Stability and Instability for Switching Jump-Diffusion Processes

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Contents

References

This talk is based on the following joint papers with George Yin:

- Stability of Regime-Switching Jump Diffusions, *SIAM J. Contr. Optim.* 48(2010), 4525–4549.
- Jump-diffusions with state-dependent switching: existence and uniqueness, Feller property, linearization, and uniform ergodicity, *Sci. China Math.*, 54(2011), 2651–2667.
- Almost Sure Stability and Instability for Switching-Jump-Diffusion Systems with State-Dependent Switching, *J. Math. Anal. Appl.* 400(2013), 460–474.

Background

We consider a class of **switching jump-diffusion processes**.

The underlying process can be thought of as a number of jump-diffusion processes modulated by a random switching device.

It is a two-component process (X, Λ) with X delineating the jump-diffusion behavior and Λ describing the switching involved.

One of the main ingredients is that the switching component depends on the jump-diffusion component.

To give a better visualization of the process, we consider the following scenario.

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Suppose that we have a number of jump-diffusion processes sitting on several parallel planes, respectively, with the switching component residing at plane k initially. It sojourns in this plane for a random duration. During this random period, the component X follows a jump-diffusion process, in which the drift, diffusion, and jump coefficients are determined by the discrete-event state k.

At a random instance, the switching process Λ changes state from k to $l \neq k$, and sojourns in the new state l for a random duration. Consequently, X follows another jump-diffusion process with the drift, diffusion, and jump coefficients determined by the discrete-event state l.

Please see an Illustration with $\Lambda = \alpha$ being continuous for simplicity.

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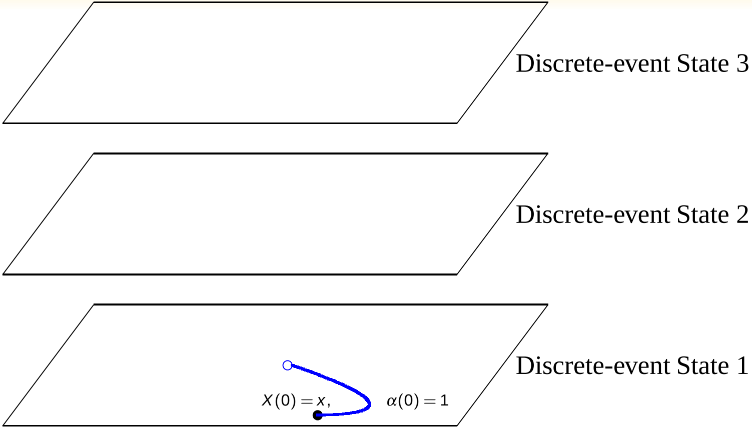
Hybrid Regime-Switching Diffusion: An Illustration



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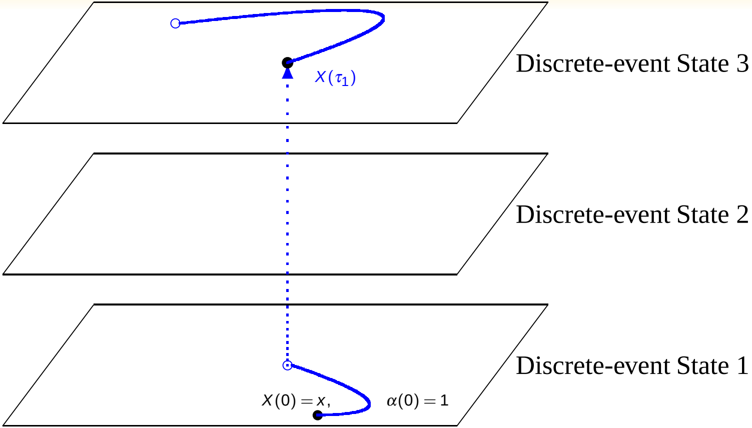

