Mixing Time of Random Walk on Poisson Geometry Small World

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1 Small world effect

Small world effect, the fact that the diameters of most networks are considerably smaller than their sizes, is one of the most important features of real-world complex networks.

See graph $G = (V, E)$ as a network, and suppose that $|V|$ is large enough. We say G exhibits the small world effect, if the diameter of G is at most polynomially large in $\lg |V|$. Namely, for some polynomial function f, one has

$$\mathrm{diam}(G) \leq f(\lg |V|).$$

History

1. Karinthy 1929, ······ a remarkable short story in his book: *Chains*.

2. Milgram 1960s, "six degrees of separation": Milgram carried out his famous "small-world" experiments, in which letters passed from person to person were able to reach a designated target individual within six steps.

3. Watts and Strogatz 1998, *Collective dynamics of 'small-world' networks*, Nature **393**, pp 440-442: WS small world.

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2 Why small world?—Modeling!

Three important models

1. BC small world 1988: B. Bollobás and F. Chung *The dimeter of a cycle plus a random matching*, SIAM J. Discrete Math. **1**, pp 328-333

2. WS small world 1998: D. J. Watts and S. H. Strogatz *Collective dynamics of 'small-world' networks*, Nature **393**, pp 440-442

3. NW small world 1999: M. E. J. Newman and D. J. Watts *Renormalization group analysis of the small-world network model*, Phys. Lett. A **263**, pp 341-346

Other models

LCD Model 2004: B. Bollobás and O. Riordan *The diameter of a scale-free random graph*, Combinatorica **24**, pp 5-34

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Remarks

1. 1-3 small world models revealed a common fact: that is, adding "long edges" to a regularly constructed (lattice-like) graph will make the resulted graph a small world—"adding long edges " mechanism.

The 1-dimensional lattice ring with n vertices is chosen to be the regularly constructed graph in all the three models.

2. Only the BC small world provided rigorous mathematical results.

3. A **NEW** mechanism other than the one working in BC, NW and WS small world makes the 'LCD' model a small world—we will discuss this aspect in another paper.

3 Our model: Poisson geometry small world.

We consider the supercritical Poisson continuous percolation on d-dimensional torus T_n^d with volume n^d.

By adding "long edges" randomly to the largest percolation cluster, we obtain a random graph $\mathscr{G}_n$.

It can be proved that the diameter of $\mathscr{G}_n$ grows at most polynomially fast in $\lg n$ and we call $\mathscr{G}_n$ the *Poisson geometry small world*.

Poisson continuous percolation on $\mathbb{R}^d$

Let $\mathscr{P}$ denote the homogeneous Poisson point process of rate 1 on $\mathbb{R}^d$. Given $r > 0$. We define a random graph $G = (V, E)$, where $V = \mathscr{P}$ and E consists of the edges which connect all Poisson point pairs lying in distance $2r$.

Theorem A[R.Meester and R.Roy (1995)] Suppose $d \geq 2$. Let $\psi(r)$ denote the probability that there exists a unique infinite connected component, write as C_∞, in G, let $\theta(r)$ denote the probability that the distance between the origin 0 and some vertex of C_∞ is less than $2r$. Then there exists $0 < r_c < \infty$ such that

$$\psi(r)\begin{cases} = 1, & \text{if } r > r_c \\ = 0, & \text{if } r < r_c \end{cases}, \quad \theta(r)\begin{cases} > 0, & \text{if } r > r_c \\ = 0, & \text{if } r < r_c \end{cases}.$$

Where $\theta(r)$ is called the percolation probability.

Continuous percolation on $B_n := [0, n]^d$

Let $\mathscr{P}_n$ denote the homogeneous Poisson point process of rate 1 on B_n.

Given $r > r_c$. We define a random graph $G_n = (V_n, E_n)$, where $V_n = \mathscr{P}_n = B_n \cap \mathscr{P}$ and E_n consists of the edges which connect all $\mathscr{P}_n$ point pairs lying in distance $2r$.

A connected component C of G_n is called crossing for B_n, if the distance between C and any of the face of B_n is less than r.

