Weighted Poincaré Inequalities for Non-local Dirichlet Forms

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Outline

1. **Introduction**
   - Lévy Type Dirichlet Forms
   - Fractional Dirichlet Forms
2. **Main Results**
   - Weighted Poincaré Inequalities
3. **Remark and Application**
   - Remark
   - Application

Outline

Local Dirichlet Forms

- Let $\mu_V(dx) = e^{-V(x)}\,dx$ be a probability measure.

$$D_V(f,f) = \int |\nabla f(x)|^2\,\mu_V(dx).$$

-

$$A = \triangle - \nabla V \cdot \nabla$$

$$dX_t = \sqrt{2}dB_t - \nabla V(X_t)\,dt$$

- Chen, M.-F. (2005); Wang, F.-Y. (2005)

Non-Local Dirichlet Forms

- Let $\mu_V(dx) = e^{-V(x)}\,dx$ be a probability measure. Let $\alpha \in (0,2)$.

$$D_{\alpha,V}(f,f) = \int\int \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}}\mu_V(dy)\,\mu_V(dx)$$

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- $$A_\alpha = -(-\triangle)^{\alpha/2} - b\cdot\nabla$$

$$dX_t = dZ_t - b(X_t)\,dt$$

- Questions: Forms? Drift?

$$L_\alpha = -(-\Delta)^{\alpha/2} - x \cdot \nabla, \qquad \alpha \in (0,2)$$

- L_α has an invariant (but not symmetric) measure μ_α such that

$$\int e^{ix\cdot\xi}\,\mu_\alpha(dx) = e^{-|\xi|^\alpha/\alpha}.$$

- $$D_\alpha(f,f) := \langle f, -L_\alpha f\rangle_{L^2(\mu_\alpha)} = \iint \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}}\,dy\,\mu_\alpha(dx)$$

- $$\mathrm{Var}_{\mu_\alpha}(f) \leqslant C \iint \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}}\,dy\,\mu_\alpha(dx),$$

see Röckner, M. and Wang, F.-Y. (2003)

Lévy-type Dirichlet Forms

- $$D_\alpha(f,f) = \iint \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}}\, dy\, \mu_\alpha(dx)$$

- $$D_{\alpha,V}(f,f) = \iint \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}}\, dy\, \mu_V(dx),$$

  where
  $$\mu_V(dx) = e^{-V(x)}dx.$$

- $$D_{\rho,V}(f,f) = \iint (f(y)-f(x))^2 \rho(|x-y|)\, dy\, \mu_V(dx),$$

  where
  $$\int \rho(r)(1\wedge r^2)r^{d-1}\, dr < \infty.$$

Results from Harmonic Analysis

Theorem (Mouhot, C., Russ, E. and Sire, Y. (2011))

If $V \in C^2(\mathbb{R}^d)$ *such that for some constant* $\varepsilon > 0$,

$$\frac{(1-\varepsilon)|\nabla V(x)|^2}{2} - \Delta V(x) \to \infty, \qquad |x| \to \infty,$$

then

$$\int (f - \mu_V(f))^2 \left(1 + |\nabla V|^\alpha\right) d\mu_V \leqslant C D_{\alpha,V,\delta}(f,f),$$

where

$$D_{\alpha,V,\delta}(f,f) = \iint \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}} e^{-\delta|y-x|}\, dy\, \mu_V(dx).$$

Results from Harmonic Analysis

Corollary (Mouhot, C., Russ, E. and Sire, Y. (2011))

If $V \in C^2(\mathbb{R}^d)$ *such that for some constant* $\varepsilon > 0$,

$$\frac{(1-\varepsilon)|\nabla V(x)|^2}{2} - \Delta V(x) \to \infty, \qquad |x| \to \infty,$$

then

$$\int (f - \mu_V(f))^2 (1 + |\nabla V|^\alpha)\, d\mu_V \leqslant C D_{\alpha,V}(f,f),$$

where

$$D_{\alpha,V}(f,f) = \iint \frac{(f(y) - f(x))^2}{|y-x|^{d+\alpha}}\, dy\, \mu_V(dx).$$

