Maximum principles for parabolic Waldenfels operators

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Outline

1. Lévy type generators
2. Weak and strong maximum principles
3. Boundary point lemma for elliptic Waldenfels operators

Based on a joint work with Jinqiao Duan.

A fairly large class of Markov processes on $\mathbb{R}^d$ are governed by Lévy type generators, either via martingale problem (cf e.g. D W Stroock, “Markov Processes from K. Itô’s Perspectives”, Princeton Univ Press 2003 or V.N. Kolokoltsov, “Markov Processes, Semigroups and Generators”, de Gruyter, 2011)

$$\begin{aligned} Lu(x) \quad := \quad & \frac{1}{2}\sum_{i,j} a_{i,j}(x)\partial_i\partial_j u(x) + b(x)\nabla u(x) \\ & + \int_{\mathbb{R}^d\setminus\{0\}} \{u(x+z) - u(x) - \frac{z\nabla u(x)}{1+|z|^2}\}\nu(x,dz) \end{aligned}$$

for $u \in Dom(L)$, where $a(x) = (a_{i,j}(x))_{d\times d}$ is non-negative definite symmetric, $b(x)$ is d-dimensional, and $\nu(x, dz)$ is a Lévy kernel, i.e., $\forall x \in \mathbb{R}^d$, $\nu(x, \cdot)$ is a σ-finite measure on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$ such that

$$\int_{\mathbb{R}^d \setminus \{0\}} \frac{|z|^2}{1 + |z|^2} \nu(x, dz) < \infty.$$

Such L is linked to a jump SDE (cf. e.g. Bennett, Wu: Stoch Anal Appl 08)

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t \\ &\quad + \int_{U \setminus U_0} c_1(X_{t-}, z)\tilde{N}(dt, dz) \\ &\quad + \int_{U_0} c_2(X_{t-}, z)N(dt, dz). \end{aligned}$$

The operator $(L, Dom(L))$ is referred as an integro-differential operators or pseudo-differential operators associated with negative definite symbols, consisting of a combination of second order elliptic differential operators with integral operators of Lévy type. Moreover, Lévy type generators fulfill the (global) positive maximum principle in the sense that for any $u \in Dom(L)$

$$0 \leq u(x_0) = \sup_{x \in \mathbb{R}^d} u(x) \text{ for some } x_0 \in \mathbb{R}^d \Longrightarrow Lu(x_0) \leq 0 .$$

Further, it's known that the positive maximum principle implies the dissipativity of the operator $(L, Dom(L))$, which is a crucial condition of the Hille-Yosida theorem for $(L, Dom(L))$ to generate a strongly contraction semigroup on certain function spaces over $\mathbb{R}^d$. The Hille-Yosida theorem induces the semigroup approach towards constructing Markov processes on $\mathbb{R}^d$ which makes an intrinsic link between Markov processes and second order elliptic integro-differential operators. In the same vein, K. Taira [*Semigroups, Boundary Value Problems and Markov Processes*. Springer Monographs in Mathematics, 2004] explores the functional analytic approach to constructing Markov processes in a prescribed region $D \subset \mathbb{R}^d$ via the elliptic boundary value problems for associated Lévy-type generators.

Due to pseudo-differential operators (involving integral operators) in its nature, the Lévy-type generators are nonlocal operators. Thus, the boundary conditions are significantly different from those for second order elliptic partial differential operators. In this context, the second order elliptic Waldenfels operators appear naturally towards the well posedness of the boundary value problems. Such kind of inegro-differential operators was initiated by W. von Waldenfels in 1964. It was elucidated by Taira that a Markov process obtained in this manner could be interpreted with a physical picture that a Markovian particle moves both by jumps and continuously in certain region of the state space $\mathbb{R}^d$ until it reaches the (boundary) set where the particle is definitely absorbed or it jumps outside the region forever.

