Exponential Mixing of SFDEs

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Outline

- Exponential Ergodicity for Retarded SDEs
- Exponential Ergodicity fo Neutral SDEs
- Exponential Ergodicity for Retarded SDEs with Jumps

We start with some notation. For each integer $n \geq 1$, let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle, |\cdot|)$ be the n-dimensional Euclidean space and $\mathbb{R}^n \otimes \mathbb{R}^m$ denote the totality of all $n \times m$ matrices. For a fixed constant $\tau > 0$, $\mathscr{C} := C([-\tau, 0]; \mathbb{R}^n)$ stands for the family of all continuous mappings $\zeta : [-\tau, 0] \mapsto \mathbb{R}^n$ equipped with the uniform norm $\|\zeta\|_\infty := \sup_{-\tau \leq \theta \leq 0} |\zeta(\theta)|$. For any continuous function $f : [-\tau, \infty) \mapsto \mathbb{R}^n$ and $t \geq 0$, let $f_t \in \mathscr{C}$ be such that $f_t(\theta) = f(t + \theta)$ for each $\theta \in [-\tau, 0]$. Let $W(t)$ be an m-dimensional Wiener process defined on a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $\mathcal{P}(\mathscr{C})$ denote the collection of all probability measures on $(\mathscr{C}, \mathscr{B}(\mathscr{C}))$, $\mathscr{B}_b(\mathscr{C})$ means the set of all bounded measurable functions $F : \mathscr{C} \to \mathbb{R}$ endowed with the uniform norm $\|F\|_0 := \sup_{\phi \in \mathscr{C}} |F(\phi)|$, and $\mu(\cdot)$ stands for a probability measure on $[-\tau, 0]$. For any $F \in \mathscr{B}_b(\mathscr{C})$ and $\pi(\cdot) \in \mathcal{P}(\mathscr{C})$, let $\pi(F) := \int_{\mathscr{C}} F(\phi)\pi(\mathrm{d}\phi)$.

We consider a retarded SDE on $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, |\cdot|)$ in the framework

$$\mathrm{d}X(t) = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W(t), \quad t > 0 \tag{1}$$

with the initial data $X_0 = \xi \in \mathscr{C}$, where $b : [0,\infty) \times \mathscr{C} \mapsto \mathbb{R}^n$ and $\sigma : [0,\infty) \times \mathscr{C} \mapsto \mathbb{R}^n \otimes \mathbb{R}^m$ are measurable and locally Lipschitz with respect to the second variable. We assume that the initial value $\xi \in \mathscr{C}$ is independent of $\{W(t)\}_{t\geq 0}$.

For any $\phi, \psi \in \mathscr{C}$ and $t, p \geq 0$, we assume that

(H1) There exist $\alpha_1 > \alpha_2 > 0$ such that

$$\mathbb{E}\{|\phi(0)-\psi(0)|^p(2\langle\phi(0)-\psi(0), b(t,\phi)-b(t,\psi)\rangle + \|\sigma(t,\phi)-\sigma(t,\psi)\|^2$$
$$\leq -\alpha_1\mathbb{E}|\phi(0)-\psi(0)|^{2+p} + \alpha_2 \sup_{-\tau\leq\theta\leq 0}\mathbb{E}\{|\phi(0)-\psi(0)|^p|\phi(\theta)-\psi(\theta)|^2\}$$

(H2) There exists $\alpha_3 > 0$ such that

$$\mathbb{E}\|\sigma(t,\phi)-\sigma(t,\psi)\|^{2+p} \leq \alpha_3 \sup_{-\tau\leq\theta\leq 0}\mathbb{E}(|\phi(\theta)-\psi(\theta)|^{2+p}).$$

The following remark shows that there are some examples such that (H1) and (H2).

Let $b(t,\phi) = b(t,\phi(0),\phi(-\delta(t)))$ and $\sigma(t,\phi) = \sigma(t,\phi(0),\phi(-\delta(t)))$ with $\phi \in \mathscr{C}$, where $\delta : [0,\infty) \mapsto [0,\tau]$ is a measurable function. For any $\phi \in \mathscr{C}$ and $t \geq 0$, if

$$\begin{aligned} &2\langle \phi(0)-\psi(0), b(t,\phi(0),\phi(-\delta(t))) - b(t,\psi(0),\psi(-\delta(t)))\rangle \\ &\qquad + \|\sigma(t,\phi(0),\phi(-\delta(t))) - \sigma(t,\psi(0),\psi(-\delta(t)))\|^2 \\ &\leq -\alpha_1|\phi(0)-\psi(0)|^2 + \alpha_2|\phi(-\delta(t)) - \psi(-\delta(t))|^2, \end{aligned}$$

and

$$\|\sigma(t,\phi)-\sigma(t,\psi)\|^2 \leq \alpha_3(|\phi(0)-\psi(0)|^2 + |\phi(-\delta(t))-\psi(-\delta(t))|^2),$$

then (H1) and (H2) hold respectively for some constants $\alpha_1, \alpha_2, \alpha_3 > 0$.