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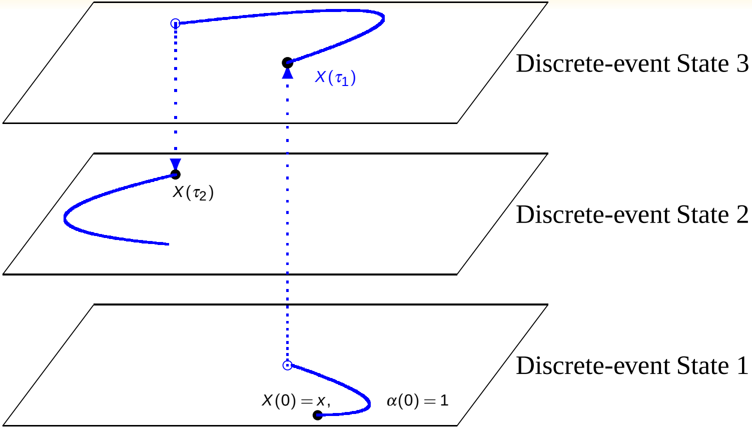

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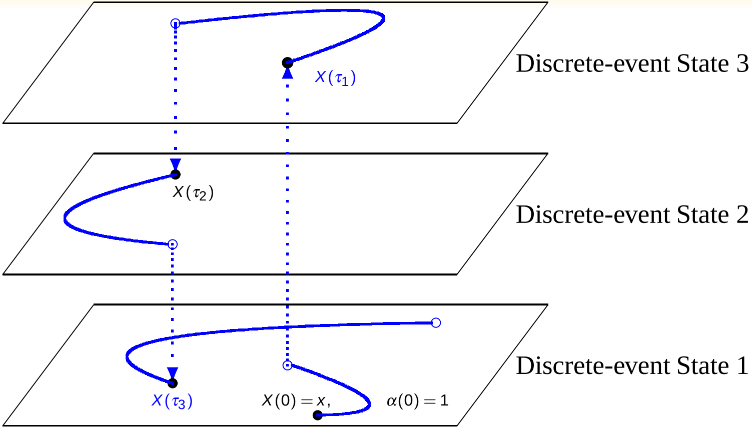

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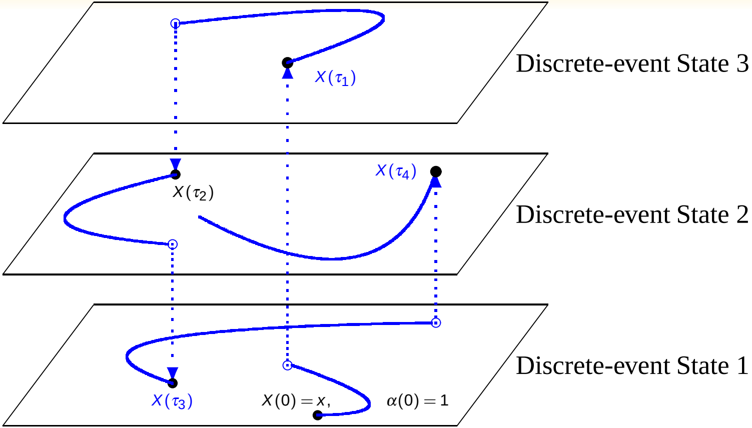

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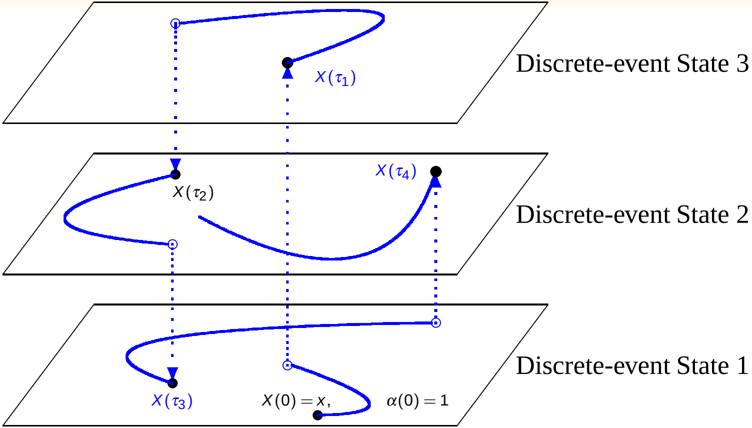

Figure: A “Sample Path” of the Switching Diffusion $(X(t), \alpha(t))$.

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The **switching jump-diffusion process** makes the formulation more versatile with wider range of applications. Nevertheless, it makes the analysis more difficult.

The motivation of our study stems from a wide range of applications in communication systems, flexible manufacturing and production planning, financial engineering, and economics.

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Problems

Stability theory: stability in probability, **stability almost surely**, Lyapunov exponent, stability in distribution, stability in total variation norm, $\cdots\cdots$.

