# Laplacian Perturbed by Non-local Operators

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$$p(t,x,y) = (4\pi t)^{-d/2} e^{-|x-y|^2/4t}.$$

- The study of heat kernel estimates for perturbation of Laplace operator $\Delta$ by gradient operator has a long history and this subject has been studied in many literatures. In general, the heat kernel of $\Delta + b(x)\nabla$ under an appropriate Kato condition on the drift function $b$ is comparable with Gaussion kernel in short time (see Q.Zhang (1996), Q.S.Zhang (1997), Kim-Song (2006)).
- In this talk, we are concerned with the existence and uniqueness of fundamental solution of Laplacian operator perturbed by non-local operator. Furthermore, we will give two sided kernel estimates.

# Background and Motivation

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# Symmetric Stable Processes and $\Delta^{\beta/2}$

- A stochastic process $Z = (Z_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ is called a (rotationally) symmetric $\beta$-stable process ($\beta \in (0,2)$) on $\mathbb{R}^d$ if it is a Lévy process such that its characteristic function is given by

$$\mathbb{E}_x\left[e^{i\xi\cdot(Z_t - Z_0)}\right] = e^{-t|\xi|^\beta} \qquad \text{for every } x \in \mathbb{R}^d.$$

- The infinitesimal generator for the symmetric $\beta$-stable process on $\mathbb{R}^d$ is $\Delta^{\beta/2}$ as follows:

$$\Delta^{\beta/2} f(x) = \mathcal{A}(d,-\beta)\int_{\mathbb{R}^d}\left(f(x+z) - f(x) - \nabla f(x)\cdot z 1_{\{|z|\le 1\}}\right)\frac{dz}{|z|^{d+\beta}}$$

for $f \in C_b^2(\mathbb{R}^d)$. Lévy measure of $Z_t$: $\nu(dz) = \mathcal{A}(d,-\beta)\frac{1}{|z|^{d+\beta}}\,dz$.

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## Truncated Symmetric Stable Processes and $\overline{\Delta}^{\beta/2}$

- $\{\overline{Z}_t, t \geq 0\}$ is called a truncated (also called finite range) isotropically symmetric $\beta$-stable process in $\mathbb{R}^d$ if it is a symmetric $\beta$-stable process with jumps of size larger than 1 removed.
- The infinitesimal generator for the truncated symmetric $\beta$-stable process on $\mathbb{R}^d$ is $\overline{\Delta}^{\beta/2}$ as follows:

$$\overline{\Delta}^{\beta/2} f(x) = \mathcal{A}(d, -\beta) \int_{\{|z| \leq 1\}} \left( f(x+z) - f(x) - \langle \nabla f(x) \cdot z \rangle \right) \frac{dz}{|z|^{d+\beta}}$$

Lévy measure of $\overline{Z}_t$:

$$\overline{\nu}(dz) = 1_{\{|z| \leq 1\}}(z) \nu(dz) = 1_{\{|z| \leq 1\}}(z) \frac{\mathcal{A}(d, -\beta)}{|z|^{d+\beta}} \, dz.$$

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# Question

In this talk ,we consider Laplacian operator perturbed by a class of non-symmetric and non-local operator:

$$\mathcal{L}^b f(x) = \Delta f(x) + \mathcal{S}^b f(x),$$

where

$$\mathcal{S}^b f(x) := \mathcal{A}(d, -\beta) \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \langle \nabla f(x), z 1_{\{|z| \leq 1\}} \rangle\right) \frac{b(x,z)}{|z|^{d+\beta}} dz$$

for $f \in C_b^2(\mathbb{R}^d)$, where $0 < \beta < 2$ and $b(x,z)$ is a real-valued bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ with

$$b(x,z) = b(x,-z) \qquad \text{for every } x, z \in \mathbb{R}^d.$$

The operator we consider:

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Remarks:

- The operator $\mathcal{L}^b$ is in general non-symmetric.
- $\mathcal{L}^b = \Delta$ when $b \equiv 0$ and $\mathcal{L}^b = \Delta + \Delta^{\beta/2}$ when $b \equiv 1$.
- $\mathcal{L}^b = \Delta + \overline{\Delta}^{\beta/2}$ when $b(x,z) = 1_{\{|z| \le 1\}}(z)$.
- The imposed condition $b(x,z) = b(x,-z)$ can ensure that the truncation $|z| \le 1$ can be replaced by $|z| \le \lambda$ for any $\lambda > 0$ in the definition of $\mathcal{S}^b$.

