Functional Inequalities on path and loop spaces over Riemannian manifolds

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Outline

Introduction

Framework

Let M be a d-dimensional connected complete Riemannian manifold, and $x \in M$ a fixed reference point. Let $\mathbb{P}_x$ denote the distribution of Brownian motion X. starting at x. Thus, $\mathbb{P}_x$ is a probability measure on the canonical **Path space**:

$$W_x(M) := \{\gamma \in C([0,1]; M) | \gamma(0) = x\}$$

Cameron-Martin space

$$\mathbb{H} := \left\{ h \in C([0,1]; \mathbb{R}^d) \middle| h \text{ is absolutely continuous}, \right.$$
$$\left. h_0 = 0, \| h \|_{\mathbb{H}} = \int_0^1 |h'_s|^2 ds < \infty \right\}$$

Horizontal lift

Let U_t be the horizontal lift of $X_\cdot$; that is,

$$\mathrm{d}U_t = \sum_{i=1}^{d} H_i(U_t) \circ \mathrm{d}B_t^i, \quad t \geq 0,$$

where U_0 is an othornormal basis of $T_x M$, $B_t^1, \cdots, B_t^d$ are independent one dimensional Brownian motions, $\{H_i\}_{i=1}^d$ is the standard othornormal basis of horizontal vector fields.

Cylindrical function

$\mathcal{F}C_b =$ bounded Lipschitz cylindrical functions on $W_x(M)$,

$F \in \mathcal{F}C_b \Longleftrightarrow \exists\ 0 \leqslant t_1 < \cdots < t_n \leqslant 1$ and $f \in C_0^{Lip}(M^n)$

$$F(\gamma) = f(\gamma_{t_1}, \cdots, \gamma_{t_n}), \quad \gamma \in W_x(M).$$

Define

$\rho(\gamma) := \sup_{t\in[0,1]} d_M(\gamma_t, x), B_R := \{\gamma \in W_x(M) | \rho(\gamma) \leqslant R\}$,

$\mathcal{F}C_{b,loc} =$ "local" bounded Lipschitz cylindrical functions on $W_x(M)$,

$$\mathcal{F}C_{b,loc} := \left\{ Fl(\rho) \middle| F \in \mathcal{F}C_b,\ l \in C_0^\infty(\mathbb{R}) \right\}.$$

Gradient operator

For any $F \in \mathcal{F}C_b$ with $F(\gamma) := f(\gamma_{t_1}, \cdots, \gamma_{t_n})$ and any $h \in \mathbb{H}$, let

$$D_h F(\gamma) := \sum_{i=1}^{n} \langle \nabla_i f(\gamma), U_{t_i} h_{t_i} \rangle,$$

where ∇_i is the (distributional) gradient operator in the i-th component. Then $D^0 F(\gamma) \in \mathbb{H}, \gamma \in W_x(M)$, is well-defined via $\langle D^0 F(\gamma), h \rangle_{\mathbb{H}} = D_h F(\gamma), h \in \mathbb{H}$. Define

$$\mathcal{E}(F, G) := \int_{W_x(M)} \langle D^0 F, D^0 G \rangle_{\mathbb{H}} \mathrm{d}\mathbb{P}_x, \quad F, G \in \mathcal{F}C_b.$$

Integration by parts formula(IPF)

Let $\mathbf{Ric}_{U_t} : \mathbb{R}^d \to \mathbb{R}^d$. Assume $\mathbb{E}\int_0^1 \|\mathbf{Ric}_{U_t}\|^2 \mathrm{d}t < \infty$. We have

$$\int_{W_x(M)} F D_h G \mathrm{d}\mathbb{P}_x = \int_{W_x(M)} G D_h^* F \mathrm{d}\mathbb{P}_x, \quad F, G \in \mathcal{F}C_b,$$

where

$$D_h^* = -D_h + \int_0^1 \left\langle \dot{h}_t + \frac{1}{2}\mathbf{Ric}_{U_t} h_t, \mathrm{d}B_t \right\rangle$$

Importance:(**IPF**)$\Rightarrow$ $(\mathcal{E}, \mathcal{F}C_b)$ is closable in $L^2(W_x(M); \mathbb{P}_x)$, and its closure $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a local conservative quaasi-regular Dirichlet form on $L^2(\mathbb{P}_x)$.