Let $D \subset \mathbb{R}^d$ be a bounded, connected domain with smooth boundary ∂D with closure $\bar{D} := D \cup \partial D$. A second order (elliptic) Waldenfels operator W is defined as

$$
\begin{aligned}
Wu(x) &= Au(x) + Ku(x) \qquad (1)\\
&:= \sum_{j,l=1}^{d} a_{j,l}(x) \frac{\partial^2 u(x)}{\partial x_j \partial x_k} + b(x)\nabla u(x) + c(x)u(x) \\
&\quad + \int_{\mathbb{R}^d \setminus \{0\}} \big[u(x+z) - u(x) - z\nabla u(x)\big] k(x,z)\nu(dz)
\end{aligned}
$$

where

1. $a_{j,l} \in C^\infty(\bar{D})$, $a_{j,l}(x) = a_{l,j}(x)$, $\exists\, \lambda > 0$ such that
$$\sum_{j,l=1}^{d} a_{j,l}(x) z_j z_l \geq \lambda |z|^2$$
2. $b_j \in C^\infty(\bar{D}), c \in C^\infty(\bar{D}), c(x) \leq 0, \forall x \in \bar{D}$, but $c \not\equiv 0$.
3. $k : \bar{D} \times \mathbb{R}^d \to [0,1]$ is continuous, and $\exists C > 0$ and $\theta \in (0,1)$ such that
$$|k(x,z) - k(y,z)| \leq C|x-y|^\theta, \quad x, y \in \bar{D},\ z \in \mathbb{R}^d$$
$$k(x,z) = 0, \quad \text{if } x + z \notin \bar{D}\,.$$
4. The measure ν is a Radon measure on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$ satisfying the following moment condition
$$\int_{0<|z|\leq 1} |z|^2 \nu(dz) + \int_{|z|>1} |z| \nu(dz) < \infty\,.$$

Based on the above, we introduce the following time inhomogeneous operator

$$
\begin{aligned}
Lu(t,x) &= Au(t,x) + Ku(t,x) \qquad (2)\\
&:= \sum_{j,l=1}^{d} a_{j,l}(t,x)\frac{\partial^2 u(t,x)}{\partial x_j \partial x_k} + b(t,x)\nabla u(t,x) + c(t,x)u(t,x)\\
&\quad + \int_{\mathbb{R}^d\setminus\{0\}} \Big[u(t,x+z) - u(t,x) - \frac{z\nabla u(t,x)}{1+|z|^2}\Big]\nu(t,x,dz)
\end{aligned}
$$

with

1. $a_{j,l} \in C^{1,\infty}([0,\infty) \times \bar{D}), a_{j,l}(t,x) = a_{l,j}(t,x), \exists \lambda > 0$ s.t.

$$\sum_{j,l=1}^{d} a_{j,k}(t,x) z_j z_l \geq \lambda |z|^2 \quad \forall (t,x,z) \in [0,\infty) \times \bar{D} \times \mathbb{R}^d$$

2. $b_j \in C^{1,\infty}([0,\infty) \times \bar{D}), c \in C^{1,\infty}([0,\infty) \times \bar{D}), c(t,x) \leq 0, \forall (t,x) \in [0,\infty) \times \bar{D}$, but $c \not\equiv 0$.
3. $\nu(t,x,dz)$ is measurable on $(t,x) \in [0,\infty) \times \bar{D}$, and $\forall (t,x) \in [0,\infty) \times \mathbb{R}^d$,$\nu(t,x,\cdot)$ is a Lévy measure for $(t,x) \in [0,\infty) \times \bar{D}$. Here we further assume that the measure $\nu(t,x,dz)$ is supported in $\bar{D} - x := \{y - x : y \in \bar{D}\}$, i.e.

$$\text{supp}\nu(t,x,\cdot) = \bar{D} - x$$

which indicates that a Markovian particle cannot move by jumps from a point $x \in D$ to the outside of $\bar{D}$.

In what follows, we introduce parabolic Waldenfels operators related to the time inhomogeneous elliptic Waldenfels operators. Our parabolic Waldenfels operator is then defined as

$$-\frac{\partial}{\partial t} + L$$

and we are concerned with the maximum principles for such parabolic operators.

let $T > 0$ be arbitrarily fixed. Set further that

$$D_T := (0, T] \times D, \quad \partial^* D_T := \partial D_T \setminus \big(\{T\} \times D\big).$$

Notice that $\partial^* D_T$ is nothing but the time space boundary which can have a clear picture if for instance in one space dimensional case $D \subset \mathbb{R}$, one simply takes $D = (0, h)$ for some fixed $h > 0$. Then $D_T = (0, T] \times (0, h)$ and $\partial^* D_T$ consists of three edges of the rectangle $[0, T] \times [0, h]$ except the edge $\{T\} \times [0, h]$, i.e.,

$$\partial^* D_T = (\{0\} \times [0, h]) \cup ([0, T] \times \{0\}) \cup ([0, T] \times \{h\}).$$

Theorem

(The weak maximum principle) *Let* $u \in C^{1,2}(D_T) \cap C(\overline{D_T})$.