Results from Harmonic Analysis

Corollary (Mouhot, C., Russ, E. and Sire, Y. (2011))

If $V \in C^2(\mathbb{R}^d)$ *such that for some constant* $\varepsilon > 0$,

$$\frac{(1-\varepsilon)|\nabla V(x)|^2}{2} - \Delta V(x) \to \infty, \qquad |x| \to \infty,$$

then

$$\mathrm{Var}_{\mu_V}(f) := \int (f - \mu_V(f))^2 \, d\mu_V \leqslant C D_{\alpha,V}(f,f),$$

where

$$D_{\alpha,V}(f,f) = \iint \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}} \, dy \, \mu_V(dx).$$

Note: $\mu_p(dx) = C_p e^{-|x|^p} \, dx$ with $p > 1$.

Stable-Like Dirichlet Forms

Theorem (Wang, F.-Y. and W. (2012))

If $e^{-V} \in C^2(\mathbb{R}^d)$ *such that*

$$\liminf_{|x|\to\infty} \frac{e^{V(x)}}{|x|^{d+\alpha}} > 0,$$

then

$$\mathrm{Var}_{\mu_V}(f) := \mu_V\big((f-\mu_V(f))^2\big) \leqslant C D_{\alpha,V}(f,f);$$

if moreover

$$\liminf_{|x|\to\infty} \frac{e^{V(x)}}{|x|^{d+\alpha}} = \infty,$$

then the following super-Poincaré inequality holds

$$\mu_V(f^2) \leqslant r D_{\alpha,V}(f,f) + \beta(r)\mu_V(|f|)^2, \quad r > 0.$$

Note: $\mu_\varepsilon(dx) = C_\varepsilon(1+|x|)^{-d-\varepsilon}\,dx$ with $\varepsilon \geqslant \alpha$.

Question: Weighted Function and Kernel

Theorem (Mouhot, C., Russ, E. and Sire, Y. (2011))

$$\int (f - \mu_V(f))^2 \left(1 + |\nabla V|^\alpha\right) d\mu_V \leqslant C D_{\alpha,V,\delta}(f,f),$$

where

$$D_{\alpha,V,\delta}(f,f) = \iint \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} \, dy \, \mu_V(dx).$$

Theorem (Wang, F.-Y. and W. (2012))

$$\mu_V\left(\left(f - \mu_V(f)\right)^2\right) \leqslant C D_{\alpha,V}(f,f).$$

Non-local Dirichlet Form and its Truncation

(1)

$$D_{\alpha,V,\delta}(f,f) := \iint \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}} e^{-\delta|y-x|}\, dy\, \mu_V(dx)$$

(2)

$$\hat{D}_{\alpha,V,\delta}(f,f) := \iint_{\{|x-y|>1\}} \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}} e^{-\delta|y-x|}\, dy\, \mu_V(dx)$$

Weighted Poincaré Inequalities

Theorem (Chen, X. and W. (2013))

Suppose that for some constants $\delta \geqslant 0$ *and* $\alpha \in (0,2)$

$$\liminf_{|x|\to\infty} \frac{e^{V(x)-\delta|x|}}{|x|^{d+\alpha}} > 0.$$

Then,

$$\int \big(f(x)-\mu_V(f)\big)^2 \frac{e^{V(x)-\delta|x|}}{(1+|x|)^{d+\alpha}}\,\mu_V(dx) \leqslant C\hat{D}_{\alpha,V,\delta}(f,f).$$

In particular,

$$\int \big(f(x)-\mu_V(f)\big)^2 \frac{e^{V(x)-\delta|x|}}{(1+|x|)^{d+\alpha}}\,\mu_V(dx) \leqslant CD_{\alpha,V,\delta}(f,f).$$

$\delta > 0$: Improvement of the Work by Mouhot, C., Russ, E. and Sire, Y. (2011)

Example

(1) Let $V(x) = \varepsilon(1+|x|)$ with some $\varepsilon > \delta$. Then,

$$\int \big(f(x) - \mu_V(f)\big)^2 \frac{e^{V(x)-\delta|x|}}{(1+|x|)^{d+\alpha}} \mu_V(dx) \leqslant C\hat{D}_{\alpha,V,\delta}(f,f).$$