Functional Inequalities on Path Spaces

Known results(Path):

- (Fang, CRASP94): M **Compact +Clark-Ocone Formula** $\Rightarrow$ Poincaré inequality (Logarithmic Sobolev inequality)
- (Aida-Elworthy, CRASP95): M **Compact +Gradient Brownnian systems** $\Rightarrow$ Logarithmic Sobolev inequality
- (Hsu, CMP97): $|\mathbf{Ricc}| \leq C+$ **Bismut formula**+**Markovian property** $\Rightarrow$ Logarithmic Sobolev inequality
- (Wang, IMRS04): M **Noncompact+ Some unbounded curvature** $\Rightarrow$ Weak Poincare inequality

To state our main results, we need some preparations. Let M be a connected Riemannian Manifold. Let K, K_1 be nonnegative and increasing functions such that

- $|\mathbf{Ric}_y| \leq K(\rho(y)), \quad y \in M,$
- $\mathbf{Ric}(X, X) \geq -K_1(\rho(y)), \quad y \in M, X \in T_y M, |X| = 1$
- $\tilde{K}(R) := \sup\Big\{\|\mathrm{Ric}(y)\|_{T_y M} : \ x \in M, \ d_M(y, x) \leq R\Big\},$
- $\tilde{K}_1(R) := \inf_{v \in T_y M, \ |v|=1} \Big\{\langle \mathrm{Ric}(y) v, v\rangle_{T_y M} : \ y \in M, \ d_M(y, x) \leq R\Big\}.$

Define

$$\tau_R := \inf\{t \geq 0 : \rho(X_t) \geq R\},$$

where $X_t = \pi(U_t)$.

Let $\Phi_{t,s}(\gamma) \in L(\mathbb{R}^n \to \mathbb{R}^n)$ be the solution of the following linear ODE,

$$\frac{\mathrm{d}\Phi_{t,s}(\gamma)}{\mathrm{d}s} = -\frac{1}{2}\mathrm{Ric}^{\sharp}_{U_\cdot(\gamma)}\big(\gamma(s)\big)\Phi_{t,s}(\gamma), \quad \Phi_{t,t} = \mathbf{Id}, \; t \le s \le 1,$$

Define:

$$\big(\mathbf{B}(\gamma)^{\frac{1}{2}}h\big)(t) := h(t) +$$
$$\frac{1}{2}\int_0^t \big(\Phi_r(\gamma)^*\big)^{-1}\int_r^1 \Phi_s(\gamma)^*\mathrm{Ric}^{\sharp}_{U_\cdot(\gamma)}\big((\gamma(s))\big)h'(s)ds, \; h \in \mathbb{H}$$
$$\mathcal{E}_{\mathbf{B}}(F,G) := \int_{W_x(M)} \langle \mathbf{B}^{1/2}D^0F, \mathbf{B}^{1/2}D^0G\rangle_{\mathbb{H}_0}\mathrm{d}\mathbb{P}_x,$$
$$F, G \in Y := \{F \in \mathcal{F}C_b : D^0F \in \mathcal{D}(\mathbf{B}^{1/2})\}.$$

Theorem A(Chen-Wu) Assume that M is complete and stoch. complete. Then $(\mathcal{E}_{\mathbf{B}}, \mathcal{F}C_{b,loc})$ is closable on $(W_x(M), L^2(\mathbb{P}_x))$ and its closure $(\mathcal{E}_{\mathbf{B}}, \mathcal{D}(\mathcal{E}_{\mathbf{B}}))$ is a local conservative quasi-regular Dirichlet form on $L^2(\mathbb{P}_x)$. Moreover, the following log-Sobolev inequality holds for $(\mathcal{E}_{\mathbf{B}}, \mathcal{D}(\mathcal{E}_{\mathbf{B}}))$,

$$\mathbf{Ent}_{\mathbb{P}_x}(F^2) \leq 2\mathcal{E}_{\mathbf{B}}(F, F), \qquad F \in \mathcal{F}C_{b,loc}.$$

