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Three SPDEs from branching interacting particle systems

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Outline

1. BIPS
2. Stochastic log-Laplace equation
3. SPDE for density field
4. SPDE for “distribution” process

1. BIPS

At time $t = 0$, k_n particles at locations

$$x_i^n \in \mathbb{R}, \qquad i = 1, 2, \cdots, k_n.$$

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- When time-up, the particle will split/die w/ equal prob.
- Denote particle by a multi-index α.
- Between branching times, particle α moves according to

$$dx_t^\alpha = b(x_t^\alpha)dt + c(x_t^\alpha)dW_t + e(x_t^\alpha)dB_t^\alpha$$

W, B^α indep. B.M.

Define measure-valued process

$$\mu_t^n = \frac{1}{n} \sum_{\alpha \sim t} \delta_{x_\alpha^n(t)}$$

where $\alpha \sim t$ means particle α is alive at time t.

Convergence Theorem

$(\mu_t^n) \Rightarrow (\mu_t)$ unique sol. to MP: μ_t is $\mathcal{M}(R)$-valued

$$M_t(f) \equiv \langle \mu_t, f \rangle - \langle \mu, f \rangle - \int_0^t \langle \mu_s, bf' + af'' \rangle \, ds$$

continuous martingale with

$$\langle M(f) \rangle_t = \int_0^t \left(\langle \mu_s, f^2 \rangle + \langle \mu_s, cf' \rangle^2 \right) ds$$

where $a(x) = \frac{1}{2}(e(x)^2 + c(x)^2)$.

Studied by Adler and Skoulakis (2001) and Wang (1998) among others.

(CMP)

$$N_t(f) \equiv \langle \mu_t, f \rangle - \langle \mu, f \rangle - \int_0^t \langle \mu_s, bf' + df'' \rangle \, ds - \int_0^t \langle \mu_s, cf' \rangle \, dW_s$$

P^W-martingale w/

$$\langle N(f) \rangle_t = \int_0^t \langle \mu_s, f^2 \rangle \, ds$$

Formally,

$$\begin{aligned} N_t(f) &= \langle \mu_t, f\rangle - \langle \mu, f\rangle \\ &\quad - \int_0^t \left\langle \mu_s, (b + c\dot{W}_s) f' + d f''\right\rangle ds, \end{aligned}$$

Therefore,

$$E^W \exp\left(-\langle \mu_t, f\rangle\right) = \exp\left(-\langle \mu, y_{0,t}\rangle\right),$$

where

$$y_{s,t} = f + \int_s^t c\partial_x y_{r,t} \dot{W}_r dr + \int_s^t \left(b\partial_x y_{r,t} + a\partial_x^2 y_{r,t} - y_{r,t}(x)^2 \right) dr.$$

Stochastic integral is backward Itô integral.

Theorem (Xiong, 2004, AP)

$$E^W \exp\left(-\langle \mu_t, f\rangle\right) = \exp\left(-\langle \mu, y_{0,t}\rangle\right),$$

where $Lf = af'' + bf'$,

$$\begin{aligned} y_{s,t} &= f + \int_s^t c\partial_x y_{r,t} \hat{d} W_r \\ &\quad + \int_s^t \left(Ly_{r,t} - y_{r,t}(x)^2\right) dr. \end{aligned}$$

Idea: Existence Particle representation (Kurtz & Xiong, 1999, SPA) Conditional Laplace transform Divide $[0, t]$ into small ones. Alternatively, without W or without branching.

2. SPDE for density field

Random measure

Let $(U, \mathcal{U}, \mu)$ be a measure space. A mapping

$$W : \Omega \times (\mathcal{B}(\mathbb{R}_+) \times \mathcal{U}) \to \mathbb{R}$$

is a *random measure* if

- $W(\omega, \cdot)$ is a signed measure on $\mathbb{R}_+ \times U$ for each ω
- $W(\cdot, B)$ is a r.v. for each $B \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{U}$.

White noise random measure

r.m. W is *independently scattered* if for any disjoint $B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{U}$, r.v.'s

$$W(\cdot, B_1), \cdots, W(\cdot, B_n)$$

are independent.

An independently scattered r.m. is a *white noise random measure* if for any $B \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{U}$ with $(dtd\mu)(B) < \infty$,

$$W(\cdot, B) \sim N(0, (dtd\mu)(B)).$$

μ is the *intensity measure* of W.

Recall CMP:

$$N_t(f) \equiv \langle \mu_t, f\rangle - \langle \mu, f\rangle - \int_0^t \langle \mu_s, Lf\rangle\, ds - \int_0^t \langle \mu_s, cf'\rangle\, dW_s$$

P^W-martingale w/

$$\begin{aligned}\langle N(f)\rangle_t &= \int_0^t \langle \mu_s, f^2\rangle\, ds \\ &= \int_0^t \int_{\mathbb{R}} \left(\sqrt{X_s(x)} f(x)\right)^2 dx ds.\end{aligned}$$

Then,

$$N_t(f) = \int_0^t \int_{\mathbb{R}} \sqrt{X_s(x)} f(x) B(dx ds),$$

where B is WNRM.