For continuous percolation on $B_n := [0, n]^d$, we have the following two propositions.(Penrose and Pisztora (1996))

Proposition 1 *Suppose* $r > r_c$. *Suppose* $\{\phi_n : n \geq 1\}$ *is increasing with* $\phi_n / \lg n \to \infty$ *as* $n \to \infty$, *and with* $\phi_n < n$ *for all* n.

Let $E_1(n)$ *be the event that*

(i) there is a unique component in G_n *that is crossing for* B_n, *and*

(ii) no other component in G_n *has diameter greater than* ϕ_n.

Then there exists a constant $c_1 > 0$ *such that for all large enough* n,

$$\mathbb{P}(E_1(n)) \geq 1 - \exp(-c_1 \phi_n).$$

Remark. *If we take* $\phi_n = \lg^{(1+\epsilon)} n$, *then it follows from Proposition 1 that, with high probability, all components except for the largest one have diameters less than* $\lg^{(1+\epsilon)} n$.

Proposition 2 *Suppose $r > r_c$, and $0 < \epsilon < 1/2$. Let $E_2(n)$ be the event that*

(i) there is a unique component C_b in G_n containing more than $\epsilon\theta(r)n^d$ points of $\mathscr{P}_n$,

(ii)

$$(1-\epsilon)\theta(r) \le n^{-d}|C_b| \le (1+\epsilon)\theta(r),$$

(iii) C_b is crossing for B_n, and

(iv) C_b is part of the infinite component C_∞ in G.

Then, there exist $c_2 > 0$ and n_2 such that

$$\mathbb{P}(E_2(n)) \ge 1 - \exp(-c_2 n^{d-1}), \quad n \ge n_2.$$

Continuous percolation on the d-dimensional torus T_n^d

Let T_n^d denote the d-dimensional torus obtained from B_n by cohering its opposite faces, and let $\mathscr{P}_n^T$ denote the Poisson process of rate 1 on T_n^d.

Given $r > r_c$. We define a random graph $G_n^T = (V_n^T, E_n^T)$, where $V_n^T = \mathscr{P}_n^T$ and E_n^T consists of the edges which connect all $\mathscr{P}_n^T$ point pairs lying in distance $2r$.

Let $C_{\max}$ denote the maximum connected component in G_n^T, let $\bar{G}_n$ denote the subgraph of G_n^T, which corresponds the maximum connected component $C_{\max}$.

Define the Poisson geometry small world $\mathscr{G}_n$

For any given constants α, β, σ and ζ satisfying $0 < \alpha < \beta < 1/2$, $\sigma > 0$ and $\zeta \geq 0$, we define a random graph $\mathscr{G}_n = \mathscr{G}_n(\alpha, \beta, \sigma, \zeta)$ from $\bar{G}_n$ as the following: for any $u, v \in C_{\max}$, if

$$\alpha n \leq d_\infty^T(u, v) \leq \beta n,$$

then we connect u and v independently by a "long edge" with probability

$$p_n = \sigma n^{-d} \lg^{-\zeta} n;$$

otherwise, we do nothing.

Where $d_\infty^T(\cdot,\cdot)$ denote the l_∞ metric on T_n^d inherited from the usual l_∞ metric $d_\infty(\cdot,\cdot)$ on $\mathbb{R}^d$ defined by

$$d_\infty(x,y) := \max_{1\le s\le d} |x_s - y_s|$$

for any $x, y \in \mathbb{R}^d$.

Let $\mathcal{E}_n$ denote the new edge set with long edges, and let

$$\mathscr{G}_n = (C_{\max}, \mathcal{E}_n).$$

1. $\mathscr{G}_n$ is a small world!

Theorem 1 *Suppose $r > r_c$. Then for any $0 < \alpha < \beta < 1/2$ with*

$$(2\beta)^d - (2\alpha)^d > 1/2,$$

$\sigma > 0$ and $\zeta \geq 0$, there exists constant $C_1 > 0$ such that for any $\nu > 0$,

$$\lim_{n\to\infty} \mathbb{P}(\operatorname{diam}(\mathscr{G}_n) \leq C_1 \lg^{3+\zeta+\nu} n) = 1.$$

Remark *It seems that our setting on "long edge" is REASONABLE! Obviously, if only shorter edges, for example with length $n^{1-\epsilon}$, are added, then the diameter of the resulted graph grows at least fast in n^{ϵ}, and the resulted graph does not exhibit the small world effect. On the other hand, Theorem 1 indicates that, to make the resulted graph a small world, adding such shorter edges is not necessary.*

2. Random Walk on graph

In a graph $G = (V, E)$, for any $u, v \in V$, let $d_G(u)$ be the the degree of u in G, and write $u \sim v$ if u and v are neighbors in G.