Ergodicity theory: ordinary ergodicity, exponential ergodicity, strong (or uniform) ergodicity, ergodic rate estimate, $\cdots\cdots$.

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Construction of the Process

Let (X, Λ) be a Markov process on $\mathbb{R}^d \times \mathbb{S}$, where $\mathbb{S} := \{1, 2, \cdots, n_0\}$. The first component X satisfies

$$\begin{aligned} dX(t) &= b(X(t), \Lambda(t))dt + \sigma(X(t), \Lambda(t))dB(t) \\ &\quad + \int_U c(X(t-), \Lambda(t-), u)N(dt, du), \end{aligned} \tag{1}$$

and the second component Λ satisfies

$$P\{\Lambda(t+\Delta) = l | \Lambda(t) = k, X(t) = x\} = \begin{cases} q_{kl}(x)\Delta + o(\Delta), & \text{if } l \neq k, \\ 1 + q_{kk}(x)\Delta + o(\Delta), & \text{if } l = k. \end{cases} \tag{2}$$

Standing Hypothesis

Assumption 1

(i) The functions $b(\cdot)$ and $\sigma(\cdot)$ satisfy $b(0,k)=0$ and $\sigma(0,k)=0$ for each $k\in\mathbb{S}$; and $\sigma(\cdot,k)$ vanishes only at $x=0$ for each $k\in\mathbb{S}$.

(ii) For each $k\in\mathbb{S}$, both $b(\cdot,k)$ and $\sigma(\cdot,k)$ satisfy the Lipschitz condition.

(iii) For any $k\neq l\in\mathbb{S}$, $q_{kl}(\cdot)$ is $\mathcal{B}(\mathbb{R}^d)$ measurable and for some constant $H>0$, $\sup_{x\in\mathbb{R}^d,\ k\neq l\in\mathbb{S}} q_{kl}(x)\leq H<+\infty$.

(iv) For each $k\in\mathbb{S}$, $c(\cdot,k,\cdot)$ is $\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(U)$ measurable and satisfy that $c(0,k,u)=0$ for each $u\in U$. In addition, assume that there is $c_1(u)>0$ such that for each $x\in\mathbb{R}^d$, $k\in\mathbb{S}$ and $u\in U$,

$$|x+c(x,k,u)|\geq c_1(u)|x| \quad \text{and} \quad \int_U \frac{1}{c_1(u)}\Pi(du)<\infty. \tag{3}$$

SDE's Representation

For $x \in \mathbb{R}^d$ and $k, l \in \mathbb{S}$ with $k \neq l$, let $\triangle_{kl}(x)$ be the consecutive (with respect to the lexicographic ordering on $\mathbb{S} \times \mathbb{S}$), left-closed, right-open intervals of $\mathbb{R}_+$, each having length $q_{kl}(x)$.

Define a function h: $\mathbb{R}^d \times \mathbb{S} \times [0, n_0(n_0-1)H] \to \mathbb{R}$ by

$$h(x, k, r) = \sum_{l \in \mathbb{S}} (l-k) \mathbf{1}_{\triangle_{kl}(x)}(r).$$

Then, (2) is equivalent to

$$d\Lambda(t) = \int_{\mathbb{R}} h(X(t-), \Lambda(t-), r) N_1(dt, dr), \tag{4}$$

where $N_1(dt, dr)$ is another Poisson random measure.

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Switching Jump-Diffusion Process

The (X, Λ) associated with system (1) and (2) can be thought of as the solution to system (1) and (4), the driving forces being the Brownian motion $B(\cdot)$ and the Poisson random measures $N(\cdot, \cdot)$ and $N_1(\cdot, \cdot)$

Theorem 2

System (1) and (4) has a unique (non-explosive) strong solution (X, Λ).