(2) Let $V(x) = 1 + |x|^2$. Then,

$$\int (f(x) - \mu_V(f))^2 \exp\left(\frac{1}{2}(1+|x|^2)\right) \mu_V(dx) \leqslant c\hat{D}_{\alpha,V,\delta}(f,f).$$

However,

$$\int (f(x) - \mu_V(f))^2 \left(1 + |x|^\alpha\right) \mu_V(dx) \leqslant cD_{\alpha,V,\delta}(f,f).$$

$\delta > 0$: Qualitatively Sharp

Proposition (Chen, X. and W. (2013))

Let $\delta > 0$. *Suppose that*

$$\mu_V\left(\left(f - \mu_V(f)\right)^2\right) \leqslant C D_{\alpha,V,\delta}(f,f).$$

Then, there is a constant $\lambda > 0$ *such that*

$$\int e^{\lambda|x|}\mu_V(dx) < \infty.$$

Note:

$$\liminf_{|x|\to\infty} \frac{e^{V(x)-\delta|x|}}{|x|^{d+\alpha}} > 0.$$

$\delta = 0$: Completeness of the Work by Wang, F.-Y. and W. (2012)

Theorem (Chen, X. and W. (2013))

Suppose that

$$\liminf_{|x|\to\infty} \frac{e^{V(x)}}{|x|^{d+\alpha}} > 0.$$

Then,

$$\int \big(f(x) - \mu_V(f)\big)^2 \frac{e^{V(x)}}{(1+|x|)^{d+\alpha}} \mu_V(dx) \leqslant C\hat{D}_{\alpha,V}(f,f).$$

$\delta = 0$: First Interesting Point

Proposition (Chen, X. and W. (2013))

Suppose

$$\liminf_{|x|\to\infty} \frac{e^{V(x)}}{|x|^{d+\alpha}} > 0.$$

For any continuous positive function ω, there is a constant $C(\omega) > 0$ such that

$$\int \big(f(x) - \mu_V(f)\big)^2 \frac{e^{V(x)}}{(1+|x|)^{d+\alpha}} \mu_V(dx)$$
$$\leqslant C(\omega) \int \omega(x) \int_{\{|x-y|>1\}} \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}} \, dy \, \mu_V(dx).$$

Note:

$$\mathrm{Var}_{\mu_V}(f) \leqslant C(\omega) \int \omega(x) |\nabla f(x)|^2 \, \mu_V(dx).$$

$\delta = 0$: Second Interesting Point

Proposition (Chen, X. and W. (2013))

Suppose that

$$\int \big(f(x) - \mu_V(f)\big)^2 \omega(x)\, \mu_V(dx) \leqslant C D_{\alpha,V}(f,f)$$

where $\lim_{|x|\to\infty} \omega(x) = \infty$. *Then the following super Poincaré inequality*

$$\mu_V(f^2) \leqslant r D_{\alpha,V}(f,f) + \beta(r)\mu_V(|f|)^2, \quad r > 0$$

holds with

$$\beta(r) = \inf\left\{C_2 H(t)^{2+d/\alpha} h(t)^{-1-d/\alpha}\left(1 + s^{-d/\alpha}\right) : \ldots\right\}.$$

Wrong !!!

Suppose that

$$\int \big(f(x) - \mu_V(f)\big)^2 \omega(x)\, \mu_V(dx) \leqslant C\hat{D}_{\alpha,V}(f,f)$$

where $\lim_{|x|\to\infty} \omega(x) = \infty$. Then the following super Poincaré inequality holds

$$\mu_V(f^2) \leqslant r\hat{D}_{\alpha,V}(f,f) + \beta(r)\mu_V(|f|)^2, \quad r > 0,$$

where

$$\beta(r) = \ldots$$

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Three Different Dirichlet Forms

(1)

$$D_{\alpha,V,\delta}(f,f) := \iint \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}} e^{-\delta|y-x|}\,dy\,\mu_V(dx).$$