Theorem (Li, Wang, Xiong, Zhou, 2012, PTRF)

μ_t has density $X_t(x)$ satisfying SPDE

$$\partial_t X_t(x) = L^* X_t(x) - \partial_x(cX_t(x))\dot{W}_t + \sqrt{X_t(x)}\dot{B}_{tx}. \tag{2.1}$$

How about continuity?

Using Krylov's L_p theory
Lee-Mueller-Xiong (AIHP, 2008) proved, for a.e. t, $X_t \in C(\mathbb{R})$

When $c = 0$, X_t Dawson-Watanabe process, the joint continuity studied by Konno-Shiga (PTRF, 1988) using a convolution technique.

If we apply the same technique here, then the density can be represented as

$$X_t(x) = \int \varphi(t,x,y)\mu(y)dy$$
$$+ \int_0^t \int_{\mathbb{R}} c(z)X_s(z)\partial_z\varphi(t-s,x,z)dzdW_s$$
$$+ \int_0^t \int_{\mathbb{R}} \sqrt{X_s(z)}\varphi(t-s,x,z)B(dsdz). \qquad (2.2)$$

The second term is roughly equal to

$$\int_0^t (t-s)^{-1/2} dW_s$$

divergent.

Therefore, the convolution argument of Konno-Shiga does not apply to our model. Actually, this means that the SPDE (2.1) does not have a mild solution.

Let $\{T_{r,t}(x):\ r\leq t\}$ be the unique solution to

$$\begin{aligned} T_{r,t}(x) &= f(x)+\int_r^t LT_{s,t}(x)ds \\ &\quad +\int_r^t \partial_x T_{s,t}(x)c(x)\hat{d}W_s. \end{aligned} \tag{2.3}$$

Lemma 1

$$\langle X_t, f\rangle = \langle X_0, T_{0,t}\rangle + \int_0^t\int_{\mathbb{R}} T_{s,t}(x)\sqrt{X_s(x)}B(dsdx).$$

Lemma 2

Let $p^W(s,x;t,y)$ be the sol. of (2.3) with $f=\delta_y$. Then,

$$\begin{aligned} X_t(y) &= \int_{\mathbb{R}} X_0(x)p^W(s,x;t,y)dx \\ &\quad + \int_0^t \int_{\mathbb{R}} p^W(s,x;t,y)\sqrt{X_s(x)}B(dsdx). \end{aligned} \tag{2.4}$$

Theorem (Li-Wang-X-Zhou, PTRF, 2012)

$$\begin{aligned} &\mathbb{E}|X_{t_1}(y_1) - X_{t_2}(y_2)|^{2p} \\ \leq \quad & K\left(|t_1 - t_2| + |y_1 - y_2|\right)^{2+\epsilon}. \end{aligned}$$

Remark

Our Holder exponent is not optimal. Improved by Hu, Lu and Nualart (2012, PTRF) using Malliavin's calculus.

4. SPDE for "distribution" process

How about strong uniqueness?

Open even for $c = 0$

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5. signed solution, non-uniqueness, $\frac{1}{2} \leq \alpha < \frac{3}{4}$, Mueller-Mytnik-Perkins (2012)

Define

$$\tilde{u}_t(y) = \mu_t((-\infty, y]), \qquad \forall y \in \mathbb{R}. \tag{3.5}$$

Consider SPDE

$$\begin{aligned} \tilde{u}_t(y) \quad = \quad & F(y) + \int_0^t \int_0^{\tilde{u}_s(y)} \tilde{B}(dsdu) \\ & + \int_0^t L_2 \tilde{u}_s(y) ds + \int_0^t L_1 \tilde{u}_s(y) d\tilde{W}_s, \end{aligned} \tag{3.6}$$

where

$$F(y) = \int_{-\infty}^{y} \mu(x)dx,$$

and $\tilde{B}(dsdu)$ is white noise on $(0,\infty)^2$,

$$L_2 f = af'' + (a' - b)f', \qquad L_1 f = -cf'.$$

When $c = 0$, L_2 is replaced by the bounded operator

$$Af(x) = (\gamma(x) - f(x))b,$$

the equation (3.6) is studied by Dawson-Li (2010, AP).
When $c = 0$ and $L_2 = \Delta$, studied by Xiong (2013, AP).

Theorem

$\{\tilde{u}_t\}$ is a weak solution to the SPDE (3.6) if and only if $\{\mu_t\}$, defined by (3.5), is a SBMRE.