Lemma 3

$$P\{X^{(x,k)}(t) \neq 0, t \geq 0\} = 1 \text{ for any } x \neq 0,\ k \in \mathbb{S}. \quad (5)$$

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Almost Sure Stability Definition

Defintion 4

The equilibrium point $x = 0$ of system (1) and (2) is said to be almost surely exponentially stable if for any $(x, k) \in \mathbb{R}^d \times \mathbb{S}$,

$$\limsup_{t\to\infty} \frac{1}{t} \ln |X^{(x,k)}(t)| < 0 \quad \text{w.p.1.} \tag{6}$$

It is said to be almost surely exponentially unstable if for any $(x, k) \in \mathbb{R}^d \times \mathbb{S}$ with $x \neq 0$,

$$\liminf_{t\to\infty} \frac{1}{t} \ln |X^{(x,k)}(t)| > 0 \quad \text{w.p.1.} \tag{7}$$

$$\begin{aligned}
&\ln|X(t)| - \ln|x| \\
&\quad = \int_0^t \frac{X^T(s)b(X(s),\Lambda(s))}{|X(s)|^2}ds \\
&\qquad + \frac{1}{2}\int_0^t \Big[\frac{|\sigma(X(s),\Lambda(s))|^2}{|X(s)|^2} - \frac{2|X^T(s)\sigma(X(s),\Lambda(s))|^2}{|X(s)|^4}\Big]ds \\
&\qquad + \int_0^t \int_U [\ln|X(s-) + c(X(s-),\Lambda(s-),u))| - \ln|X(s-)|]\Pi(du)ds \\
&\qquad + M_1(t) + M_2(t),
\end{aligned}$$

where

$$\begin{aligned}
M_1(t) &= \int_0^t \frac{X^T(s)}{|X(s)|^2}\sigma(X(s),\Lambda(s))dB(s), \\
M_2(t) &= \int_0^t \int_U [\ln|X(s-) + c(X(s-),\Lambda(s-),u))| - \ln|X(s-)|]\widetilde{N}(ds,du).
\end{aligned}$$

Order-Preserving Coupling

The coupling methods have been used for a wide variety of applications (see Chen M. F. (2004)). One application is that the study of complex systems can be converted to the study of some simple ones. For this the order-preserving couplings usually play an important role.

Let $\Lambda^{(1)}$ and $\Lambda^{(2)}$ be two continuous-time Markov chains defined by Q-matrices $Q^{(1)} = \left(q_{kl}^{(1)}\right)$ and $Q^{(2)} = \left(q_{kl}^{(2)}\right)$ on the finite state space $\mathbb{S}$, respectively.

On the product space $\mathbb{S} \times \mathbb{S}$, an order-preserving coupling $\widetilde{Q}$ of $Q^{(1)}$ and $Q^{(2)}$ yields that

$$\widetilde{P}^{(k_1,k_2)}(\Lambda^{(1)}(t) \le \Lambda^{(2)}(t)) = 1, \quad t \ge 0, \quad k_1 \le k_2 \in \mathbb{S}, \qquad (8)$$

where $\left(\Lambda^{(1)}, \Lambda^{(2)}\right)$ is the Markov chain generated by $\widetilde{Q}$.

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Order-Preserving Coupling (cont)

For more general case, the construction of order-preserving couplings was studied in Zhang Y.H. (1996, 1998). In particular, we have the following lemma.

Lemma 5

If the generators $Q^{(1)}$ and $Q^{(2)}$ on $\mathbb{S}$ satisfy that

$$\begin{aligned} &\sum_{l\geq m} q^{(1)}_{k_1 l} \leq \sum_{l\geq m} q^{(2)}_{k_2 l} \quad \text{for all} \quad k_1 \leq k_2 < m \quad \text{and} \\ &\sum_{l\leq m} q^{(1)}_{k_1 l} \geq \sum_{l\leq m} q^{(2)}_{k_2 l} \quad \text{for all} \quad m < k_1 \leq k_2, \end{aligned} \tag{9}$$

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An Assumption

Assumption 6

Assume that there exists a generator $Q^* = \big(q^*(k,l)\big)$ on $\mathbb{S}$ such that the following bounds hold:

$$\begin{aligned} &\sup_{x\in\mathbb{R}^d} \sum_{l\geq m} q_{k_1 l}(x) \leq \sum_{l\geq m} q^*(k_2,l) \quad \text{for all} \quad k_1 \leq k_2 < m \quad \text{and} \\ &\inf_{x\in\mathbb{R}^d} \sum_{l\leq m} q_{k_1 l}(x) \geq \sum_{l\leq m} q^*(k_2,l) \quad \text{for all} \quad m < k_1 \leq k_2, \end{aligned} \tag{10}$$

where the matrix $\big(q_{kl}(x)\big)$ is given in (2)

Order-Preserving Coupling (cont)

By Lemma 5, for each $x \in \mathbb{R}^d$, there exists an order-preserving coupling of $Q(x)$ and Q^* given in Assumption 6. In fact, such an order-preserving coupling $\widetilde{Q}^*(x) = \left(\widetilde{q}^*(k, l; m, n)(x)\right)$ was constructed explicitly in Zhang Y.H. (1996, 1998); see also Chen M. F. (2004).