In particular, the following weighted log-Sobolev inequality also holds,

$$\mathbf{Ent}_{\mathbb{P}_x}(F^2) \leq \int_{W_x(M)} \left(4 + K(\gamma)^2 \exp^{-K_1(\gamma)}\right) \|DF\|_{\mathbb{H}}^2 \mathrm{d}\mathbb{P}_x.$$

Theorem B(Chen-Wu) If

$$\lim_{R\to\infty} \frac{1}{\sqrt{\mathbb{P}_x(\rho > R)}} \int_R^\infty \frac{\mathrm{d}s}{\sqrt{4 + \exp^{-\tilde{K}_1(s)} \tilde{K}(s)^2}} = \infty, \qquad (1)$$

then the following weak log-Sooblev inequality holds,

$$\mathbf{Ent}_{\mathbb{P}_x}(F^2) \le \alpha(r)\mathcal{E}(F,F) + r\|F\|_\infty^2, \quad F \in \mathcal{F}C_b^\infty, 0 < r < r_0,$$

for some $r_0 > 0$ with

$$\alpha(r) := \inf_{R\in\Lambda_r} \left\{2\left(4 + \tilde{K}(R)^2 \exp^{-\tilde{K}_1(R)}\right)\right\} < \infty, \quad r > 0,$$

where $\Lambda_r :=$

$$\left\{R > 0 : \inf_{R_1\in(0,R)} \left\{ \frac{2\mathbb{P}_x(\rho > R_1)}{\left(\int_{R_1}^R \frac{\mathrm{d}s}{\sqrt{4+\exp^{-\tilde{K}_1(s)} \tilde{K}(s)^2}}\right)^2} + 3\mathbb{P}_x(\rho > R_1)^{1/2} \right\} \le r \right\},$$

Example: Suppose

$$\tilde{K}(s) \le c_1(1+s^{\delta_1}), \quad \tilde{K}_1(s) \ge -c_2 - \delta_2 \log(1+s), \quad s > 0,$$

for some positive constants $c_1, c_2, \delta_1, \delta_2$.
(1) If $2\delta_1 + \delta_2 < 1$, then the following super Poincaré inequality holds,

$$\mathbb{P}_x(F^2) \le r\mathcal{E}(F,F) + \beta(r)\mathbb{P}_x(|F|)^2, \quad F \in \mathcal{D}(\mathcal{E}),\ r > 0,$$

where $\beta(r) = \exp\left(c_3\left(1 + r^{-\frac{1}{1-2\delta_1-\delta_2}}\right)\right)$ for some $c_3 > 0$.
(2) If $2\delta_1 + \delta_2 \le 1$, then the following Poincaré inequality

$$\mathbb{P}_x(F^2) \le c_4\mathcal{E}(F,F) + \mathbb{P}_x(F)^2, \quad F \in \mathcal{D}(\mathcal{E}),$$

holds for some $c_4 > 0$.

In order to prove that **Theorem A**, firstly, we prove the quasi-regularity of D.-F. $(\mathcal{E}_{\mathbf{B}}, \mathcal{D}(\mathcal{E}_{\mathbf{B}}))$. In the following, we shall provide a more general result.

Let $\mathbf{B} : L(W_x(M) \times \mathbb{H}) \to \mathbb{H}$ be a measurable operator.