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Key in the proof

$$\langle \mu_t, f \rangle = -\langle \tilde{u}_t, f' \rangle$$

and

$$\int_0^\infty f\left(\tilde{u}_s^{-1}(u)\right)^2 du = \langle \mu_s, f^2 \rangle .$$

Consider general SPDE

$$\tilde{u}_t(y) = F(y) + \int_0^t \int_0^\infty G(u, \tilde{u}_s(y)) \tilde{B}(dsdu) + \int_0^t L_2 \tilde{u}_s(y) ds + \int_0^t L_1 \tilde{u}_s(y) d\tilde{W}_s. \qquad (3.7)$$

Special case,

$$G(u, v) = 1_{u \le v}.$$

Conditions:

$$\int_0^\infty |G(u,y_1) - G(u,y_2)|^2 du \le K|y_1 - y_2|, \tag{3.8}$$

and

$$\int_0^\infty |G(u,y)|^2 du \le K(1 + |y|^2). \tag{3.9}$$

Backward SPDE

$$u_t(y) = F(y) + \int_t^T \int_0^\infty G(u, u_s(y)) B(\hat{d}s du) + \int_t^T L_2 u_s(y) ds + \int_t^T L_1 u_s(y) \hat{d}W_s. \quad (3.10)$$

For simplicity of notation, take $a = \frac{1}{2}$ and $b = 0$.

Consider the following BDSDE: $t \leq s \leq T$

$$Y_s = f(X_T) + \int_s^T \int_0^\infty G(u, Y_r) B(\hat{d}r du) - \int_s^T c Z_r \hat{d} W_r - \int_s^T Z_r dB_r^1, \tag{3.11}$$

where

$$X_s = X_s^{t,x} = x + B_s^1 - B_t^1.$$

Denote solution as $(Y_s^{t,x}, Z_s^{t,x})$.

Pardoux-Peng (1994, PTRF), Xiong (2013, AP)

Definition

The pair (X_t, Y_t, Z_t) solution to (3.11) if they are $\mathcal{G}_t$-adapted and $\forall t \in [0, T]$, (3.11) holds a.s., where

$$\mathcal{G}_t = \sigma\left(B_s^1,\ s \le t;\ B([r, T] \times A),\ W_T - W_r,\ r \in [t, T],\ A \in \mathcal{B}(\mathbb{R})\right).$$

Note that $\mathcal{G}_t$ is the σ-algebra generated by B before time t and by W after time t.

The following backward Itô's formula is useful.

Lemma (Pardoux-Peng)

Suppose

$$y_t = \xi + \int_t^T \int_U \alpha(s,u) W(\hat{d}sdu) - \int_t^T z_s dB_s,$$

where $\alpha : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a $\mathcal{G}_t$-adapted random field. Then, $\forall f \in C_b^2(\mathbb{R})$,

$$\begin{aligned} f(y_t) &= f(\xi) - \int_t^T \int_U f'(y_s)\alpha(s,u) W(\hat{d}sdu) - \int_t^T z_s f'(y_s) dB_s \\ &+ \frac{1}{2}\int_t^T \int_U f''(y_s)\alpha(s,u)^2 duds - \frac{1}{2}\int_t^T z_s^2 f''(y_s) ds. \end{aligned}$$

Here is one of the main results.

Theorem

The FBDSDE (3.11) has at most one solution.

Key to proof Yamada-Watanabe argument.

By Pardoux-Peng's formula, we get

$$\begin{aligned}
&\phi_k(Y_t^1 - Y_t^2) \\
= \quad & -\int_t^T \int_0^\infty \phi_k'(Y_s^1 - Y_s^2)\left(G(u, Y_s^1) - G(u, Y_s^2)\right) W(\hat{d}sdu) \\
& -\int_t^T \phi_k'(Y_s^1 - Y_s^2)\left(Z_s^1 - Z_s^2\right) dB_s \\
& +\frac{1}{2}\int_t^T \int_0^\infty \phi_k''(Y_s^1 - Y_s^2)\left(G(u, Y_s^1) - G(u, Y_s^2)\right)^2 duds \\
& -\frac{1}{2}\int_t^T \phi_k''(Y_s^1 - Y_s^2)\left(Z_s^1 - Z_s^2\right)^2 ds. \qquad (3.12)
\end{aligned}$$

Theorem

If $\{u_t(x)\}$ is a solution to (3.10), then

$$u_t(x) = Y_t^{t,x},$$

where $Y_s^{t,x}$ is a solution to the BDSDE (3.11).

Sketch of proof Let

$$Y_s^{t,x} = u_s(X_s^{t,x}) \text{ and } Z_s^{t,x} = \nabla u_s(X_s^{t,x}), \qquad t \le s \le T. \quad (3.13)$$

Uniqueness

Theorem

Under (3.8) and (3.9), the SPDE (3.7) has at most one solution such that $u \in C^{0,1}([0,T] \times \mathbb{R})$ a.s.

Thanks!