For any $u, v \in V$, we define a transition kernel by $P(u, u) = 1/2$, $P(u, v) = 1/2d_G(u)$ if $u \sim v$ and $P(u, v) = 0$ otherwise. A discrete time Markov chain $\{X_t : t \geq 0\}$ on V with transition kernel $(P(u, v))$ is called the *lazy random walk* on G.

Note that $\pi(u) := d_G(u)/D$ where $D = \sum_{v \in V} d_G(v) = 2|E|$, defines a reversible stationary distribution of $\{X_t\}$ since

$$\pi(u)P(u, v) = 1/2D = \pi(v)P(v, u).$$

Mixing time of Random Walk

By the basic theory of Markov chains, for any initial state $u \in V$, the distribution of X_t, i.e. $P^t(u, \cdot) := \mathbb{P}(X_t \in \cdot \mid X_0 = u)$, converges weakly to π as $t \to \infty$. To measure convergence to equilibrium, we will use the *total variation distance*

$$||P^t(u, \cdot) - \pi||_{TV} := \sum_{v \in V} |p^t(u, v) - \pi(v)|.$$

The mixing time of $\{X_t : t \geq 0\}$ is defined by

$$T_{\text{mix}} := \min\left\{t : \max_{u \in V} ||P^t(u, \cdot) - \pi||_{TV} < 1/e\right\}.$$

Random Walk on $\mathscr{G}_n$ is rapid mixing

Denote by T_{mix} the mixing time of lazy random walk on $\mathscr{G}_n$, then we have

Throrem 2 *Suppose $r > r_c$. Then,*

(i) for any $0 < \alpha < \beta < 1/2$, $\zeta \geq d-1$ and for $\sigma > 0$ small enough, there exists constant $C_2 > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}(T_{\text{mix}} \geq C_2 \lg n) = 1;$$

(ii) for $0 < \alpha < \beta < 1/2$ with $(2\beta)^d - (2\alpha)^d > 1/2$, $\zeta \geq 0$ and $\sigma > 0$, there exists constant $C_3 > 0$ such that for any $\nu > 0$,

$$\lim_{n\to\infty} \mathbb{P}(T_{\text{mix}} \leq C_3 \lg^{4\zeta+5+\nu} n) = 1.$$

1. Small world effect

There is a known result in algebraic graph theory, see N. Alon and V. D. Milman (1985) and A. Berman and X. D. Zhang (2000), that

Lemma 1 *For any connected graph $G = (V, E)$, let $\Delta(G)$ denote its maximum degree, $\iota(G)$ denote its edge isoperimetric constant and let $\mathrm{diam(G)}$ denote its diameter. Then*

$$\mathrm{diam(G)} \le \frac{4\Delta(G)}{\iota(G)} \lg |V|.$$

The *edge isoperimetric constant* $\iota(G)$ is defined by

$$\iota(G) := \min_{S:|S|\le|V|/2} \frac{e(S, S^c)}{|S|}.$$

Where $e(S, S^c)$ be the number of edges between S and S^c.

Theorem 1 follows from Lemma 1, Proposition 2 and the following Lemmas 2 and 3.

Lemma 2 *Suppose $r > r_c$. Then*

$$\lim_{n\to\infty} \mathbb{P}(\Delta(\mathscr{G}_n) \leq \lg n) = 1.$$

Lemma 3 Suppose $r > r_c$, $0 < \alpha < \beta < 1/2$ with $(2\beta)^d - (2\alpha)^d > 1/2$, $\zeta \geq 0$ and $\sigma > 0$. Then there exists $C_4 > 0$ such that for any $\nu > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\iota(\mathscr{G}_n) \geq C_4 \mathrm{lg}^{-[1+\zeta+\nu]}\, n\right) = 1.$$

2. Lower bound of the mixing time

To bound the mixing time from below, it suffices to prove that, with high probability (tends to 1 as $n \to \infty$), there exists a box B in T_n^d satisfying

1. The side length of B is $K \lg n$;

2. If we denote B' as the box with side length $K \lg n/2$ and centered at the same center of B. Then $B' \cap C_{\max} \neq \phi$. Note that $C_{\max}$ is the vertex set of $\mathscr{G}_n$;

3. There is no long edge between B and B^c.

Then the random walk started in B' can not escape from B in time $K \lg n/2r$.