- (**A**1) For a.s. $\gamma \in W_x(M), \mathbf{B}(\gamma) : \mathbb{H} \to \mathbb{H}$ densely defined self-adjoint with the domain $\mathcal{D}(\mathbf{B}(\gamma))$
- (**A2**) For every $v \in L^{\infty}_{loc}(W_x(M) \to \mathbb{R}^n; \mathbb{P}_x)$, $(t \wedge \cdot)v(\gamma) \in \mathcal{D}(\mathbf{B}(\gamma)^{\frac{1}{2}})$ a.s.. For any $R > 1$, $D\phi_R(\gamma) \in \mathcal{D}(\mathbf{B}(\gamma)^{\frac{1}{2}})$ a.s. and $C(R) > 0$ s.t.

$$\int_{B_R} ||\mathbf{B}^{\frac{1}{2}} \Psi_{t,v}||^2_{\mathbb{H}} \mathrm{d}\mathbb{P}_x \leq C(R)||v1_{B_R}||^2_{L^\infty(W_x(M) \to \mathbb{R}^n; \mathbb{P}_x)},$$

$$\int_{B_R} \left|\left|\mathbf{B}^{\frac{1}{2}}(D\phi_R)\right|\right|^2_{\mathbb{H}} \mathrm{d}\mathbb{P}_x \leq C(R),$$

- (**A**3) For every $R > 0, \exists\ \varepsilon(R) > 0$ s.t.

$$\mathbf{B}(\gamma) \geqslant \varepsilon(R)\mathbf{Id}, \quad \gamma \in B_R$$

Theorem C(Chen-Wu) Assume $(\mathbf{A}1), (\mathbf{A}2)$ and $(\mathbf{A}3)$. Then $(\mathcal{E}_{\mathbf{B}}, \mathcal{F}C_{b,loc})$ is closable on $(W_x(M), L^2(\mathbb{P}_x))$ and its closure $(\mathcal{E}_{\mathbf{B}}, \mathcal{D}(\mathcal{E}_{\mathbf{B}}))$ is a local conservative quasi-regular Dirichlet form on $L^2(\mathbb{P}_x)$.

Remark:

(1) Driver-Röckner(92,M=compact, $\mathbf{B} = \mathbf{Id}$); Elworthy-Ma(97, M=compact); Löbus(04,$\mathbf{B} = nonrandom$, M=Compact); Wang-Wu(08, $\mathbf{B} > \varepsilon Id$);

(2) Wang F. Y.-Wu.(08) showed that **Theorem C** holds under **IPF**, $\mathbf{B} \geq \varepsilon \mathbf{Id}$ for some $\varepsilon > 0$ and

$$\int_{W_x(x)} \|\mathbf{B}^{\frac{1}{2}}\Psi_{t,v}\|_{\mathbb{H}}^2 \mathrm{d}\mathbb{P}_x \leq C||v1_{B_R}||^2_{L^\infty(W_x(M)\to\mathbb{R}^n;\mathbb{P}_x)}.$$

Sketch of Proof of Theorem C

We only show that $(\mathcal{E}, \mathcal{F}C_b)$ is closable.
Taking a sequence of operators $\{L_k\}_{k\geq 1}$ s.t. L_k converges to L.
Let $g_k \in C_0^\infty(M)$ such that $g_k|_{\{d_M \leq k\}} = 1$. Let $L_k := g_k^2 L$ and $M_k := \{y \in M : g_k(y) > 0\}$. Consider the metric

$$\langle \cdot, \cdot \rangle_k := g_k^{-2} \langle \cdot, \cdot \rangle$$

on M_k. Then $(M_k, \langle \cdot, \cdot \rangle_k)$ is a complete Riemannian manifold and

$$\sup_{M_k} \left(\|\mathbf{Ric}^{(k)}\|_k + \|\nabla^{(k)} Z^{(k)}\|_k \right) < \infty.$$

Let $\mathbb{P}_x^k$ be the distribution of the L_k-diffusion process on M_k.

According to the construction of L_k, we known that

$$\mathcal{E}_k(F,F) := \int_{W_x(M_k)} \langle D_k F, D_k F\rangle_{\mathbb{H}} \mathrm{d}\mathbb{P}_x^k, \quad F \in \mathcal{F}C_b(M_k)$$

is closable, where D_k is the (closed) gradient operator on $L^2(\mathbb{P}_x^k)$.

Finally, by the approximation procedure, it is not difficulty to show that $(\mathcal{E}, \mathcal{F}C_b)$ is closable.