1. *If* $-\frac{\partial u}{\partial t} + Lu > 0$ *in* D_T, *then* u *may take its nonnegative maximum only on* $\partial^* D_T$.
2. *If* $c(t,x) < 0$ *and* $-\frac{\partial u}{\partial t} + Lu \geq 0$ *in* D_T, *then* u *may take its positive maximum only on* $\partial^* D_T$. *Consequently,*

$$\max_{(t,x)\in D_T} u(t,x) \leq \max_{(t,x)\in\partial^* D_T} u^+(t,x)$$

where, as usual, $u^+ := max\{u,0\}, u^- := max\{-u,0\}$ *and* $u = u^+ - u^-$.

Proof We assume the contrary that there is a point $(t_0, x_0) \in D_T$ such that $u(t_0, x_0)$ is the nonnegative maximum of u, that is,

$$0 \leq u(t_0, x_0) = \sup_{(t,x)\in D_T} u(t,x) = \max_{(t,x)\in \overline{D_T}} u(t,x) .$$

Then we have

$$\frac{\partial u(T, x_0)}{\partial t} \geq 0, \quad \text{and} \quad \frac{\partial u(t_0, x_0)}{\partial t} = 0 \quad \text{for } 0 < t_0 < T$$

and

$$\frac{\partial u(t_0, x_0)}{\partial x_j} = 0, \text{ for } 1 \leq j \leq d, \text{ with } \sum_{j,l=1}^{d} a_{j.l}(t_0, x_0) \frac{\partial^2 u(t_0, x_0)}{\partial x_j \partial x_l} \leq 0$$

and

$$\int_{\mathbb{R}^d \setminus \{0\}} [u(t_0, x_0 + z) - u(t_0, x_0)] \nu(t_0, x_0, dz) \leq 0 .$$

Hence we get

$$-\frac{\partial u(t_0,x_0)}{\partial t} + Lu(t_0,x_0) \le c(t_0,x_0)u(t_0,x_0). \qquad (3)$$

Now for Assertion 1, we have $-\frac{\partial u}{\partial t} + Lu > 0$ in D_T which gives that

$$-\frac{\partial u(t_0,x_0)}{\partial t} + Lu(t_0,x_0) > 0\,.$$

On the other hand, we have by (3)

$$-\frac{\partial u(t_0,x_0)}{\partial t} + Lu(t_0,x_0) \le c(t_0,x_0)u(t_0,x_0) \le 0$$

as $c(t_0,x_0) \le 0$ and $u(t_0,x_0) \ge 0$. Thus, we end with a contradiction

$$0 < -\frac{\partial u(t_0,x_0)}{\partial t} + Lu(t_0,x_0) \le c(t_0,x_0)u(t_0,x_0) \le 0\,.$$

Hence, u may take its nonnegative maximum only on $\partial^* D_T$. An immediate consequence is then that

$$-\frac{\partial u}{\partial t} + Lu \geq 0 \text{ in } D_T, \max_{(t,x)\in D_T} u(t,x) \geq 0$$

$$\Rightarrow \max_{(t,x)\in D_T} u(t,x) \leq \max_{(t,x)\in\partial^* D_T} u(t,x).$$

Towards our Assertion 2, we note that $u^+(t_0, x_0) > 0$ is the positive maximum of u. By (3), we have the following derivation

$$
\begin{aligned}
0 &\leq -\frac{\partial u(t_0, x_0)}{\partial t} + Lu(t_0, x_0) \\
&\leq c(t_0, x_0)u(t_0, x_0) = c(t_0, x_0)u^+(t_0.x_0) - c(t_0, x_0)u^+(t_0.x_0) \\
&\leq c(t_0, x_0)u^+(t_0.x_0) \\
&< 0
\end{aligned}
$$

since $c(t_0, x_0) \leq 0$ and $u^+(t_0, x_0) > 0$. This is yet again a contradiction. Thus, u may take its positive maximum only on $\partial^* D_T$ and consequently,