3. Upper bound of the mixing time

Conductance: For the lazy random walk $\{X_t : t \geq 0\}$ on $\mathscr{G}_n$, let $Q(u,v) := \pi(u)P(u,v)$ and $Q(S,S^c) := \sum_{u\in S}\sum_{v\in S^c} Q(u,v)$. Define

$$h := \min_{S:\pi(S)\leq 1/2} \frac{Q(S,S^c)}{\pi(S)}$$

to be the *conductance* of $\{X_t : t \geq 0\}$. Letting $e(S,S^c)$ be the number of edges between S and S^c, we have

$$h = \frac{1}{2} \min_{S:\pi(S)\leq 1/2} \frac{e(S,S^c)}{\mathrm{Vol}(S)},$$

where $\mathrm{Vol}(S) = \sum_{u\in S} d_{\mathscr{G}_n}(u)$.

For our lazy random walk $\{X_t : t \geq 0\}$ on $\mathscr{G}_n$, matrix theory tell us that the transition kernel (P(u,v)) has nonnegative real eigenvalues

$$1 = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{|C_{\max}|-1} \geq 0.$$

Note that $1 - \lambda_1$ is called the spectral gap of $(P(u,v))$. Let $\pi_{\min} = \min_{u \in C_{\max}} \pi(u)$.

It was given in A. Sinclair and M. Jerrum (1989) that

$$(*) \qquad \sup_{u \in C_{\max}} ||P^t(u,\cdot) - \pi||_{TV} \leq \frac{\lambda_1^t}{\pi_{\min}}.$$

On the other hand the spectral gap $1 - \lambda_1$ can be bounded from above and below by the conductance h in the following way (see Theorem 6.2.1] in R. Durrett (2007)),

$$(**) \qquad \frac{h^2}{2} \leq 1 - \lambda_1 \leq 2h.$$

By equations (*) and (**) and the definition of mixing time, the upper bound for T_{mix} follows from the following lemma.

Lemma 4 *Suppose* $r > r_c$, $0 < \alpha < \beta < 1/2$ *with*

$$(2\beta)^d - (2\alpha)^d > 1/2,$$

$\zeta \geq 0$ *and* $\sigma > 0$. *Then there exists* $C_5 > 0$ *such that for any* $\nu > 0$

$$\lim_{n \to \infty} \mathbb{P}\left(h \geq C_5 \mathrm{lg}^{-[2(\zeta+1)+\nu]} n\right) = 1.$$

Geometry of $\bar{G}_n$

With high probability, we have

$$1).\ (1-\epsilon)\theta(r) \leq n^{-d}|C_{\max}| \leq (1+\epsilon)\theta(r),$$

$$2).\ (1-\epsilon)\Gamma \leq n^{-d}|\Lambda_n(u)| \leq (1+\epsilon)\Gamma \text{ for all } u \in C_{\max},$$

$$3).\ \Delta(\bar{G}_n) \leq \lg n/2.$$

Where $\Gamma = [(2\beta)^d - (2\alpha)^d]\theta(r)$ and

$$\Lambda_n(u) = \left\{v \in C_{\max} : \alpha n \leq d_\infty^T(u,v) \leq \beta n\right\}.$$

The proof of Lemma 4 follows from the following two lemmas.