On the other hand, for arbitrary $\phi \in \mathscr{C}$ and $t \geq 0$, if

$$2\langle \phi(0) - \psi(0), b(t,\phi) - b(t,\psi)\rangle + \|\sigma(t,\phi) - \sigma(t,\psi)\|^2 \\ \leq -\alpha_1|\phi(0) - \psi(0)|^2 + \alpha_2 \int_{-\tau}^{0} |\phi(\theta) - \psi(\theta)|^2 \mu(\mathrm{d}\theta),$$

and

$$\|\sigma(t,\phi) - \sigma(t,\psi)\|^2 \leq \alpha_3\Big(|\phi(0) - \psi(0)|^2 + \int_{-\tau}^{0} |\phi(\theta) - \psi(\theta)|^2 \mu(\mathrm{d}\theta)\Big),$$

where $\mu(\cdot)$ is a probability measure on $[-\tau, 0]$, then (H1) and (H2) are also fulfilled for some $\alpha_1, \alpha_2, \alpha_3 > 0$.

From the previous discussions, we deduce that our framework cover SDEs with constant/variable/distributed delays

Lemma

Let $u, v : [0, \infty) \mapsto \mathbb{R}_+$ be continuous functions and $\beta > 0$. If

$$u(t) \leq u(s) - \beta \int_s^t u(r)\mathrm{d}r + \int_s^t v(r)\mathrm{d}r, \quad 0 \leq s < t < \infty,$$

then

$$u(t) \leq u(0) + \int_0^t \epsilon^{-\beta(t-r)} v(r)\mathrm{d}r.$$

Lemma

Let $u : [0, \infty) \mapsto \mathbb{R}_+$ be a continuous function and $\delta > 0, \alpha > \beta > 0$. If

$$u(t) \leq \delta + \beta \int_0^t \epsilon^{-\alpha(t-s)} u(s) \mathrm{d}s, \quad t \geq 0,$$

then $u(t) \leq (\delta\alpha)/(\alpha - \beta)$.

Lemma

For $a, b > 0$, let $u(\cdot)$ be a nonnegative function such that

$$u'(t) \leq -au(t) + b \sup_{t-\tau \leq s \leq t} u(s), \quad t > 0$$

Then, for $a > b > 0$, there exists $\lambda > 0$ such that

$$u(t) \leq \Big(\sup_{-\tau \leq s \leq 0} u(s) \Big) \epsilon^{-\lambda t}, \quad t \geq 0.$$

Lemma

Assume that (H1) and (H2) hold. Then there exists a sufficiently small $\kappa > 0$ such that

$$\sup_{t \geq -\tau} \mathbb{E}\|X_t(\xi)\|_\infty^{2+\kappa} < \infty. \quad (2)$$

Sketch of Proof

For any $\kappa > 0$, by the Itô formula, we obtain that

$$\begin{aligned}\rho(t) &:= \mathbb{E}|X(t)|^{2+\kappa}\\ &\le \frac{2+\kappa}{2}\mathbb{E}\int_0^t |X(s)|^\kappa\{2\langle X(s), b(s,X_s)\rangle + \|\sigma(s,X_s)\|^2\}\mathrm{d}s\\ &\quad + |\xi(0)|^{2+\kappa} + \frac{\kappa(2+\kappa)}{2}\mathbb{E}\int_0^t |X(s)|^\kappa \cdot \|\sigma(s,X_s)\|^2\mathrm{d}s\\ &=: I_1(t) + I_2(t).\end{aligned} \tag{3}$$