$$
\max_{(t,x)\in D_T} u(t,x) \leq \max_{(t,x)\in\partial^* D_T} u^+(t,x)\,.
$$

This completes the proof of our Theorem 1. □

Some consequences of the weak maximum principle. Let u solve

$$\frac{\partial u(t,x)}{\partial t} = Lu(t,x).$$

1. Applying weak MP to u and $-u$ yields
$$\max_{\overline{D_T}} |u| \leq \max_{\partial^* D_T} |u|.$$
2. If $\max_{\overline{D_T}} c(t,x) \leq \beta < 0$, then applying weak MP to $\pm u e^{-\beta t}$ yields
$$\max_{\overline{D_T}} |u| \leq e^{\beta T} \max_{\partial^* D_T} |u|.$$
3. Let $v := u e^{\max_{\overline{D_T}} c(t,x)}$. Applying weak maximum principle to $\pm v$, one can derive the uniqueness of
$$-\frac{\partial u(t,x)}{\partial t} + Lu(t,x) = f(t,x)$$
for given f, initial $u(0,x)$ and $u|_{(0,T]\times\partial D}$.

Comparison principle

Let $f : [0, T] \times D \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfies one-sided (uniform) Lipschitz condition

$$f(t,x,u_1) - f(t,x,u_2) \leq C(u_1 - u_2), \quad \text{for } u_1 > u_2.$$

If $u, v \in C^{1,2}(D_T) \cap C(\overline{D_T})$ such that

$$-\frac{\partial v(t,x)}{\partial t} + Lv(t,x) + f(t,x,v(t,x)) \leq 0$$

$$-\frac{\partial u(t,x)}{\partial t} + Lu(t,x) + f(t,x,u(t,x)) \geq 0$$

in D_T and $u \leq v$ on ∂D_T, then $u \leq v$ in D_T.

Theorem

(The strong maximum principle) *Let D be open, bounded, and connected in $\mathbb{R}^d$. Let $u \in C^{1,2}(D_T) \cap C(\overline{D_T})$ such that*

$$-\frac{\partial u}{\partial t} + Lu \geq 0 \text{ in } D_T\,.$$

1. *Let $c \equiv 0$. If u attains maximum at an interior point of D_T, then u is a constant in D_T.*
2. *For the general case of $c(t,x) \leq 0, \forall (t,x) \in D_T$, if u attains a non-negative maximum at an interior point of D_T, then u is a constant in D_T.*

Proof Let $(t_0, x_0) \in D_T$ be a point such that $u(t_0, x_0)$ attains its maximum

$$u(t_0, x_0) = \max_{(t,x)\in\overline{D_T}} u(t, x).$$

We assume the contrary that u is not a constant in D_T. Define

$$S := \{(t, x) \in D_T : u(t, x) = u(t_0, x_0)\} \subset D_T = (0, T] \times D \subset \mathbb{R}^{1+d}.$$

Then, $S \neq D_T$ unless $u \equiv u(t_0, x_0)$ is a constant in D_T. Thus, $S \subsetneqq D_T$. Notice that $u \in C^{1,2}(D_T) \cap C(\overline{D_T})$, so the set S is closed in D_T.
Before proceeding further, let us compute $-\frac{\partial u(t_0,x_0)}{\partial t} + Lu(t_0, x_0)$.
We have on one hand

$$-\frac{\partial u(t_0, x_0)}{\partial t} + Lu(t_0, x_0) \geq 0 \tag{4}$$

which is clear from the assumption on u. On the other hand,we have

$$\frac{\partial u(T, x_0)}{\partial t} \geq 0, \quad \text{and} \quad \frac{\partial u(t_0, x_0)}{\partial t} = 0 \quad \text{for } 0 < t_0 < T$$

$$\frac{\partial u(t_0, x_0)}{\partial x_j} = 0, \ \text{for } 1 \leq j \leq d, \ \text{with} \ \sum_{j,l=1}^{d} a_{j.l}(t_0, x_0) \frac{\partial^2 u(t_0, x_0)}{\partial x_j \partial x_l} \leq 0$$

$$Ku(t_0, x_0) = \int_{\mathbb{R}^d \setminus \{0\}} [u(t_0, x_0 + z) - u(t_0, x_0)] \nu(t_0, x_0, dz) \leq 0 \,.$$

Hence we get

$$\begin{aligned} & -\frac{\partial u(t_0, x_0)}{\partial t} + Lu(t_0, x_0) \\ \leq \ & Lu(t_0, x_0) = Au(t_0, x_0) + Ku(t_0, x_0) \\ \leq \ & Au(t_0, x_0) \leq c(t_0, x_0) u(t_0, x_0) \leq 0 \qquad (5) \end{aligned}$$

since $c(t_0, x_0) \leq 0$ and $u(t_0, x_0) \geq 0$.