$$\mathcal{S}^b f(x) = \mathcal{A}(d, -\beta) \int_{\mathbb{R}^d} \left( f(x+z) - f(x) - \langle \nabla f(x), z\, 1_{\{|z| \le \lambda\}} \rangle \right) \frac{b(x,z)}{|z|^{d+\beta}} dz.$$

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# Heat Kernel Estimates in the symmetric case

Let $a \geq 0$ be a constant. Let $p_0(t,x,y)$ and $p_a(t,x,y)$ denote the kernel for $\Delta$ and $\Delta + a\Delta^{\beta/2}$ correspondingly.

- Song and Vondracek (2007) obtained the two sided estimates of $p_1(t,x,y)$ for the independent sum of B.M. and $\beta-$symmetric stable process

$$c_1 \left(t^{-d/2} \wedge t^{-d/\beta}\right) \wedge \left(p_0(t,c_2x,c_2y) + \frac{t}{|x-y|^{d+\beta}}\right)$$
$$\leq p_1(t,x,y) \leq c_3 \left(t^{-d/2} \wedge t^{-d/\beta}\right) \wedge \left(p_0(t,c_4x,c_4y) + \frac{t}{|x-y|^{d+\beta}}\right).$$

- For each $a > 0$, $p_a(t,x,y) \asymp p_1(t,cx,cy)$.
- Chen and Kumagai (2010) generalized the result of Song and Vondracek (2007) and established the two sided heat kernel estimates for diffusions with jumps in the symmetric framework.
- Heat kernel estimates for $\Delta + a\overline{\Delta}^{\beta/2}$ is unknown until now.

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# Duhamel's formula

Since $\mathcal{L}^b = \Delta + \mathcal{S}^b$ is a lower order perturbation of $\Delta$ by $\mathcal{S}^b$, heuristically the fundamental solution (or kernel) $q^b(t,x,y)$ of $\mathcal{L}^b$ should satisfy the following Duhamel's formula:

$$q^b(t,x,y) = p_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q^b(t-s,x,z)\mathcal{S}_z^b p_0(s,z,y)dzds \quad (D)$$

for $t > 0$ and $x, y \in \mathbb{R}^d$. Here the notation $S_z^b p_0(s,z,y)$ means the non-local operator $\mathcal{S}^b$ is applied to the function $z \mapsto p_0(s,z,y)$.

Let $q_0^b(t,x,y) := p_0(t,x,y)$ and

$$q_n^b(t,x,y) := \int_0^t \int_{\mathbb{R}^d} q_{n-1}^b(t-s,x,z)\mathcal{S}_z^b p_0(s,z,y)dzds \quad \text{for } n \geq 1.$$

Applying the Duhamel's formula recursively,

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## Scaling Property of $q_n^b(t,x,y)$

Note that the Gaussian kernel $p_0(t,x,y)$ has the following scaling property: for $\lambda > 0$,

$$p_0(t,x,y) = \lambda^{d/2} p_0(\lambda t, \lambda^{1/2}x, \lambda^{1/2}y), \quad t>0,\ x,y\in\mathbb{R}^d.$$

Under the condition $b(x,z) = b(x,-z)$, we have

**Scaling property**

For every integer $n \geq 0$ and $\lambda > 0$,

$$q_n^b(t,x,y) = \lambda^{d/2}\, q_n^{b^{(\lambda)}}(\lambda t, \lambda^{1/2}x, \lambda^{1/2}y), \qquad t>0, x,y\in\mathbb{R}^d,$$

where $b^{(\lambda)}(x,z) = \lambda^{\beta/2-1} b(\lambda^{-1/2}x, \lambda^{-1/2}z)$.

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# Theorem 1: (Existence and Uniqueness)

There is a continuous function $q^b(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ such that

- (i) There is a constant $A_0 = A_0(d,\beta) > 0$ so that

$$q^b(t,x,y) = \sum_{n=0}^{\infty} q_n^b(t,x,y)$$

  on $(0, (A_0/\|b\|_\infty)^{2/(2-\beta)}] \times \mathbb{R}^d \times \mathbb{R}^d$.
- (ii) $q^b(t,x,y)$ satisfies Duhamel's formula on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.
- (iii) For every $t, s > 0$ and $x, z \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} q^b(t,x,y) q^b(s,y,z) dy = q^b(t+s,x,z)$$

- (iv) For each $t > 0$ and $x \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} q^b(t,x,y)dy = 1$.

