A_{25} can be bounded as follows.

$$\begin{aligned}
& E[A_{25}] \\
\leq \ & CE[\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k-1}{2^n}}^{s} d|L|(u)) 2^n |W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})| ds] \\
\leq \ & CE[\sum_k (|L|(\frac{k}{2^n}) - |L|(\frac{k-1}{2^n})) |W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})|] \\
\leq \ & 2CE[|L|(t) \sup_{|u-v| \leq \frac{1}{2^n}} (|W(u) - W(v)|)] \\
\leq \ & 2C(E[|L|^2(t)])^{\frac{1}{2}} (\frac{1}{2^n})^{\frac{1}{2}} \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \qquad (26)
\end{aligned}$$

The main steps of the proof of Lemma 3

Collecting the estimates for all the remaining terms we arrive at

$$
\begin{aligned}
& E[A_2] \\
\leq\ & C(\frac{1}{2^n})^{\frac{1}{2}} + r^2 E[\int_0^t f_n(s) < \sigma^*\nabla\phi(X^n(s)), \sigma^*\nabla\phi(X(s)) > ds] \\
& + \frac{1}{2} r^2 E[\int_0^t f_n(s) |\sigma^*\nabla\phi|^2 (X^n(s)) ds \\
& + r \int_0^t < g_n(s)\sigma^*(X^n(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds \\
& - 2r \int_0^t < g_n(s)\sigma^*(X(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds.
\end{aligned}
\tag{27}
$$

The main steps of the proof of Lemma3

Now we turn to A_3. By the chain rule, we have

$$
\begin{aligned}
& A_3 \\
= \ & r\int_0^t f_n(s_n^-)\sum_{i=1}^m [(\sigma^*\nabla\phi)_i(X^n(s)) - (\sigma^*\nabla\phi)_i(X^n(s_n^-))]dW_i^n(s) \\
= \ & r\int_0^t f_n(s_n^-)\sum_{i=1}^m \int_{s_n^-}^s [< \nabla(\sigma^*\nabla\phi)_i(X^n(u)) - \nabla(\sigma^*\nabla\phi)_i(X^n(s_n^-)), \\
& \quad \sigma(X^n(u))dW^n(u) >]dW_i^n(s) \\
+ \ & r\int_0^t f_n(s_n^-)\sum_{i=1}^m \int_{s_n^-}^s < \nabla(\sigma^*\nabla\phi)_i(X^n(s_n^-)), (\sigma(X^n(u)) \\
& \quad -\sigma(X^n(s_n^-)))dW^n(u) > dW_i^n(s)
\end{aligned}
$$

$$+r\int_0^t f_n(s_n^-)\sum_{i=1}^m \int_{s_n^-}^s < \nabla(\sigma^*\nabla\phi)_i(X^n(s_n^-)), \sigma(X^n(s_n^-))dW^n(u) > d$$

$$+r\int_0^t f_n(s_n^-)\sum_{i=1}^m \int_{s_n^-}^s < \nabla(\sigma^*\nabla\phi)_i(X^n(u)), \nu(X^n(u))d|L^n|(u) > dW_i$$

$$+r\int_0^t f_n(s_n^-)\sum_{i=1}^m \int_{s_n^-}^s < \nabla(\sigma^*\nabla\phi)_i(X^n(u)), b(X^n(u))du > dW_i^n(s)$$

$$:= A_{31} + A_{32} + A_{33} + A_{34} + A_{35}$$

Similar to the estimates for A_{214}, A_{22} and the term $A_{21,2}$, it can be shown that

$$E[A_{3i}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}, \quad i = 1, 2, 4, 5. \tag{29}$$

Now,

$$\begin{aligned} & A_{33} \\ = \ & r\sum_k (2^n)^2 \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} f_n(\frac{k-1}{2^n}) \sum_{i=1}^m \sum_{j=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_j (X^n(\frac{k-1}{2^n} \\ & \times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k-1}{2^n} - W_j(\frac{k-2}{2^n}))dsdu \\ + \ & r\sum_k (2^n)^2 \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \int_{\frac{k}{2^n}}^{s} f_n(\frac{k-1}{2^n}) \sum_{i=1}^m \sum_{j=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_j (X^n(\frac{k-1}{2^n}) \\ & \times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n}))dsdu \end{aligned}$$

The main steps of the proof of Lemma3

$$
\begin{aligned}
= \quad & r\sum_k f_n(\frac{k-1}{2^n})\sum_{i=1}^m\sum_{j=1}^m(\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_j(X^n(\frac{k-1}{2^n})) \\
& \times(W_i(\frac{k}{2^n})-W_i(\frac{k-1}{2^n}))(W_j(\frac{k-1}{2^n})-W_j(\frac{k-2}{2^n})) \\
+ \quad & \frac{1}{2}r\sum_k f_n(\frac{k-1}{2^n})\sum_{i=1}^m\sum_{j=1}^m(\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_j(X^n(\frac{k-1}{2^n})) \\
& \times(W_i(\frac{k}{2^n})-W_i(\frac{k-1}{2^n}))(W_j(\frac{k}{2^n})-W_j(\frac{k-1}{2^n})) \\
:= \quad & A_{331}+A_{332} \qquad (30)
\end{aligned}
$$