By (H1) and (H2), it is readily to see that there exist $\nu_1 > \nu_2 > 0$ such that

$$\mathbb{E}\{|\phi(0)|^\kappa(2\langle\phi(0), b(t,\phi)\rangle + \|\sigma(t,\phi)\|^2)\} \leq -\nu_1\mathbb{E}|\phi(0)|^{2+\kappa} + \nu_2 \sup_{-\tau\leq\theta\leq 0} \mathbb{E}(|\phi(0)|^\kappa \cdot |\phi(\theta)|^2) + c \tag{4}$$

for any $t \geq 0$ and $\phi \in \mathscr{C}$. This, together with the Young inequality:

$$a^\beta b^{1-\beta} \leq \beta a + (1-\beta)b, \quad a, b > 0, \beta \in (0,1), \tag{5}$$

gives that

$$I_1(t) \leq \frac{2+\kappa}{2}\int_0^t \{-\nu_1\rho(s) + \nu_2 \sup_{-\tau\leq\theta\leq 0} \mathbb{E}(|X(s)|^\kappa \cdot |X(s+\theta)|^2) + c\}\mathrm{d}s$$
$$\leq -\frac{(2+\kappa)\nu_1}{2}\int_0^t \rho(s)\mathrm{d}s + \frac{(2+\kappa)\nu_2}{2}\int_0^t \Big\{\frac{\kappa}{2+\kappa}\rho(s) + \frac{2}{2+\kappa}\sup_{-\tau\leq\theta\leq s}\rho(r) + c\Big\}\mathrm{d}s$$
$$\leq -\frac{(2+\kappa)}{2}\Big(\nu_1 - \frac{\nu_2\kappa}{2+\kappa} - \kappa\Big)\int_0^t \rho(s)\mathrm{d}s + \int_0^t \{c + \nu_2 r(s)\}\mathrm{d}s,$$

where $r(t) := \sup_{0\leq s\leq t}\rho(s)$.

Hence, we arrive at

$$\rho(t) \leq \|\xi\|_\infty^{2+\kappa} - \lambda_1 \int_0^t \rho(s)\mathrm{d}s + \int_0^t \{c + \lambda_2 r(s)\}\mathrm{d}s, \tag{6}$$

where, for a sufficiently small $\kappa \in (0,1)$,

$$\lambda_1 := \frac{(2+\kappa)}{2}\Big(\nu_1 - \frac{\nu_2\kappa}{2+\kappa} - (c+1)\kappa\Big) > \lambda_2 := \nu_2 + \frac{c\kappa(2+\kappa)}{2}$$

due to $\nu_1 > \nu_2$. Combining (6) with Lemma gives that

$$\rho(t) \leq \|\xi\|_\infty^{2+\kappa} + \int_0^t \epsilon^{-\lambda_1(t-s)}\{c + \lambda_2 r(s)\}\mathrm{d}s. \tag{7}$$

We therefore infer from (7) that

$$r(t) \leq \|\xi\|_\infty^{2+\kappa} + \int_0^t \epsilon^{-\lambda_1(t-s)}\{c + \lambda_2 r(s)\}\mathrm{d}s \leq c + \lambda_2 \int_0^t \epsilon^{-\lambda_1(t-s)} r(s)\mathrm{d}s.$$

Thanks to $\lambda_1 > \lambda_2$, Lemma leads to $\sup_{t\geq -\tau}\rho(t) < \infty$.

Next, for any $t \geq \tau$, applying the Itô formula, together with the Burkhold-Davis-Gundy inequality and the Young inequality (5), we deduce from (4) that

$$\begin{aligned} \mathbb{E}\|X_t\|_\infty^{2+\kappa} &\leq \rho(t-\tau) + c\int_{t-\tau}^{t}\{1+\rho(s)+r(s)\}\mathrm{d}s \\ &\quad + (2+\kappa)\mathbb{E}\Big(\sup_{-\tau\leq\theta\leq 0}\Big|\int_{t-\tau}^{t+\theta}|X(s)|^{\kappa}\langle X(s),\sigma(s,X_s)\mathrm{d}W(s)\rangle\Big|\Big) \\ &\leq \frac{1}{2}\mathbb{E}\|X_t\|_\infty^{2+\kappa} + \rho(t-\tau) + c\int_{t-\tau}^{t}\{1+\rho(s)+r(s)\}\mathrm{d}s. \end{aligned}$$

That is,

$$\mathbb{E}\|X_t\|_\infty^{2+\kappa} \leq 2\rho(t-\tau) + c\int_{t-\tau}^{t}\{1+\rho(s)+r(s)\}\mathrm{d}s, \quad t \geq \tau. \qquad (8)$$

Definition

A probability measure $\pi(\cdot) \in \mathcal{P}(\mathscr{C})$ is called an invariant measure of (1) if, for arbitrary $F \in \mathscr{B}_b(\mathscr{C})$,

$$\pi(P_t F) = \pi(F), \quad t \geq 0,$$

where $P_t F(\xi) := \mathbb{E} F(X_t(\xi))$.