Functional Inequalities on Loop Spaces

Define:

$$W(M) := C([0,1]; M)$$
$$W_{x,y}(M) := \{\gamma \in W_x(M) : \gamma(1) = y\}.$$

Let $\mathbb{P}_{x,y}$ be the Brownian bridge measure on $W_{x,y}(M)$. In particular, for any $F \in \mathcal{F}C_0^\infty$ with $F(\gamma) := f(\gamma_{t_1}, \cdots, \gamma_{t_n})$

$$\int_{W_{x,y}(M)} F \mathrm{d}\mathbb{P}_{x,y} = \int_{M^n} f(z_1, \cdots, z_n) p_{t_1}(x, z_1) p_{t_2 - t_1}(z_1, z_2) \cdots$$
$$p_{t_n - t_{n-1}}(z_{n-1}, z_n) p_{1-t_n}(z_n, y) \prod_i^n \mathrm{d}z_i / p_1(x, y),$$

here $p_t(x,y)$ is the heat kernel of $\frac{1}{2}\Delta$ on M.

Let $\mathbb{H}_0 := \{h \in \mathbb{H} : h_1 = 0\}$. Then there exist $DF(\gamma) \in \mathbb{H}$, $D^0F(\gamma) \in \mathbb{H}_0$ such that $\langle D^0F(\gamma), h\rangle_{\mathbb{H}_0} = D_hF(\gamma)$. Let

$$\mathcal{E}(F, G) := \int_{W_{x,y}(M)} \langle D^0F, D^0G\rangle_{\mathbb{H}_0} \mathrm{d}\mathbb{P}_{x,y}, \quad F, G \in \mathcal{F}C_0^\infty.$$

Note that $(\mathcal{E}, \mathcal{F}C_b)$ is closable and its closure $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a quasi-regular Dirichlet Form under some reasonable conditions(Compact Manifold or Hyperbolic space).

Known results(Loop):

- $M=\mathbb{R}^n$, $\mathbf{Ent}_{\mathbb{P}_{x,y}}(F^2)\leqslant 2\int_{W_{x,y}(M)}\| DF \|_{\mathbb{H}_0}^2\,\mathrm{d}\mathbb{P}_{x,y}$.
- (Gross, JFA91) M=Compact Lie group,

$$\mathbf{Ent}_{\mathbb{P}_{x,y}}(F^2)\leqslant C\int_{W_{x,y}(M)}\| DF \|_{\mathbb{H}_0}^2 +(c_1|B(1)|+c_2)F^2\mathrm{d}\mathbb{P}_{x,y}.$$

- (Gong-Ma, JFA98) M=Compact manifold, $\nabla = Levi-Civitaconnection$,

$$\mathbf{Ent}_{\mathbb{P}_{x,y}}(F^2)\leqslant C\int_{W_{x,y}(M)}\| DF \|_{\mathbb{H}_0}^2 +VF^2\mathrm{d}\mathbb{P}_{x,y},$$

where V is $L^p(\mathbb{P}_{x,y})$-integrable for any $p>0$.
- (Gong-Röckner-Wu, JFA01) M=Compact manifold, $\nabla = torsionskewsymmetricconnection$,

- (Aida, JFA00) M=Hyperbolic space,

$$\mathbf{Ent}_{\mathbb{P}_{x,y}}(F^2) \leqslant C \int_{W_{x,y}(M)} V \parallel DF \parallel_{\mathbb{H}_0}^2 \, \mathrm{d}\mathbb{P}_{x,y},$$

  where aV is exponential integrable for some enough small constant $a > 0$.

- (Chen-Li-Wu, JFA10) M=Hyperbolic space, Poincaré inequality holds:

$$\mathbf{Var}_{\mathbb{P}_{x,y}}(F) \leqslant C \int_{W_{x,y}(M)} V \parallel DF \parallel_{\mathbb{H}_0}^2 \, \mathrm{d}\mathbb{P}_{x,y}.$$

- (Chen-Li-Wu, PTRF11) M=Compact and $\mathbf{Ric} > 0$, weak Poincaré inequality holds:

$$\mathbf{Var}_{\mathbb{P}_{x,x}}(F) \leqslant \frac{1}{s^\alpha} \int_{W_{x,x}(M)} \parallel DF \parallel_{\mathbb{H}_0}^2 \, \mathrm{d}\mathbb{P}_{x,x} + s\|F\|_\infty^2, \quad s \in (0, s_0).$$