Let Λ^* be the Markov chain generated by Q^*. We now construct a coupling process $\left(X, \Lambda, \Lambda^*\right)$ as follows.

Let the first component X satisfy

$$\begin{aligned} dX(t) &= b(X(t), \Lambda(t))dt + \sigma(X(t), \Lambda(t))dB(t) \\ &\quad + \int_U c(X(t-), \Lambda(t-), u)N(dt, du), \end{aligned}$$

and let the second and third components together satisfy

$$\begin{aligned} &P\{(\Lambda(t+\Delta), \Lambda^*(t+\Delta)) = (m, n) | (\Lambda(t), \Lambda^*(t)) = (k, l), X(t) = x\} \\ &= \begin{cases} \widetilde{q}^*(k, l; m, n)(x)\Delta + o(\Delta), & \text{if } (m, n) \neq (k, l), \\ 1 + \widetilde{q}^*(k, l; m, n)(x)\Delta + o(\Delta), & \text{if } (m, n) = (k, l). \end{cases} \end{aligned}$$

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$$\begin{aligned} dX(t) &= b(X(t),\Lambda(t))dt + \sigma(X(t),\Lambda(t))dB(t) \\ &\quad + \int_U c(X(t-),\Lambda(t-),u)N(dt,du), \end{aligned}$$

and let the second and third components together satisfy

$$P\{(\Lambda(t+\Delta),\Lambda^*(t+\Delta)) = (m,n) | (\Lambda(t),\Lambda^*(t)) = (k,l), X(t) = x\}$$
$$= \begin{cases} \widetilde{q^*}(k,l;m,n)(x)\Delta + o(\Delta), & \text{if } (m,n) \neq (k,l), \\ 1 + \widetilde{q^*}(k,l;m,n)(x)\Delta + o(\Delta), & \text{if } (m,n) = (k,l). \end{cases}$$

Order-Preserving Coupling (cont)

Theorem 7

Suppose that Assumption 6 holds. For the coupling process (X, Λ, Λ^*) *constructed above, we have*

$$\widetilde{P}^{(x,k,l)}\big(\Lambda(t) \le \Lambda^*(t)\big) = 1, \quad t \ge 0, \quad x \in \mathbb{R}^d, \quad k \le l \in \mathbb{S}, \tag{11}$$

where $\widetilde{P}^{(x,k,l)}$ *denotes the distribution of* (X, Λ, Λ^*) *starting from* (x, k, l)*. Moreover, suppose that* Q^* *is irreducible, then for each monotonic function* h *on* $\mathbb{S}$ *and each* $(x, k) \in \mathbb{R}^d \times \mathbb{S}$,

$$P^{(x,k)}\left(\limsup_{t\to\infty} \frac{1}{t}\int_0^t h(\Lambda(s))ds \le \sum_{m\in\mathbb{S}} h(m)\mu_m^*\right) = 1, \tag{12}$$

where $P^{(x,k)}$ *denotes the distribution of* (X, Λ) *starting from* (x, k) *and* $\mu^* = (\mu_1^*, \cdots, \mu_{n_0}^*)$ *is the invariant probability measure associated with* Λ^*.

An Assumption

Assumption 8

(i) For each $k \in \mathbb{S}$, there exist positive constants $H^b(k)$, $H^\sigma(k)$, and $H^d(k)$ such that for all $x \in \mathbb{R}^d$,

$$x^T b(x,k) \leq H^b(k)|x|^2, |\sigma(x,k)|^2 \leq H^\sigma(k)|x|^2, |x^T \sigma(x,k)| \geq H^d(k)|x|^2 \tag{13}$$

(ii) For each $k \in \mathbb{S}$, there exists positive constant $H^c(k)$ such that for all $x \in \mathbb{R}^d$ and $u \in U$,

$$|c(x,k,u)| \leq H^c(k)|x|. \tag{14}$$

(iii) The following function on $\mathbb{S}$,

$$H(k) := H^b(k) + \frac{1}{2}H^\sigma(k) - \left(H^d(k)\right)^2 + \Pi(U)\ln\left(1 + H^c(k)\right) \tag{15}$$

is monotonic.