The main steps of the proof of Lemma 3

Conditioning on $\mathcal{F}_{\frac{k-1}{2^n}}$, it is easy to see $E[A_{331}] = 0$. For the second term we have

$$
\begin{aligned}
& A_{332} \\
= \;& \frac{1}{2} r \sum_k f_n(\frac{k-1}{2^n}) \sum_{i \neq j}^{m} (\sigma^*(\nabla(\sigma^* \nabla \phi)_i))_j (X^n(\frac{k-1}{2^n})) \\
& \times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n})) \\
+ \;& \frac{1}{2} r \sum_k f_n(\frac{k-1}{2^n}) \sum_{i=1}^{m} (\sigma^*(\nabla(\sigma^* \nabla \phi)_i))_i (X^n(\frac{k-1}{2^n})) \\
& \times \{ |W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n})|^2 - \frac{1}{2^n} \}
\end{aligned}
$$

The main steps of the proof of Lemma 3

$$
\begin{aligned}
+ \quad & \frac{1}{2} r \int_0^t \{ f_n(s_n^-) \sum_{i=1}^m (\sigma^*(\nabla(\sigma^* \nabla \phi)_i))_i (X^n(s_n^-)) \\
& - f_n(s) \sum_{i=1}^m (\sigma^*(\nabla(\sigma^* \nabla \phi)_i))_i (X^n(s)) \} ds \\
+ \quad & \frac{1}{2} r \int_0^t f_n(s) \sum_{i=1}^m (\sigma^*(\nabla(\sigma^* \nabla \phi)_i))_i (X^n(s)) ds \\
:= \quad & A_{3321} + A_{3322} + A_{3323} + A_{3324} \qquad (31)
\end{aligned}
$$

The main steps of the proof of Lemma 3

Using the martingale property and the independence of W_i, W_j for $i \neq j$, we find that $E[A_{3321}] = 0$ and $E[A_{3322}] = 0$. In view of (7) and (8), we have $E[A_{3323}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}$. Thus, we deduce from (30), (31) that

$$\begin{aligned} & E[A_{33}] \\ \leq \quad & C(\frac{1}{2^n})^{\frac{1}{2}} + \frac{1}{2} r \int_0^t f_n(s) \sum_{i=1}^m (\sigma^*(\nabla(\sigma^* \nabla \phi)_i))_i (X^n(s)) ds. \end{aligned} \tag{32}$$

Finally it follows from (28), (29), (29) that

$$\begin{aligned} & E[A_3] \\ \leq \quad & C(\frac{1}{2^n})^{\frac{1}{2}} + \frac{1}{2} r \int_0^t f_n(s) \sum_{i=1}^m (\sigma^*(\nabla(\sigma^* \nabla \phi)_i))_i (X^n(s)) ds. \end{aligned} \tag{33}$$

Combining (27) with (33), we complete the proof of Lemma.

The sketch of the proof

lemma[4].We have

$$
\begin{aligned}
& rE[\int_0^t g_n(s) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) >] \\
\leq\ & rE[\int_0^t g_n(s) < \sigma^*\nabla\phi(X^n(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds] \\
+\ & 2rE[\int_0^t g_n(s) < \sigma^*\nabla\phi(X(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds] \\
+\ & E[\int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X^n(s))ds] \\
+\ & E[\int_0^t g_n(s) \sum_{i=1}^d (X_i^n(s) - X_i(s)) \sum_{j=1}^m (\sigma^*\nabla\sigma_{ij})_j(X^n(s))ds] \\
-\ & 2E[\int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}(X(s))\sigma_{ij}(X^n(s))ds] + C(\frac{1}{2^n})^{\frac{1}{2}}. \qquad (34)
\end{aligned}
$$

Proof of the main result: (Continued). Choose $r < -\frac{2C_0}{\alpha}$, where α, C_0 are the constants appeared in the assumptions (D.1) and (D.2). By the Lipschitz continuity of the coefficients and boundedness of ϕ, ϕ'', $\nabla\phi$, $\sigma\sigma'$ on the domain $\bar{D}$, it follows from (12) that

$$
\begin{aligned}
& E[f_n(t)] \\
\leq\; & C_r E[\int_0^t f_n(s)ds] \\
+\; & E[\int_0^t \{< rf_n(s)\nabla\phi(X(s)) - 2g_n(s)(X^n(s) - X(s)), \nu(X(s)) >\}d|L|
\end{aligned}
$$

$$
\begin{aligned}
+\; & rE[\int_0^t f_n(s) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >] \\
+\; & E[\int_0^t \{< rf_n(s)\nabla\phi(X^n(s)) + 2g_n(s)(X^n(s) - X(s)), \\
& \nu(X^n(s)) >\}d|L^n|(s)]
\end{aligned}
$$