# Theorem 1: (continued)

- (v) For every $f \in C_b^2(\mathbb{R}^d)$,

$$T_t^b f(x) - f(x) = \int_0^t T_s^b \mathcal{L}^b f(x) ds,$$

where $T_t^b f(x) = \int_{\mathbb{R}^d} q^b(t,x,y) f(y) dy$.

- (vi) There are positive constants $C_k, k = 1, 2$ so that

$$|q^b(t,x,y)| \le C_1 p_{\|b\|_\infty}(t, C_2 x, C_2 y) \quad \text{on } (0,1) \times \mathbb{R}^d \times \mathbb{R}^d,$$

where $p_{\|b\|_\infty}(t,x,y)$ is the kernel for the operator $\Delta + \|b\|_\infty \Delta^{\beta/2}$.

Moreover, suppose that $\overline{q}^b(t,x,y)$ is any continuous kernel that
(1)satisfies C-K equation on $(0,\infty] \times \mathbb{R}^d \times \mathbb{R}^d$;

(2)satisfies Duhamel's formula (D) on $(0,\varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$;

(3)$|\overline{q}^b(t,x,y)| \le c_1 p_1(t, c_2 x, c_2 y)$ on $(0,\varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$ for some $\varepsilon, c_1, c_2 > 0$,

then $\overline{q}^b = q^b$ on $[0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

## Theorem 1: (continued)

- (v) For every $f \in C_b^2(\mathbb{R}^d)$,

$$T_t^b f(x) - f(x) = \int_0^t T_s^b \mathcal{L}^b f(x) ds,$$

where $T_t^b f(x) = \int_{\mathbb{R}^d} q^b(t,x,y) f(y) dy$.

- (vi) There are positive constants $C_k, k = 1, 2$ so that

$$|q^b(t,x,y)| \le C_1 p_{\|b\|_\infty}(t, C_2 x, C_2 y) \quad \text{on } (0,1) \times \mathbb{R}^d \times \mathbb{R}^d,$$

where $p_{\|b\|_\infty}(t,x,y)$ is the kernel for the operator $\Delta + \|b\|_\infty \Delta^{\beta/2}$.

Moreover, suppose that $\overline{q}^b(t,x,y)$ is any continuous kernel that
(1)satisfies C-K equation on $(0,\infty] \times \mathbb{R}^d \times \mathbb{R}^d$;

(2)satisfies Duhamel's formula (D) on $(0,\varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$;

(3)$|\overline{q}^b(t,x,y)| \le c_1 p_1(t, c_2 x, c_2 y)$ on $(0,\varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$ for some $\varepsilon, c_1, c_2 > 0$,

then $\overline{q}^b = q^b$ on $[0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

The kernel $q^b(t,x,y)$ in Theorem 1 can be negative.

**Theorem 2: Positivity**

Suppose that

$$x \mapsto b(x,z) \text{ is continuous for a.e. } z \in \mathbb{R}^d.$$

Then $q^b(t,x,y) \geq 0$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ if and only if for each $x \in \mathbb{R}^d$,

$$b(x,z) \geq 0 \quad \text{for a.e. } z \in \mathbb{R}^d. \quad (P)$$

- The positivity condition (P) is equivalent that the jump kernel of $\mathcal{L}^b$ is nonnegative.
- By some approximation arguments, we can remove the continuity condition on $b$.

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# Theorem 3: Feller Process and Heat Kernel Estimates

Suppose $b$ satisfies the positivity condition (P). Then

(i) For every $A > 0$, there are positive constants $C_k = C_k(d, \beta, A), k = 1, \cdots, 4$ such that for any bounded $b$ on $\mathbb{R}^d \times \mathbb{R}^d$ with $\|b\|_\infty \leq A$,

$$C_1 p_{m_b}(t, C_2 x, C_2 y) \leq q^b(t, x, y) \leq C_3 p_{M_b}(t, C_4 x, C_4 y)$$

for $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$, where $m_b := \operatorname{essinf}_{x,z \in \mathbb{R}^d} b(x, z)$, $M_b = \operatorname{esssup}_{x,z \in \mathbb{R}^d} b($