Lemma 5 *Let* $a = a(n) = \lg^{-(1+\zeta)} n$. *Then*

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcap_{S:|S|\geq(1-a)|C_{\max}|} \left\{ \pi(S) > \frac{1}{2} \right\} \right) = 1.$$

Let

$$\hat{\imath}(\mathscr{G}_n) = \min_{S:|S|\leq(1-a)|C_{\max}|} \frac{e(S,S^c)}{\mathrm{Vol}(S)}.$$

Then by Lemma 5,

$$\mathbb{P}\left(h \geq \hat{\imath}(\mathscr{G}_n)/2\right) \to 1, \quad \text{as } n \to \infty.$$

So, to bound h from below, it suffices to bound $\hat{\imath}(\mathscr{G}_n)$ from bellow. In fact, we have

Lemma 6 *Suppose* $r > r_c$, $0 < \alpha < \beta < 1/2$ *with*

$$(2\beta)^d - (2\alpha)^d > 1/2,$$

$\zeta \geq 0$ *and* $\sigma > 0$. *Then there exists* $C > 0$ *such that for any* $\nu > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\hat{\imath}(\mathscr{G}_n) \geq C \lg^{-[2(\zeta+1)+\nu]} n\right) = 1.$$

The sketch of the proof of Lemma 6

Let

$$\mathcal{B}_1 := \{S \subset C_{\max} : |S| \leq M/a\}, \text{ and}$$

$$\mathcal{B}_2 := \{S \subset C_{\max} : M/a < |S| \leq a|C_{\max}|\}, \text{ and}$$

$$\mathcal{B}_3 := \{S \subset C_{\max} : a|C_{\max}| < |S| \leq (1-a)|C_{\max}|\},$$

where $M > 0$ is a large constant and $a = a(n) = \lg^{-(\zeta+1)} n$.

We will prove Lemma 6 for $S \in \mathcal{B}_i,\ i = 1, 2, 3$ respectively. Let

$$\mathcal{B} = \bigcup_{i=1}^{3} \mathcal{B}_i.$$

For any $S \in \mathcal{B}$,

$$\frac{e(S,S^c)}{\mathrm{Vol}(S)} \geq \frac{c(S)+L(S,S^c)}{\Delta(\bar{G}_n)|S|+2L(S)}$$

Where

$c(S)$ is the number of connected components of S in $\bar{G}_n$.

$L(S,S^c)$ is the number of "long edges" between S and S^c.

$L(S)$ is the number of "long edges" associated to S.

Clearly

$$L(S) = L(S,S^c) + L(S,S).$$

Let $N(S, S^c)$ be the number of non-ordered pairs $u \in S, v \in S^c := C_{\max} \setminus S$ such that $u \in \Lambda_n(v)$, $N(S, S)$ be the number of non-ordered pairs $u, v \in S$ such that $u \in \Lambda_n(v)$. Let

$$N(S) = N(S, S^c) + N(S, S).$$

Then $L(S, S^c)$ is independent to $L(S, S)$, and

$$L(S, S^c) \sim b(N(S, S^c), p_n), \quad L(S, S) \sim b(N(S, S), p_n),$$

$$L(S) \sim b(N(S), p_n).$$

Large deviation inequality for binomial distribution.

Lemma 7:*Suppose $Z \sim b(n,p)$. Then*

$$\mathbb{P}(Z \geq zn) \leq \exp(-I(z)n),\ z > p;\ \ \mathbb{P}(Z \leq zn) \leq \exp(-I(z)n),\ z < p,$$

where $I(z)$ is the common rate function defined by

$$I(z) := z \lg \frac{zq}{(1-z)p} - \lg \frac{q}{1-z},\ \ p \neq z \in (0,1).$$

Especially for small p, the above inequalities can be rewritten as

$$\mathbb{P}(Z \geq zpn) \leq \exp(-\gamma(z)pn)\ \mathit{for}\ z > 1,\ \text{ and}$$

$$\mathbb{P}(Z \leq zpn) \leq \exp\left(-\tfrac{1}{2}\gamma(z)pn\right)\ \ \mathit{for}\ 0 < z < 1,$$

with $\gamma(z) = z \lg z - z + 1$.

Now, let

$$\mathcal{B}_3 = \mathcal{B}_3^{>b} \cup \mathcal{B}_3^{<b}$$

Where $\mathcal{B}_3^{>K}$ is the set of S in $\mathcal{B}_3$ such that $c(S) \geq |S|/\lg^b n$ and $\mathcal{B}_3^{<b} = \mathcal{B}_3 \setminus \mathcal{B}_3^{>b}$.