Remarks: Actually, by the martingale representation theorem, we have

$$F = \mathbb{E}_{\mathbb{P}_{x,y}}(F) + \int_0^1 \langle H_t^F, d\beta_t \rangle,$$

where

$$H_t^F = \mathbb{E}_{\mathbb{P}_{x,y}}\Big[\frac{d}{dt}DF(t) + \frac{DF(t)}{1-t} + F\frac{\int_t^1 A_s d\beta_s}{1-t}|\mathcal{F}_t\Big]$$

$$A_s = I_{\mathbb{R}^d} - \frac{1-s}{2}\mathbf{Ric}_{U_s} + Hess_{U_s}\mathbf{Log}\, p_{1-s}(x_s, y).$$

Let $G = F^2$and G_t be a right continuous version of $\mathbb{E}_{\mathbb{P}_{x,y}}[G|\mathcal{F}_t], 0 \leqslant t \leqslant 1$, then $dG_t = \langle H_t^G, d\beta_t \rangle$. Applying Ito's formula we obtain,

$$\begin{aligned} d(G_t \mathbf{Log}\, G_t) &= (1 + \mathbf{Log}\, G_t) dG_t + \frac{|H_t^G|^2}{2G_t} dt \\ &= \langle (1 + \mathbf{Log}\, G_t) H_t^G, d\beta_t \rangle + \frac{|H_t^G|^2}{2G_t} dt. \end{aligned}$$

Thus

$$\mathbf{Ent}_{\mathbb{P}_{x,y}}(G) = \frac{1}{2} \mathbb{E}_{\mathbb{P}_{x,y}} \left[\int_0^1 \frac{|H_t^G|^2}{2G_t} dt. \right]$$

- (Eberle, JMPA02, Counterexample) M=Compact manifold which contains a non-trivial closed geodesic ζ such that the curvature is constant and strictly negative in a neighbourhood of ζ, **(PI)** does not hold.
- **Example**. Suppose that $dim(M) = 2$, and M contains an open subset U that is isometric to the surface of revolution in $\mathbb{R}^3$ given as the image of the map $g : (-A, A) \times \mathbb{R} \to \mathbb{R}^3$:

  $$g(s, \phi) = (R \cosh s \cosh \phi, R \cosh s \sin \phi, \int_0^s (1 - R^2 \sinh^2 t)^{1/2} dt)$$

  for some $R, A, > 0$ with $\sinh A < 1/R$.

Free Loop:

Let μ be a probability measure on M such that $d\mu(x) = v(x)dx, v \geq 0, v \in W^{1,2}(dx)$. Define $d\mathbb{P}_\mu = \mathbb{P}_{x,x} d\mu$ and

$$\mathbb{E}_{\mathbb{P}_\mu}(F, G) := \int_{W(M)} \langle \nabla F, \nabla G \rangle_{\mathbb{H}_0} d\mathbb{P}_\mu.$$

Assumptions: For all $x, y \in M$ and $t > 0$

(*A*1) $$|p_t(x, y)| \leqslant \frac{1}{f_t} \exp^{-C_1 \frac{d(x,y)^2}{t}}$$

(*A*2) $$|\nabla_x \log p_t(x, y)| \leqslant C \Big[\frac{d(x, y)}{t} + \frac{1}{\sqrt{t}} \Big],$$

here C_1, C are positive constants and $f \in C((0, \infty))$.

$$(A3) \qquad \underline{\mathbf{Ric}}_z \geqslant -K(\rho_o(z)^2 + 1),$$

for some positive constants K.
(A4) There exists a bounded closed neighborhood $W_0 \subset M$ of x such that

$$\sup_{y \in W_0} \int_0^1 \mathbb{E}_y |\mathbf{Ric}_{U_s}|^2 \mathrm{d}s \mathrm{d}y < \infty,$$