(ii)The kernel $q^b(t, x, y)$ uniquely determines a conservative Feller process $X^b = (X_t^b, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d)$ on the canonical Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ such that

$$\mathbb{E}_x \left[ f(X_t^b) \right] = \int_{\mathbb{R}^d} q^b(t, x, y) f(y) dy$$

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(iii) The Feller process $X^b$ has a Lévy system $(J^b(x,y)dy, t)$, where the jump kernel

$$J^b(x,y) = \frac{\mathcal{A}(d,-\beta)\, b(x,y-x)}{|x-y|^{d+\beta}}.$$

(iv) Moreover, for each $x \in \mathbb{R}^d$, $(X^b, \mathbb{P}_x)$ is the unique solution to the martingale problem $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ with initial value $x$. Here $\mathcal{S}(\mathbb{R}^d)$ denotes the space of tempered functions on $\mathbb{R}^d$.

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# Perturbation by finite range operator

Recall that $\mathcal{L}^b = \Delta + \overline{\Delta}^{\beta/2}$ when $b(x,z) = 1_{\{|z|\le 1\}}(z)$. Next we will give the refined upper and lower bound estimates for $\mathcal{L}^b = \Delta + \mathcal{S}^b$ with finite range non-local operator $\mathcal{S}^b$.

Let $\overline{p}_\beta(t,x,y)$ denote the heat kernel for $\overline{\Delta}^{\beta/2}$. Chen-Kim-Kumagai (2008) proves that $\overline{p}_\beta(t,x,y)$ is jointly continuous and enjoys the following two sided estimates for $t \in (0,1]$:

$$\overline{p}_\beta(t,x,y) \asymp t^{-d/\beta} \wedge \frac{t}{|x-y|^{d+\beta}}, \quad |x-y| \le 1.$$

and

$$c_1 \left(\frac{t}{|x-y|}\right)^{c_2|x-y|} \le \overline{p}_\beta(t,x,y) \le c_3 \left(\frac{t}{|x-y|}\right)^{c_4|x-y|}, \quad |x-y| > 1.$$

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## Theorem 4: Perturbation by finite range operator

(i) For every $\lambda > 0$ and $M \geq 1$, there are positive constants $C_k = C_k(d, \beta, M, \lambda), k = 5, 6$ such that for any bounded $b$ satisfying the positivity condition and

$$\sup_x b(x,z) \leq M 1_{|z| \leq \lambda}(z),$$

we have

$$q^b(t,x,y) \leq C_5 \left[ t^{-d/2} \wedge \left( p_0(t, C_6 x, C_6 y) + \overline{p}_\beta(t, C_6 x, C_6 y) \right) \right]$$

for $t \in (0, 1]$, $x, y \in \mathbb{R}^d$.

## Theorem 4: Continued

(ii) Furthermore, if the function $b$ satisfies

$$\inf_{x\in\mathbb{R}^d} b(x,z) > \varepsilon 1_{\{|z|\le\lambda\}}(z) \quad (\Longleftrightarrow \text{the jump measure of } \mathcal{S}^b > \varepsilon\overline{\nu}_\lambda(dz))$$

for some $\lambda > 0$ and $\varepsilon > 0$, then there are positive constants $C_k, k = 7, 8$ so that

$$q^b(t,x,y) \ge C_7 \left( t^{-d/2} \wedge (p_{m_b}(t, C_8 x, C_8 y) + \overline{p}_\beta(t, C_8 x, C_8 y)) \right)$$

for $t \in (0,1]$ and $x, y \in \mathbb{R}^d$.

The following follows immediately from Theorem 4.

**Corollary:**

For every $\lambda > 0$ and $M \geq 1$, there are positive constants $c_k = c_k(d, \beta, M, \lambda), k = 1, \cdots, 4$ such that for any bounded $b$ with

$$M^{-1} 1_{|z| \leq \lambda}(z) \leq \inf_x b(x,z) \leq \sup_x b(x,z) \leq M 1_{|z| \leq \lambda}(z),$$

we have

$$\begin{aligned} & c_1 \left[ t^{-d/2} \wedge \left( p_0(t, c_2 x, c_2 y) + \overline{p}_\beta(t, c_2 x, c_2 y) \right) \right] \\ \leq & q^b(t,x,y) \leq c_3 \left[ t^{-d/2} \wedge \left( p_0(t, c_4 x, c_4 y) + \overline{p}_\beta(t, c_4 x, c_4 y) \right) \right] \end{aligned}$$

for $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$.

Thank you!