Theorem

Under (H1) and (H2), (1) has a unique invariant measure $\pi(\cdot) \in \mathcal{P}(\mathscr{C})$ and is exponentially mixing. More precisely, there exists $\lambda > 0$ such that

$$|P_t F(\xi) - \pi(F)| \leq c\epsilon^{-\lambda t}, \quad t \geq 0, \; F \in \mathscr{B}_b(\mathscr{C}), \; \xi \in \mathscr{C}. \tag{9}$$

Sketch of Proof

Step 1: Existence of an Invariant Measure. The proof on existence of an invariant measure is due to the classical Arzelà–Ascoli tightness characterization of the space $\mathscr{C}$. For arbitrary integer $n \geq 1$, set

$$\mu_n(\cdot) := \frac{1}{n} \int_0^n \mathbb{P}_t(\xi, \cdot) \mathrm{d}t,$$

where $\mathbb{P}_t(\xi, \cdot)$ is the Markovian transition kernel of $X_t(\xi)$. By the Krylov-Bogoliubov theorem, to show existence of an invariant measure, it is sufficient to verify that $\{\mu_n(\cdot)\}_{n\geq 1}$ is relatively compact. Note that the phase space $\mathscr{C}$ for the segment process $X_t(\xi)$ is a complete separable space under the uniform metric $\|\cdot\|_\infty$.

We only need to show that $\{\mu_n(\cdot)\}_{n\geq 1}$ is tight. It suffices to claim that

$$\lim_{\delta\downarrow 0}\sup_{n\geq 1}\mu_n(\varphi\in\mathscr{C}:w_{[-\tau,0]}(\varphi,\delta)\geq\varepsilon)=0 \tag{10}$$

for any $\varepsilon>0$, where $w_{[-\tau,0]}(\varphi,\delta)$, the modulus of continuity of $\varphi\in\mathscr{C}$, is defined by

$$w_{[-\tau,0]}(\varphi,\delta):=\sup_{|s-t|\leq\delta,s,t\in[-\tau,0]}|\varphi(s)-\varphi(t)|,\quad \delta>0.$$

$$
\begin{aligned}
I(t,\delta) :&= \sup_{t\le v\le u\le t+\tau, 0\le u-v\le\delta} |X(u)-X(v)| \\
&\le \sup_{t\le v\le u\le t+\tau, 0\le u-v\le\delta} \int_v^u |b(s,X_s)|\mathrm{d}s \\
&+ \sup_{t\le v\le u\le t+\tau, 0\le u-v\le\delta} \Big| \int_v^u \sigma(s,X_s)\mathrm{d}W(s) \Big| \\
&=: I_1(t,\delta) + I_2(t,\delta), \quad t\ge\tau,
\end{aligned}
$$

one has

$$
\mathbb{P}(I(t,\delta)\ge\varepsilon) \le \mathbb{P}(I_1(t,\delta)\ge\varepsilon/2) + \mathbb{P}(I_2(t,\delta)\ge\varepsilon/2).
$$

For any $\widetilde{\varepsilon} \in (0,1)$, by the Chebyshev inequality and Lemma 4, there exists an $R_0 > 0$ sufficiently large such that

$$\begin{aligned} &\mathbb{P}(\|X_t\|_\infty > R_0) + \mathbb{P}(\|X_{t+\tau}\|_\infty > R_0) \\ &\leq R_0^{-2} \sup_{t \geq -\tau} (\mathbb{E}\|X_{t+\tau}\|_\infty^2 + \mathbb{E}\|X_t\|_\infty^2) \leq \widetilde{\varepsilon}. \end{aligned} \tag{11}$$

Moreover, since b enjoys locally bounded property, there exists a sufficiently small $\delta_0 > 0$ such that

$$\mathbb{P}(I_1(t,\delta) \geq \varepsilon/2| \quad \|X_t\|_\infty \leq R_0, \|X_{t+\tau}\|_\infty \leq R_0) = 0, \quad \delta < \delta_0. \tag{12}$$

Accordingly, we obtain from (11) and (27) that

$$\begin{aligned} \mathbb{P}(I_1(t,\delta) \geq \varepsilon/2) &\leq \mathbb{P}(I_1(t,\delta) \geq \varepsilon/2| \quad \|X_t\|_\infty \leq R_0, \|X_{t+\tau}\|_\infty \leq R_0) \\ &\quad + \mathbb{P}(\|X_t\|_\infty \geq R_0) + \mathbb{P}(\|X_{t+\tau}\|_\infty \geq R_0) \\ &\leq \widetilde{\varepsilon}. \end{aligned} \tag{13}$$