The sketch of the proof

$$
\begin{aligned}
&+ \quad 2E[\int_0^t g_n(s) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) >] \\
&- \quad E[\int_0^t g_n(s) < X^n(s) - X(s), \sigma\sigma'(X(s)) > ds] \\
&+ \quad E[\int_0^t g_n(s) tr(\sigma\sigma^*(X(s)))ds] \\
&- \quad 2rE[\int_0^t g_n(s) < \sigma^*(X(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X(s)) > ds].
\end{aligned} \tag{35}
$$

The sketch of the proof

In view of $r < 0$ and the assumptions (D.1) and (D.2), we deduce that

$$\begin{aligned} & < rf_n(s)\nabla\phi(X(s)) - 2g_n(s)(X^n(s) - X(s)), \nu(X(s)) > \\ = \; & g_n(s)[r < \nabla\phi(X(s)), \nu(X(s)) > |x^n(s) - X(s)|^2 \\ & -2 < X^n(s) - X(s), \nu(X(s)) >] \\ \leq \; & g_n(s)[r\alpha|x^n(s) - X(s)|^2 + 2C_0|X^n(s) - X(s)|^2] \leq 0, \quad (36) \end{aligned}$$

and similarly

$$\begin{aligned} & < rf_n(s)\nabla\phi(X^n(s)) + 2g_n(s)(X^n(s) - X(s)), \nu(X^n(s)) > \\ \leq \; & 0. \quad (37) \end{aligned}$$

Thus, using Lemma 3 and Lemma 4, taking into account (36) and (37) we obtain from (35) that

Sketch of the proof

$$
\begin{aligned}
& E[f_n(t)] \\
\leq \;& C_r E[\int_0^t f_n(s)ds] + C(\frac{1}{2^n})^{\frac{1}{2}} \\
- \;& E[\int_0^t g_n(s) < X^n(s) - X(s), \sigma\sigma'(X(s)) > ds] \\
+ \;& E[\int_0^t g_n(s) \sum_{i=1}^{d} \sum_{j=1}^{m} \sigma_{ij}^2(X(s))ds] \\
- \;& 2rE[\int_0^t g_n(s) < \sigma^*(X(s))(X^n(s) - X(s)), \sigma^* \nabla\phi(X(s)) > ds] \\
+ \;& 2r \int_0^t < g_n(s)\sigma^*(X^n(s))(X^n(s) - X(s)), \sigma^* \nabla\phi(X^n(s)) > ds \\
- \;& 2r \int_0^t < g_n(s)\sigma^*(X(s))(X^n(s) - X(s)), \sigma^* \nabla\phi(X^n(s)) > ds
\end{aligned}
$$

Sketch of the proof

$$
\begin{aligned}
+ \quad & 2rE[\int_0^t g_n(s) < \sigma^* \nabla\phi(X(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds] \\
+ \quad & E[\int_0^t g_n(s) \sum_{i=1}^{d} \sum_{j=1}^{m} \sigma_{ij}^2(X^n(s))ds] \\
+ \quad & E[\int_0^t g_n(s) < X^n(s) - X(s), \sigma\sigma'(X^n(s)) > ds] \\
-2 \quad & E[\int_0^t g_n(s) \sum_{i=1}^{d} \sum_{j=1}^{m} \sigma_{ij}(X(s))\sigma_{ij}(X^n(s))ds]
\end{aligned}
$$

Sketch of the proof

$$
\begin{aligned}
&\leq && C_r E[\int_0^t f_n(s)ds] + C(\frac{1}{2^n})^{\frac{1}{2}} \\
&+ && E[\int_0^t g_n(s) \sum_{i=1}^{d} \sum_{j=1}^{m} (\sigma_{ij}(X(s)) - \sigma_{ij}(X^n(s)))^2 ds] \\
&+ && 2r \int_0^t < g_n(s)(\sigma^*(X^n(s)) - \sigma^*(X(s)))(X^n(s) - X(s)), \\
& && \qquad \sigma^* \nabla \phi(X^n(s)) > ds \\
&+ && 2rE[\int_0^t g_n(s) < \sigma^* \nabla \phi(X(s)), \\
& && \qquad (\sigma^*(X^n(s)) - \sigma^*(X(s)))(X^n(s) - X(s)) > ds] \\
&+ && E[\int_0^t g_n(s) < X^n(s) - X(s), \sigma\sigma'(X^n(s)) - \sigma\sigma'(X(s)) > ds] \\
&\leq && CE[\int_0^t f_n(s)ds] + C(\frac{1}{2^n})^{\frac{1}{2}}, \qquad (38)
\end{aligned}
$$

Sketch of the proof

where the Lipschitz continuity of the coefficients and the fact that $f_n(s) = g_n(s)|X^n(s) - X(s)|^2$ have been used. Finally by the Gronwall's inequality, we obtain

$$E[f_n(t)] \leq C(\frac{1}{2^n})^{\frac{1}{2}} \to 0 \tag{39}$$

as $n \to \infty$, completing the proof of (11), hence the theorem.

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