Theorem D(Chen-Li-Wu) Under conditions $(A1), (A2), (A3)$ and $(A4)$, Assume $\exists\ \varepsilon > 0$ s.t. $\forall\ \ n, x \in M$

$$\int_{B_n} \int_0^1 \mathbb{E}_x |\mathbf{Ric}_{U_s}|^2 \mathrm{d}s\mathrm{d}x < \infty, \quad \int_0^1 \mathbb{E}_{x,x} |\mathbf{Ric}_{U_s}|^{2(1+\varepsilon)} \mathrm{d}s < \infty.$$

Then for any vector field Y on M,
$h_s(\gamma) := Y(\gamma_0) + s(U_1^{-1}(\gamma) - I)Y(\gamma_0) + y_s, y \in \mathbb{H}_0$, we have

$$\mathbb{E}_{\mathbb{P}_\mu}(D_h F) = \mathbb{E}_{\mathbb{P}_\mu}(F D_h^* 1), \quad F \in \mathcal{F}C_0.$$

where

$$D_h^* 1 = -\mathrm{div}(Y) + \langle Y, \nabla \log v \rangle + D_y^{0,*} 1$$
$$- \left\langle \int_0^u \left[h' - \frac{1}{2} \mathbf{Ric}_{U((\mathbf{J}B))_s} h_s \right] \mathrm{d}b_s + \int_0^u \left[h' - \frac{1}{2} \mathbf{Ric}_{\tilde{U}((\mathbf{J}B))_s} h_s \right] \mathrm{d}b_s, \right.$$
$$\left. Y(x) \right\rangle - \langle \nabla_x p_1(x,x), Y(x) \rangle.$$

WPI–Free:

Theorem E(Chen-Li-Wu) Assume M is compact and **Ric** $> 0, v \in W^{2,1}(\mathrm{d}x)$. If

$$\mathbf{Var}_\mu(f) \leq \lambda(t)\mu(|\nabla f|^2) + t\|f\|_\infty^2, \quad t \in (0, t_0).$$

Then the following weak Poincaré inequality holds,

$$\mathbf{Var}_{\mathbb{P}_\mu}(F) \leq \Phi(r)\mathbb{E}_{\mathbb{P}_\mu}(F, F) + r\|F\|_\infty^2, \quad F \in \mathcal{F}C_b^\infty, r \in (0, r_0)$$

where

$$\Phi(r) = \frac{3^\delta C^{1+\delta}(1 + \lambda(r/3))}{r^\delta}, \quad r \in (0, r_0)$$

for small enough positive number δ and some constants $C, r_0 > 0$.

PI–Free:

Theorem E(Chen-Li-Wu) Assume $M = \textit{hyperbolic}$ and μ satisfies (PI). Then the following Poincaré inequality holds:

$$\mathbf{Var}_{\mathbb{P}_\mu}(F)) \leq C_2 \mathbb{E}_{\mathbb{P}_\mu}(F, F)$$

holds for some constant $C_1 > 0$.

Main references

- (Aida, JFA00) Logarithmic derivatives of heat kernels and logarithmic Sobolev inequalities with unbounded diffusion coefficients on loop spaces
- (Eberle, JMPA02) Absence of spectral gaps on a class of loop spaces
- (Eberle, IDAQ03) Spectral gaps on discretized loop spaces
- (Gong-Ma, JFA98) The log-Sobolev inequality on loop space over a compact Riemannian manifold
- (Gross, JFA91) Logarithmic Sobolev inequalities on loop groups

Summary

- M is complete and stoch. complete and under some conditions w.r.t. **B**, $(\mathcal{E}_{\mathbf{B}}, \mathcal{D}(\mathcal{E}_{\mathbf{B}}))$ is a **quasi-regular** D.-F.;
- Under some unbounded curvature conditions, **Super Poincare inequality** and **Poincare inequality** may be derived respectively.
- Under some conditions on M and initial distribution, we obtain **(IPF)** on loop and free loop spaces respective.
- we construct **(PI)** and **(WPI)** on free loop spaces over hyperbolic and some compact manifolds respective.

The End

Thank you for your attention!