On the other hand, for $\kappa \in (0,1)$ arbitrary $0 \le s \le t$, by the Burkhold-Davis-Gundy inequality, (H2), it follows that

$$\begin{aligned}\mathbb{E}\Big|\int_s^t \sigma(r, X_r)\mathrm{d}W(r)\Big|^{2+k} &\le c(t-s)^{\kappa/2}\int_s^t \{1+\mathbb{E}\|X_r\|_\infty^{2+\kappa}\}\mathrm{d}r \\ &\le c(t-s)^{1+\kappa/2}.\end{aligned}$$

This, combining with the Kolmogrov tightness criterion, implies that

$$\lim_{\delta\downarrow 0}\sup_{t\ge\tau}\mathbb{P}(I_2(t,\delta)\ge\varepsilon/2)=0. \tag{14}$$

Consequently, (10) follows from (13), (14), the arbitrariness of $\widetilde{\varepsilon}$, and by noticing that

$$\mu_n(\varphi\in\mathscr{C}: w_{[-\tau,0]}(\varphi,\delta)\ge\varepsilon)\le\frac{2\tau}{n}+\frac{1}{n}\int_\tau^n \mathbb{P}(I(t,\delta)\ge\varepsilon)\mathrm{d}t$$

for $n > \tau$.

Step 2: Uniqueness of Invariant Measures.

By the Itô formula, it is easy to see that

$$\begin{aligned} u(t) &:= \mathbb{E}|X(t,\xi) - X(t,\eta)|^2 \\ &= |\xi(0) - \eta(0)|^2 + \int_0^t \mathbb{E}\{2\langle X(s,\xi) - X(s,\eta), b(s,X_s(\xi)) - b(s,X_s(\eta))\rangle \\ &+ \|\sigma(s,X_s(\xi)) - \sigma(s,X_s(\eta))\|^2\}\mathrm{d}s. \end{aligned} \quad (15)$$

Differentiating with respect to t on both sides of (15), one has from (H1) with $p = 0$ that

$$u'(t) \leq -\alpha_1 u(t) + \alpha_2 \sup_{t-\tau \leq s \leq t} |u(s)|.$$

Then

$$\mathbb{E}|X(t,\xi) - X(t,\eta)|^2 \leq \|\xi - \eta\|_\infty^2 \epsilon^{-\lambda t}, \quad t \geq 0 \quad (16)$$

Exponential Mixing for Neutral SDEs

Consider a neutral SDE on $\mathbb{R}^n$

$$\mathrm{d}\{X(t) - G(X_t)\} = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W(t) \tag{17}$$

with the initial value $X_0 = \xi \in \mathscr{C}$ which is independent of $\{W(t)\}_{t\geq 0}$, where $G : \mathscr{C} \mapsto \mathbb{R}^n$ is measurable and continuous such that $G(0) = 0$, and $b : \mathscr{C} \mapsto \mathbb{R}^n, \sigma : \mathscr{C} \mapsto \mathbb{R}^n \otimes \mathbb{R}^m$ are measurable and locally Lipschitz. For any $\phi, \psi \in \mathscr{C}$, we assume that

(A1) There exists $\kappa \in (0, 1)$ such that

$$\mathbb{E}|G(\phi) - G(\psi)| \leq \kappa \sup_{-\tau\leq\theta\leq 0} \mathbb{E}|\phi(\theta) - \psi(\theta)|^2$$

.

(A2) There exist $\alpha_1 > \alpha_2 > 0$ such that

$$\mathbb{E}\{2\langle\phi(0)-\psi(0)-(G(\phi)-G(\psi)), b(\phi)-b(\psi)\rangle + \|\sigma(\phi)-\sigma_2(\psi)\|^2\}$$
$$\leq -\alpha_1\mathbb{E}|\phi(0)-\psi(0)|^2 + \alpha_2 \sup_{-\tau\leq\theta\leq 0} \mathbb{E}|\phi(\theta)-\psi(\theta)|^2.$$

(A3) There exists $\alpha_3 > 0$ such that

$$\mathbb{E}\|\sigma(\phi)-\sigma(\psi)\|^2 \leq \alpha_3 \sup_{-\tau\leq\theta\leq 0} \mathbb{E}|\phi(\theta)-\psi(\theta)|^2.$$

Under (A1)-(A2), (17) has a unique strong solution $\{X(t,\xi)\}_{t\geq 0}$ with the initial data $\xi \in \mathscr{C}$.