Combining (4) and (5) we have

$$-\frac{\partial u(t_0,x_0)}{\partial t}+Lu(t_0,x_0)=0$$

and further

$$Au(t_0,x_0)=Ku(t_0,x_0)=0$$

due to both $Au(t_0,x_0)\leq 0$ and $Ku(t_0,x_0)\leq 0$. On the other hand, $\nu(t_0,x_0,\cdot)$ is supported on $\bar{D}-x_0$, we then have

$$0=Ku(t_0,x_0)=\int_{\bar{D}-x_0}[u(t_0,x_0+z)-u(t_0,x_0)]v(t_0,x_0,dz).$$

Recall that $S=\{(t,x)\in D_T : u(t,x)=u(t_0,x_0)\}$, then we have the following decomposition of $\bar{D}-x_0$

$$\{z\in\mathbb{R}^d\backslash\{0\}:(t_0,z)\in S-(t_0,x_0)\}\cup\{z\in\mathbb{R}^d\backslash\{0\}:(t_0,z)\notin S-(t_0,x_0)\}$$

Notice that

$$u(t_0,x_0+z)=u(t_0,x_0),\forall(t_0,x_0+z)\in S$$

Equivalently

$$u(t_0, x_0+z) = u(t_0, x_0), \forall z \in \{z \in \mathbb{R}^d \setminus \{0\} : (t_0, z) \in S-(t_0, x_0)\}$$

$$u(t_0, x_0+z) < u(t_0, x_0), \forall z \notin \{z \in \mathbb{R}^d \setminus \{0\} : (t_0, z) \in S-(t_0, x_0)\}.$$

Hence,

$$\begin{aligned} & Ku(t_0, x_0) \\ = & \int_{\{z\in\mathbb{R}^d\setminus\{0\}:(t_0,z)\notin S-(t_0,x_0)\}} [u(t_0, x_0 + z) - u(t_0, x_0)] v(t_0, x_0, dz). \end{aligned}$$

Now in order that $Ku(t_0, x_0) = 0$, the following must hold

$$\nu(t_0, x_0, \{z \in \mathbb{R}^d \setminus \{0\} : (t_0, z) \notin S - (t_0, x_0)\}) = 0. \quad (6)$$

Next, since S is closed in D_T, we can choose a point $(s, z) \in D_T \setminus S$ such that the following open set B is entirely contained in $D_T \setminus S$

$$B := (t_0 \wedge s, t_0 \vee s) \times \{x \in D : |x - z| < r\} \subset D_T \setminus S\,.$$

with $r := |x_0 - z|$, where as usual $t_0 \wedge s := \min\{t_0, s\}$, $t_0 \vee s := \max\{t_0, s\}$. Obviously, we have $(t_0, x_0) \in \partial B$. Moreover, we set a function

$$w(t, x) := \exp\{-\theta(t + |x - z|^2)\} - \exp\{-\theta(t + |x_0 - z|^2)\}$$

that is

$$w(t, x) = e^{-\theta t}\left(e^{-\theta(|x-z|^2} - e^{-\theta r^2}\right)$$

where $\theta > 0$ is a constant to be determined later.

Clearly, we have the following

$$w(t,x) > 0, \quad \forall (t,x) \in B$$

$$w(t,x) = 0, \quad \forall (t,x) \in \partial B$$

$$w(t,x) < 0, \quad \forall (t,x) \in \overline{D_T} \setminus \bar{B}.$$

In what follows, let us compute $-\frac{\partial w(t_0,x_0)}{\partial t} + Lw(t_0,x_0)$. For the sake of clarity in computation, let us use vector formulation. Denote that

$$a(t,x) := (a_{j.l}(t,x))_{1 \le j,l \le d} \; b(t,x) := (b_1(t,x), ..., b_d(t,x))$$