By the large deviation inequality, we have

$$\lim_{n\to\infty} \mathbb{P}_\epsilon \left(L(S) \leq |S| \lg n,\ \forall\ S \in \mathcal{B}_3\right) = 1.$$

Where $\mathbb{P}_\epsilon$ is the conditional probability function that the three items of the geometry of $\bar{G}_n$ hold. This implies

$$\lim_{n\to\infty} \mathbb{P}_\epsilon \left\{ \frac{e(S,S^c)}{\mathrm{Vol}(S)} \geq \frac{2/3}{\lg^{b+1} n},\ \forall\ S \in \mathcal{B}_3^{>b} \right\} = 1.$$

Now, let's choose ϵ small enough such that

$$\frac{1-\epsilon}{(1+\epsilon)\tilde{\theta}(r)}\Gamma = \frac{(1-\epsilon)[(2\beta)^d - (2\alpha)^d]}{(1+\epsilon)} > 1/2.$$

For any $S \in \mathcal{B}_3$, if $a|C_{\max}| \leq |S| \leq |C_{\max}|/2$, then

$$\begin{aligned} N(S,S^c) &= \sum_{u\in S} \sum_{v\in S^c \cap \Lambda_n(u)} 1 \geq |S| \left(\min_{u\in S} |\Lambda_n(u)| - |S| \right) \\ &\geq |S| \left(\frac{1-\epsilon}{(1+\epsilon)\theta(r)} \Gamma |C_{\max}| - |S| \right) \\ &\geq a|C_{\max}| \left(\frac{1-\epsilon}{(1+\epsilon)\theta(r)} \Gamma |C_{\max}| - a|C_{\max}| \right) \\ &\geq a \left(\frac{1-2\epsilon}{(1+\epsilon)\theta(r)} \Gamma \right) |C_{\max}|^2. \end{aligned}$$

On the other hand, if $a|C_{\max}| \le |S^c| \le |C_{\max}|/2$, then

$$\begin{aligned} N(S,S^c) &= N(S^c,S) = \sum_{u\in S^c}\sum_{v\in S\cap\Lambda_n(u)} 1 \ge |S^c|\left(\min_{u\in S^c}|\Lambda_n(u)| - |S^c|\right) \\ &\ge a|C_{\max}|\left(\frac{1-\epsilon}{(1+\epsilon)\theta(r)}\Gamma|C_{\max}| - a|C_{\max}|\right) \\ &\ge a\left(\frac{1-2\epsilon}{(1+\epsilon)\theta(r)}\Gamma\right)|C_{\max}|^2. \end{aligned}$$

So

$$\begin{aligned} N(S) &\le (1+\epsilon)\Gamma n^d|S| \le (1+\epsilon)\Gamma n^d(1-a)|C_{\max}| \\ &\le \frac{(1+\epsilon)\Gamma(1-a)}{(1-\epsilon)\theta(r)}|C_{\max}|^2 \le \frac{(1+\epsilon)\Gamma}{(1-\epsilon)\theta(r)}|C_{\max}|^2 \\ &\le \frac{1}{a}\cdot\frac{(1+\epsilon)^2}{(1-\epsilon)(1-2\epsilon)}N(S,S^c). \end{aligned}$$

Let $f(\epsilon) = (1-\epsilon)(1-2\epsilon)/(1+\epsilon)^2$, then

$$N(S, S^c) \geq f(\epsilon) a N(S).$$

By the large deviation inequality, we have

$$\begin{aligned}
&\mathbb{P}_\epsilon \left(L(S, S^c) \geq f(\epsilon) a L(S)/4 \right) \\
&\geq \mathbb{P}_\epsilon (L(S, S^c) \geq \tfrac{1}{2} f(\epsilon) a N(S) p_n \text{ and } L(S) \leq 2N(S) p_n) \\
&\geq 1 - \exp(-\tfrac{1}{4}\sigma\gamma(1/2) f(\epsilon)(1-\epsilon)\Gamma |S| \lg^{-(2\zeta+1)} n) \\
&\qquad\qquad - \exp(-\tfrac{1}{2}\sigma\gamma(2)(1-\epsilon)\Gamma |S| \lg^{-\zeta} n) \\
&\geq 1 - \exp\left(-C(\sigma, \epsilon) |S| \lg^{-(2\zeta+1)} n \right).
\end{aligned}$$