Lemma

Let (A1) hold and assume further that there exist $\delta \geq 0, \lambda > 0$ such that

$$\mathbb{E}\{2\langle\phi(0) - G(\phi), b(\phi)\rangle + \|\sigma(\phi)\|^2\} \leq \delta - \lambda\mathbb{E}|\phi(0) - G(\phi)|^2 \tag{18}$$

provided that, for some $q > (1-\kappa)^{-2}$,

$$\mathbb{E}|\phi(\theta)|^2 < q|\phi(0) - G(\phi)|^2, \quad -\tau \leq \theta \leq 0. \tag{19}$$

Then there exists $\gamma < \lambda$ sufficiently small such that

$$\mathbb{E}|X(t)|^2 \leq \frac{\delta/\lambda + \epsilon^{-\gamma t}(1+\kappa)^2\|\xi\|_\infty^2}{(1-\kappa\epsilon^{\gamma\tau/2})^2}, \quad t \geq -\tau. \tag{20}$$

Our main result in this section is presented as below.

Theorem

Let (A1)-(A3) hold and $\kappa \in (0, 1/2)$ and $\alpha_1 > \alpha_2/(1-2\kappa)^2$. Assume further that

$$|G(\phi) - G(\psi)| \le \kappa \|\phi - \psi\|_\infty, \quad \phi, \psi \in \mathscr{C}. \tag{21}$$

Then, (17) has a unique invariant measure $\pi(\cdot) \in \mathcal{P}(\mathscr{C})$ and is exponentially mixing. That is, there exists $\lambda > 0$ such that

$$|P_t F(\xi) - \pi(F)| \le c\epsilon^{-\lambda t}, \quad t \ge 0,\ F \in \mathscr{B}_b(\mathscr{C}),\ \xi \in \mathscr{C}.$$

Exponential Mixing for Retarded SDEs with Jumps

Consider a non-autonomous retarded SDE with jump

$$\mathrm{d}X(t) = b(X_t)\mathrm{d}t + \int_\Gamma \sigma(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z), \quad t \geq 0 \tag{22}$$

with the initial value $\xi \in \mathscr{D}$ which is independent of $N(\cdot,\cdot)$, where $X_{t-}(\theta) := X((t+\theta)-) := \lim_{s\uparrow t+\theta} X(s)$ for $\theta \in [-\tau, 0]$, $b : \mathscr{D} \mapsto \mathbb{R}^n$ and $\sigma\mathscr{D} \mapsto \mathbb{R}^n \times \Gamma \mapsto \mathbb{R}^n$ are progressively measurable.

For any $\phi, \psi \in \mathscr{D}$ and any $t \geq 0$, we assume that

(B1) There exist $\alpha_1 > \alpha_2 > 0$ such that

$$\mathbb{E}\Big\{2\langle \phi(0) - \psi(0), b(\phi) - b(\psi)\rangle + \int_\Gamma |\sigma(\phi, z) - \sigma(\psi, z)|^2 m(\mathrm{d}z)\Big\}$$
$$\leq -\alpha_1 \mathbb{E}|\phi(0) - \psi(0)|^2 + \alpha_2 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\phi(\theta) - \psi(\theta)|^2;$$

(B2) There exists $\alpha_3 > 0$ such that

$$\mathbb{E}|b(\phi) - b(\psi)|^2 + \mathbb{E}\int_\Gamma |\sigma(\phi, z) - \sigma(\psi, z)|^2 m(\mathrm{d}z)$$
$$\leq \alpha_3 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\phi(\theta) - \psi(\theta)|^2.$$

The main result in this section is stated as follows.

Theorem

Under (B1)-(B2), (22) has a unique invariant measure $\pi(\cdot) \in \mathcal{P}(\mathscr{D})$ and is exponentially mixing. More precisely, there exists $\lambda > 0$ such that

$$|P_t F(\xi) - \pi(F)| \leq c\epsilon^{-\lambda t}, \quad t \geq \tau, \ F \in \mathscr{B}_b(\mathscr{D}), \ \xi \in \mathscr{D}.$$

Step 1: Claim a uniform bound of X_t:

$$\sup_{t\geq -\tau} \mathbb{E}\|X_t\|_\infty^2 < \infty. \tag{23}$$

We can derive that $\delta := \sup_{t\geq -\tau} \mathbb{E}|X(t)|^2 < \infty$. By the Itô formula, for any $t \geq \tau$ and $\theta \in [-\tau, 0]$, it follows that

$$\begin{aligned} |X(t+\theta)|^2 &= |X(t-\tau)|^2 + 2\int_{t-\tau}^{t+\theta} \langle X(s), b(s, X_s)\rangle \mathrm{d}s \\ &\quad + \int_{t-\tau}^{t+\theta}\int_\Gamma |\sigma(s, X_{s-}, z)|^2 N(\mathrm{d}s, \mathrm{d}z) + 2\Pi(t, t+\theta), \end{aligned} \tag{24}$$

where

$$\Pi(t, t+\theta) := \int_{t-\tau}^{t+\theta}\int_\Gamma \langle X(s-), \sigma(s, X_{s-}, z)\rangle \tilde{N}(\mathrm{d}s, \mathrm{d}z).$$