Almost Sure Stability Result

Theorem 9

Suppose that Assumptions 6 and 8 hold, and that Q^* is irreducible. If

$$\sum_{k\in\mathbb{S}} \mu_k^* \left(H^b(k) + \frac{1}{2} H^\sigma(k) - \left(H^d(k)\right)^2 + \Pi(U) \ln\left(1 + H^c(k)\right) \right) < 0, \quad (16)$$

then the equilibrium point $x = 0$ of system (1) and (2) is almost surely exponentially stable. Here $\mu^* = (\mu_1^*, \cdots, \mu_{n_0}^*)$ is the invariant probability measure associated with Q^*.

About the Proof (again)

$$\begin{aligned}
&\ln|X(t)| - \ln|x| \\
&= \int_0^t \frac{X^T(s)b(X(s),\Lambda(s))}{|X(s)|^2}ds \\
&\quad + \frac{1}{2}\int_0^t \left[\frac{|\sigma(X(s),\Lambda(s))|^2}{|X(s)|^2} - \frac{2|X^T(s)\sigma(X(s),\Lambda(s))|^2}{|X(s)|^4}\right]ds \\
&\quad + \int_0^t \int_U [\ln|X(s-) + c(X(s-),\Lambda(s-),u))| - \ln|X(s-)|]\Pi(du)ds \\
&\quad + M_1(t) + M_2(t),
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
M_1(t) &= \int_0^t \frac{X^T(s)}{|X(s)|^2}\sigma(X(s),\Lambda(s))dB(s), \\
M_2(t) &= \int_0^t \int_U [\ln|X(s-) + c(X(s-),\Lambda(s-),u))| - \ln|X(s-)|]\widetilde{N}(ds,du).
\end{aligned} \tag{18}$$

About the Proof (and again)

Using the order-preserving coupling in (17) and (18), we can obtain the almost sure stability by virtue of the strong law of large numbers for local martingales and the ergodic property of Markov chains.

Note that Λ is not a Markov chain generally.

Example

Example 10

Take $d = 1$, $\mathbb{S} = \{1, 2, 3\}$. Consider the system (X, Λ) satisfying (1) and (2) with the following specifications. Let

$$b(x,1) = 3x + x\sin^2 x, \quad b(x,2) = x + x\sin x, \quad b(x,3) = 2x + x\sin x\cos x,$$

$$\sigma(x,1) = 10x \quad \sigma(x,2) = 3x + x\sin x, \quad \sigma(x,3) = 3x + x\cos x\sin x,$$

$$c(x,k,u) = x \quad \text{for} \quad k = 1,2,3 \quad \text{and} \quad u \in U \quad \text{with} \quad \Pi(U) = 1,$$

and let $Q(x)$ be

$$Q(x) = \begin{pmatrix} -3 - |\cos x| + \sin^2 x & 1 + |\cos x| & 2 - \sin^2 x \\ 1 + \frac{x^2}{1+x^2} & -2 - \frac{x^2}{1+x^2} & 1 \\ 2 + |\sin x| & 1 + \frac{|x|}{1+|x|} & -3 - |\sin x| - \frac{|x|}{1+|x|} \end{pmatrix}.$$

Example (cont)

Assumption 6 is satisfied with the Q^* given by

$$Q^* = (q^*(k,l)) = \begin{pmatrix} -4 & 2 & 2 \\ 1 & -3 & 2 \\ 2 & 1 & -3 \end{pmatrix}$$

and Assumption 8 is satisfied with

$$H^b(1) = 4, H^b(2) = 2, H^b(3) = 3, H^\sigma(1) = 100, H^\sigma(2) = 20, H^\sigma(3) = 20,$$

$$H^d(1) = 10, \quad H^d(2) = 2, \quad H^d(3) = 2, \quad H^c(1) = H^c(2) = H^c(3) = 1,$$

$$H(1) = H^b(1) + \frac{1}{2}H^\sigma(1) - \big(H^d(1)\big)^2 + \Pi(U)\ln\big(1 + H^c(1)\big) = -46 + \ln 2,$$

$$H(2) = H^b(2) + \frac{1}{2}H^\sigma(2) - \big(H^d(2)\big)^2 + \Pi(U)\ln\big(1 + H^c(2)\big) = 8 + \ln 2,$$

$$H(3) = H^b(3) + \frac{1}{2}H^\sigma(3) - \big(H^d(3)\big)^2 + \Pi(U)\ln\big(1 + H^c(3)\big) = 9 + \ln 2.$$