On the other hand, by the large deviation inequality, we have

$$\begin{aligned}\mathbb{P}_\epsilon\left(L(S) \geq \mu|S|\right) &\geq \mathbb{P}\left(Z_2(S) \geq \frac{2\tau}{\sigma(1-\epsilon)\Gamma} N_2(S) p_n\right) \\ &\geq 1 - \exp\left(-O\left(|S| \lg^{-\zeta} n\right)\right).\end{aligned}$$

Where $\mu = \mu(n) = \tau \lg^{-\zeta} n$ with $2\tau/\sigma(1-\epsilon)\Gamma < 1$, $N_2(S) = \frac{1}{2}(1-\epsilon)\Gamma n^d |S|$ and

$$Z_2(S) \sim b(N_2(S), p_n).$$

Note that $L(S,S^c) \geq f(\epsilon)aL(S)/4$ and $L(S) \geq \mu|S|$ imply that

$$\frac{e(S,S^c)}{\mathrm{Vol}(S)} \geq \frac{L(S,S^c)}{|S|\Delta(\bar{G}_n)+2L(S)} \geq \frac{f(\epsilon)aL(S)/4}{|S|\lg n/2+2L(S)}$$

$$\geq \begin{cases} \dfrac{\tau \lg^{-\zeta} n f(\epsilon)a|S|/4}{|S|\lg n/2+2|S|} \geq \dfrac{C_6''}{\lg^{2(\zeta+1)} n}, & \text{if } L(S) \leq |S|; \\ \dfrac{f(\epsilon)aL(S)/4}{L(S)\lg n/2+2L(S)} \geq \dfrac{C_6'''}{\lg^{\zeta+2} n}, & \text{if } |S| \leq L(S). \end{cases}$$

Thus, for any $S \in \mathcal{B}_3$

$$\mathbb{P}_\epsilon\left(\frac{e(S,S^c)}{\mathrm{Vol}(S)} \geq \frac{C_6''}{\lg^{2(\zeta+1)} n}\right) \geq 1-\exp\left(-C(\sigma,\epsilon)|S|\lg^{-(2\zeta+1)} n\right).$$

Finally, for any $a|C_{\max}| \le s \le (1-a)|C_{\max}|$, let $\mathcal{B}_{3,s}^{<b}$ denote the set of $S \in \mathcal{B}_3^{<b}$ with $|S| = s$. For any $1 \le j \le s/\lg^b n =: j_s$, let $\mathcal{B}_{3,s,j}^{<b}$ denote the set of $S \in \mathcal{B}_{3,s}^{<b}$ such that $c(S) = j$. Then

$$
\begin{aligned}
|\mathcal{B}_{3,s,j}^{<b}| &\le \binom{|C_{\max}|}{j}\binom{s-1}{j-1} \le \binom{|C_{\max}|}{j}\binom{s}{j} \\
&\le \binom{|C_{\max}|}{j_s}\binom{s}{j_s} \le \left(\frac{|C_{\max}|e}{j_s}\right)^{j_s} \cdot \left(\frac{se}{j_s}\right)^{j_s} \\
&\le \exp\left(j_s\left\{\lg\left(\frac{|C_{\max}|}{s}\right) + 2\lg\left(\frac{s}{j_s}\right) + 2\right\}\right) \\
&\le \exp\left(D \lg\lg n \cdot s \lg^{-b} n\right).
\end{aligned}
$$

Then

$$|\mathcal{B}_{3,s}^{<b}| \le s \cdot \exp\left(D \lg\lg n \cdot s \lg^{-b} n\right).$$

Provided $b = 2\zeta + 1 + \nu$ with $\nu > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}_\epsilon \left(\frac{e(S, S^c)}{\mathrm{Vol}(S)} \geq \frac{C_7'}{\lg^{2(\zeta+1)} n}, \ \forall \ S \in \mathcal{B}_3^{<b} \right) = 1.$$

Thus, we finish the proof of the lemma.

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Thanks for your attention!