Next, due to the Burkhold-Davis-Gundy inequality, and the Jensen inequality, we derive that

$$\begin{aligned} \mathbb{E}\Big(\sup_{-\tau\le\theta\le 0}|\Pi(t,t+\theta)|\Big) &\le c\mathbb{E}\sqrt{[\Pi,\Pi]_{[t-\tau,t]}} \\ &\le c\mathbb{E}\sqrt{\int_{t-\tau}^{t}\int_{\Gamma}|\langle X(s-),\sigma(s,X_{s-},z)\rangle|^2 N(\mathrm{d}s,\mathrm{d}z)} \\ &\le c\sqrt{\mathbb{E}\|X_t\|_\infty^2\mathbb{E}\int_{t-\tau}^{t}\int_{\Gamma}|\sigma(s,X_{s-},z)|^2 N(\mathrm{d}s,\mathrm{d}z)} \\ &\le \frac{1}{4}\mathbb{E}\|X_t\|_\infty^2 + c\mathbb{E}\int_{t-\tau}^{t}\int_{\Gamma}|\sigma(s,X_s,z)|^2 m(\mathrm{d}z)\mathrm{d}s, \end{aligned} \tag{25}$$

where $[\Pi,\Pi]_{[t-\tau,t]}$ stands for the quadratic variation process (square bracket process) of $\Pi(t,t-\tau)$. Then the result follows from (24)

Step 2:

Existence of an invariant measure. For $\theta \in [-\tau, 0]$ and $\widetilde{\theta} \in [0, \triangle]$, where $\triangle > 0$ is an arbitrary constant such that $\theta + \triangle \in [-\tau, 0]$. Set $\mathbb{E}_s \cdot := \mathbb{E}(\cdot|\mathscr{F}_s), s \geq 0$. By the Itô isometry, for any $t \geq \tau$, we obtain from (22) that

$$\mathbb{E}_{t+\theta}|X_t(\theta + \widetilde{\theta}) - X_t(\theta)|^2 = \mathbb{E}_{t+\theta}|X(t + \theta + \widetilde{\theta}) - X(t + \theta)|^2$$
$$\leq c \int_{t+\theta}^{t+\theta+\triangle} \mathbb{E}_{t+\theta}\Big\{|b(s, X_s)|^2 + \int_\Gamma |\sigma(s, X_{s-}, z)|^2 m(\mathrm{d}z)\Big\}\mathrm{d}s.$$

By virtue of (B1)-(B2) and (23), there is a $\gamma_0(t, \triangle)$ satisfying

$$\mathbb{E}_{t+\theta}|X(t + \theta + \widetilde{\theta}) - X(t + \theta)|^2 \leq \mathbb{E}_{t+\theta}\gamma_0(t, \triangle).$$

By virtue of (B1)-(B2) and (23), there is a $\gamma_0(t, \triangle)$ satisfying

$$\mathbb{E}_{t+\theta}|X(t+\theta+\widetilde{\theta}) - X(t+\theta)|^2 \leq \mathbb{E}_{t+\theta}\gamma_0(t, \triangle).$$

Taking expectation and $\limsup_{t\to\infty}$ followed by $\lim_{\triangle\to 0}$, we obtain from (B1)-(B2) and (23) that

$$\lim_{\triangle\to 0} \limsup_{t\to\infty} \mathbb{E}\gamma_0(t, \triangle) = 0. \tag{26}$$

Therefore, in view of (23) and (26), combining with " Kushner, H. J., Approximation and Weak Convergence Methods for Random Processes, with Applications to Stochastic Systems Theory, MIT Press, Cambridge, MA, 1984.", we conclude that X_t is tight under the Skorohod metric d_S.

frametitleThe Remote Start Method

(H1′) There exist $\nu_1 > \nu_2 > 0$, $\nu_3 > 0$ and a probability measure $\mu(\cdot)$ on $[-\tau, 0]$ such that

$$2\langle \varphi(0) - \phi(0), b(\varphi) - b(\phi) \rangle + \int_\Gamma |\sigma(\varphi, z) - \sigma(\phi, z)|^2 m(\mathrm{d}z)$$
$$\leq -\nu_1 |\varphi(0) - \phi(0)|^2 + \nu_2 \int_{-\tau}^0 |\varphi(\theta) - \phi(\theta)|^2 \mu(\mathrm{d}\theta)$$

and

$$|b(\varphi) - b(\phi)|^2 + \int_\Gamma |\sigma(\varphi, z) - \sigma(\phi, z)|^2 m(\mathrm{d}z)$$
$$\leq \nu_3 \Big(|\varphi(0) - \phi(0)|^2 + \int_{-\tau}^0 |\varphi(\theta) - \phi(\theta)|^2 \mu(\mathrm{d}\theta) \Big).$$

Theorem

Under (H1′), (22) has a unique ergodic invariant measure.