Mouhot, C., Russ, E. and Sire, Y. (2011);
Wang, F.-Y. and W. (2012); W. (2013)

(2)

$$\hat{D}_{\alpha,V,\delta}(f,f) := \iint_{\{|x-y|>1\}} \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}} e^{-\delta|y-x|}\,dy\,\mu_V(dx).$$

Chen, X. and W. (2013)

(3)

$$\check{D}_{\alpha,V,\delta}(f,f) := \iint_{\{|x-y|\leqslant 1\}} \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}} e^{-\delta|y-x|}\,dy\,\mu_V(dx).$$

Gressman, P.T. (2012); Chen, X. and W. (2012)

Three Different Dirichlet Forms

(1)

$$D_{\alpha,V}(f,f) := \iint \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}}\, dy\, \mu_V(dx).$$

(2)

$$\hat{D}_{\alpha,V}(f,f) := \iint_{\{|x-y|>1\}} \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}}\, dy\, \mu_V(dx).$$

$$\inf_{|x|\to\infty} \frac{e^{V(x)}}{|x|^{d+\alpha}} > 0$$

(3)

$$\check{D}_{\alpha,V}(f,f) := \iint_{\{|x-y|\leqslant 1\}} \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}}\, dy\, \mu_V(dx).$$

$$\mu_V(dx) = C_\lambda e^{-\lambda|x|}$$

Poincaré Inequalities for Non-local Dirichlet Forms

$$D_{\rho,V}(f,f) := \iint (f(y)-f(x))^2 \rho(|y-x|)\,dy\,\mu_V(dx).$$

- $\rho(r) = 0$ for $r > 1$; $\mu_V(dx) = e^{-V(x)}\,dx$
- $\rho(r) = 0$ for $1 \geqslant r > 0$; $\mu_V(dx)$ and ρ
- $\rho(r) > 0$ for all $r > 0$.

Other Non-local Dirichlet Forms

- $$\mu_{2V}(dx) := e^{-2V(x)}\,dx$$

- $$D^G_{\alpha,V}(f,f) := \iint \frac{\big(f(x)-f(y)\big)^2}{|y-x|^{d+\alpha}}\,e^{-V(y)}\,dy\,e^{-V(x)}\,dx$$

- Chen, Z.-Q and Zhang, T.S. (2002); Song, R. (2006)

- $$\mu_V(dx) := e^{-V(x)}\,dx$$

- $$D^*_{\alpha,V}(f,f) := \iint \frac{\big(f(x)-f(y)\big)^2}{|y-x|^{d+\alpha}}\,e^{-V(y)}\,dy\,e^{-V(x)}\,dx$$

Outline

$$dX_t = b(X_t)\,dt + dZ_t,$$

where $(Z_t)_{t\geqslant 0}$ is a symmetric α-stable process.

(1) Assume that the SDE above has the unique invariant probability measure $\mu_V(dx) = e^{-V(x)}\,dx$.

$$-(-\Delta)^{\alpha/2}(e^{-V}) = \mathrm{div}(e^{-V}b)$$

$$\langle f, -Lf\rangle_{L^2(\mu_V)} = \iint \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}}\,dy\,\mu_V(dx)$$

(2) How to apply the results about $D_{\alpha,V}$?

$$-(-\Delta)^{\alpha/2}(e^{-V}) = \mathrm{div}(e^{-V}b) \quad (???)$$

Application to SDE

$$dX_t = b(X_t)\,dt + dZ_t.$$

(3) Exponential Ergodicity (W. (2013b))

$$\langle b(x), x\rangle \leqslant -c|x|^2, \quad |x| \gg 1$$

(4) Poincaré Inequalities (.....)

$$d = 1$$

or

$$\operatorname{sgn}(x_i) b_i(x) \leqslant -c \operatorname{sgn}(x_i)\, x_i, \quad d \geqslant 2, 1 \leqslant i \leqslant d, |x| \gg 1$$

e.g. $b(x) = \nabla V(|x|)$ for $x \in \mathbb{R}^d$ with $d \geqslant 2$.

Thank you for your attention!