Example (cont)

For the invariant probability measure $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*) = (7/25, 8/25, 2/5)$ associated with the irreducible Q^*, we have

$$\mu_1^* H(1) + \mu_2^* H(2) + \mu_3^* H(3) = -\frac{168}{25} + \ln 2 < 0.$$

Thus, by virtue of Theorem 9, the equilibrium point $x = 0$ of system (1) and (2) is almost surely exponentially stable.

The regime-switching jump-diffusion X satisfying (1) can be regarded as the following n_0 single jump-diffusions

$$\begin{aligned} dX^{(k)}(t) &= b(X^{(k)}(t), k)dt + \sigma(X^{(k)}(t), k)dB(t) \\ &\quad + \int_U c(X^{(k)}(t-), k, u)N(dt, du), \quad k \in \mathbb{S} \end{aligned} \tag{19}$$

coupled by the discrete-event component Λ according the transition rates defined by (2).

The system is often only observable when it operates in some modes but not all. Let us decompose $\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2$, where for each mode $k \in \mathbb{S}_2$, the jump-diffusion process (19) is not observable and hence cannot be stabilized by feedback control, but it can be stabilized for each mode $k \in \mathbb{S}_1$.

Stochastic Stabilization

The regime-switching jump-diffusion X satisfying (1) can be regarded as the following n_0 single jump-diffusions

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Stochastic Stabilization (cont)

One question of both theoretical and practical interest is: if a regime-switching jump-diffusion is not stable, can we design suitable controls so that the controlled regime-switching jump-diffusion become stable?

To answer the question, consider the regime-switching jump-diffusion

$$dX(t) = \left[b(X(t),\Lambda(t)) + u(X(t),\Lambda(t))\right]dt + \sigma(X(t),\Lambda(t))dB(t) + \int_U c(X(t-),\Lambda(t-),u)N(dt,du), \quad (20)$$

where $u(x,k) \equiv 0$ for $k \in \mathbb{S}_2$ while $u(x,k)$ is a feedback control for $k \in \mathbb{S}_1$.

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Stochastic Stabilization (cont)

Our aim is to design the control $u(x,k)$ for $k \in \mathbb{S}_1$ only so that the controlled system (20) is stabilized.

Now we only consider linear (in x) feedback controls of the form

$$u(x,k) = -L(k)x,$$

where for each $k \in \mathbb{S}$, $L(k) \in \mathbb{R}^{d\times d}$ is a constant matrix. Moreover, if $k \in \mathbb{S}_2$, $L(k) = 0$.

Thus (20) can be rewritten as

$$\begin{aligned} dX(t) = & \big[b(X(t),\Lambda(t)) - L(\Lambda(t))X(t)\big]dt + \sigma(X(t),\Lambda(t))dB(t) \\ & + \int_U c(X(t-),\Lambda(t-),u)N(dt,du). \end{aligned} \tag{21}$$

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Stochastic Stabilization (cont)

We have a wide variety of choices for the matrices $L(k)$ with $k \in \mathbb{S}_1$ in order to make the coefficients in the system (21) with the transition rule (2) to satisfy conditions of Theorem 9 and to get the exponential stability.

Rewrite (21) as follows:

$$\begin{aligned} dX(t) &= \widehat{b}(X(t), \Lambda(t))dt + \sigma(X(t), \Lambda(t))dB(t) \\ &\quad + \int_U c(X(t-), \Lambda(t-), u)N(dt, du), \end{aligned} \tag{22}$$

where

$$\widehat{b}(x, k) = b(x, k) - L(k)x.$$

Stochastic Stabilization (cont)

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Stochastic Stabilization (cont)

Theorem 11

For each $k \in \mathbb{S}$, select the matrix $L(k)$ so that the conditions of Theorem 9 are satisfied with the modification of replacing $b(x,k)$ by $\widehat{b}(x,k)$. Then the resulting system is exponentially stabilizable almost surely.

Likewise, we also have some almost surely instability criteria and and their applications to the stochastic destabilization.

Stochastic Stabilization (cont)

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Thank You Very Much!