Sketch of Proof

We adopt the remote start method, e.g., Da Prato and Zabczyk (1996). Let $N_1(\cdot,\cdot)$ be an independent copy of $N(\cdot,\cdot)$ and $N_0(\cdot,\cdot)$ a doubled-sided Poission process defined by

$$N_0(t,\Gamma) := \begin{cases} N(t,\Gamma), & t \geq 0 \\ N_1(-t,\Gamma), & t < 0, \end{cases}$$

for all $\Gamma \in \mathscr{B}(\mathbb{Z})$, with filtration

$$\bar{\mathscr{F}}_t := \bigcap_{s>t} \bar{\mathscr{F}}_s^0,$$

where $\bar{\mathscr{F}}_s^0 := \sigma(\{N_0([r_1,r_2],\Gamma) : -\infty < r_1 \leq r_2 \leq s, \Gamma\}, \mathscr{N})$ and $\mathscr{N} := \{A \in \mathscr{F} | \mathbb{P}(A) = 0\}$.

For arbitrary $t \in \mathbb{R}$, $s \in (-\infty, t]$ and $\xi \in \mathscr{D}$, consider functional SDE

$$\mathrm{d}X(t) = b(X_t)\mathrm{d}t + \int_\Gamma \sigma(X_t, z)\tilde{N}_0(\mathrm{d}t, \mathrm{d}z), \quad X_s = \xi, \tag{27}$$

where $\tilde{N}_0(\mathrm{d}t, \mathrm{d}z) := N_0(\mathrm{d}t, \mathrm{d}z) - \mathrm{d}t \otimes m(\mathrm{d}z)$. Equation (27), under (H$1'$), has a unique strong solution $X(t; s, \xi)$ with initial data ξ at time s.

For any $\nu > 0$, by the Itô formula and (H$1'$), one has

$$\mathbb{E}(|X(t;s_1,\xi) - X(t;s_2,\xi)|^2) \le c(1 + \|\xi\|_\infty^2)\epsilon^{-\nu(t-s_2)}. \qquad (28)$$

For $s_1, s_2 \in (-\infty, t]$ such that $s_1 \le s_2 \le t - 2\tau$, we can show that

$$\mathbb{E}(\|X_t(s_1,\xi) - X_t(s_2,\xi)\|_\infty^2) \le c(1 + \|\xi\|_\infty^2)\epsilon^{-\nu(t-s_2)}. \qquad (29)$$

Taking $s_2 \to -\infty$, it follows that there exists $\eta_t \in L^2(\Omega, \mathscr{F}, \mathbb{P}; \mathscr{D})$ such that

$$\lim_{s\to-\infty} \|X_t(s,\xi) - \eta_t\|_\infty^2 = 0. \qquad (30)$$

For bounded Lipschitz $F : \mathscr{C} \to \mathbb{R}$ and $s \le t$, let

$$\mathbb{P}_{s,t}(\xi, \mathrm{d}\eta) := \mathbb{P} \circ (X_t(s,\xi))^{-1}(\mathrm{d}\eta) \quad \text{and} \quad P_{s,t}F(\xi) := \int_{\mathscr{C}} F(\eta)\mathbb{P}_{s,t}(\xi, \mathrm{d}\eta).$$

Note from (30) implies that

$$\mathbb{P}_{-s,0}(\xi, \eta) \to \pi := \mathbb{P} \circ \eta_0^{-1} \quad \text{weakly as } s \to \infty.$$

Then one has

$$\int_{\mathscr{C}} P_{0,t}F(\eta)\pi(\mathrm{d}\eta) = \lim_{s\to\infty} P_{-s,0}(P_{0,t}F)(\xi) = \lim_{s\to\infty} P_{-(t+s),0}F(\xi)$$
$$= \int_{\mathscr{C}} F(\eta)\pi(\mathrm{d}\eta)$$

This indeed gives that $\pi = \mathbb{P} \circ \eta_0^{-1}$ is an invariant measure

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