with the diagonal vector of the $d \times d$-matrix $a(t,x)$ by

$$diag\big(a(t,x)\big) := (a_{1,1}(t,x), ..., a_{d,d}(t,x)).$$

We also use τ for the transpose of a matrix and ∇ for the (spatial) gradient operator on $\mathbb{R}^d$. We first note that

$$w(t_0, x_0) = 0$$

$$\frac{\partial w(t_0, x_0)}{\partial t} = -\theta e^{-\theta t_0}\left(e^{-\theta|x_0 - z|^2} - e^{-\theta r^2}\right) = 0$$

$$\nabla w(t_0, x_0) = -2\theta(x_0 - z)e^{-\theta(t_0 + r^2)} \neq 0\,.$$

Hence, by Conditions 1 and 2 of L defined in (2), we can derive

$$\begin{aligned} Aw(t_0, x_0) &= e^{-\theta(t_0+r^2)}\big[4\theta^2(x_0 - z)a(t_0, x_0)(x_0 - z)^\tau \\ &\qquad -2\theta\big(diag(a(t_0, x_0) + b(t_0, x_0))(x_0 - z)^\tau\big)\big] \\ &\geq e^{-\theta(t_0+r^2)}\big[4\lambda\theta^2 r^2 - C_1 r\big] \end{aligned}$$

where

$$0 < C_1 := \max_{(t_0,x_0)\in\overline{D_T}} |diag(a(t_0,x_0) + b(t_0,x_0)| < \infty$$

is a constant independent of r. On the other hand, we have for the integral operator $Kw(t_0,x_0)$

$$\begin{aligned} Kw(t_0,x_0) &= \int_{\{z\in\mathbb{R}^d\setminus\{0\}:(t_0,x_0+z)\in S\}} [w(t_0,x_0+z) - w(t_0,x_0) \\ &\qquad - \frac{z(\nabla w(t_0,x_0))^\tau}{1+|z|^2}]v(t_0,x_0,dz). \end{aligned}$$

We have then

$$|Kw(t_0,x_0)| \le (C_2\theta^2 + C_3\theta)e^{-\theta r^2}$$

so that

$$-\frac{\partial w(t_0, x_0)}{\partial t} Lw(t_0, x_0) \leq Aw(t_0, x_0) - |Kw(t_0, x_0)| > 0$$

by properly chosen θ sufficiently large.
Finally, we define for $\beta > 0$

$$u_\beta(t, x) := u(t, x) + \beta w(t, x).$$

Then choosing $\beta > 0$ properly depending on the above constants C_1, C_2, C_3, there exists a neighborhood B' of (t_0, x_0) such that

$$-\frac{\partial u_\beta(t, x)}{\partial t}_L u_\beta(t, x) > 0 \quad \text{in } B'$$

$$u_\beta \leq u(t, x) \quad \text{on } \partial B'$$

$$u_\beta u(t_0, x_0) = u(t_0, x_0)$$

which contradicts the weak maximum principle (our Theorem 2.1) accordingly. We are done. □

For the elliptic Waldenfels operator

$$\begin{aligned} Wu(x) &:= Au(x) + Ku(x) \qquad (7) \\ &= \sum_{j,l=1}^{d} a_{j,l}(t,x)\frac{\partial^2 u(x)}{\partial x_j \partial x_k} + b\nabla u(x) + c(x)u(x) \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} \left[u(x+z) - u(t,x) - \frac{z\nabla u(x)}{1+|z|^2}\right]\nu(x,dz). \end{aligned}$$

we can recover Hopf's lemma – the boundary point lemma

Theorem

Let W be defined as above. For $u \in C^(\bar{D})$ with $Wu \geq 0$ in D and $\max_{\bar{D}} u \geq 0$, if there exists a point $y \in \partial D$ such that $u(y) = \max_{\bar{D}} u$, then the interior normal derivative satisfies

$$\frac{\partial u(y)}{\partial \mathbf{n}} < 0$$

unless the function u is a constant in D.

Further works

1. Maximum principles for pure jump Waldenfels operators K and $-\frac{\partial}{\partial t} + K$, in particular when $\nu(t, x, dz) = k(t, x, z)dz$, e.g., fractional Laplacian with variable order.

2. Hopf's lemma for parabolic Waldenfels operators.

3. Jump type SDEs with boundary conditions associated with elliptic Waldenfels operators W.

Thank You!