On total progeny of multitype Galton-Watson process and the first passage time of random walk on lattice

Huaming Wang

Department of Mathematics, Anhui Normal University

国学思想与大学数学

- 《易经·八卦》与概率论
- “微积分”与“阴阳学说”
- “道”与“无穷”

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古典概型与易经六十四卦的形成过程

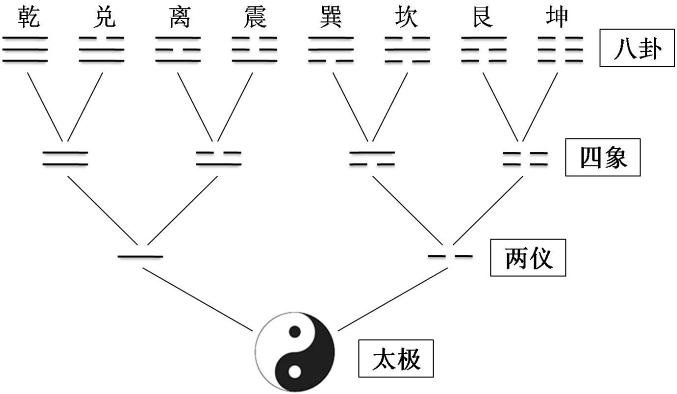

图1. 八卦形成图

Figure1. The deduction of Bagua

微积分与阴阳

Figure: 太极示意图

“道”与“无穷”

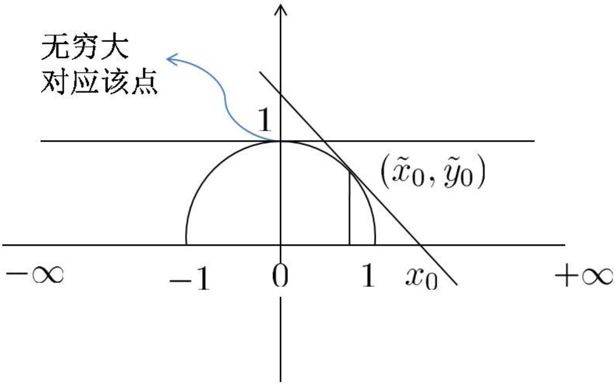

图2. O与无穷

Figure2. Zero and Infinity

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1. Motivation
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3. Sketch of proofs

An example

Let $\{X_n\}$ be a simple random walk:

$$P(X_{n+1} = x + 1 | X_n = x) = p,$$
$$P(X_{n+1} = x - 1 | X_n = x) = 1 - p.$$

Define $T = \inf\{n \geq 0 : X_n = 1\}$.

$$P(T = 2n + 1) = \frac{1}{2n+1} \binom{2n+1}{n+1} p^{n+1} (1-p)^n, \ n \geq 0.$$

Probability generating function of T :

$$\zeta(s) := E(s^T) = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs}, \ |s| < 1.$$

Some more general RW

1. RW with stay

$$
\begin{aligned}
&P(X_{n+1} = x+1 | X_n = x) = p, \\
&P(X_{n+1} = x-1 | X_n = x) = q, \quad p+q+r=1. \\
&P(X_{n+1} = x \quad\quad | X_n = x) = r.
\end{aligned}
$$

2. (2,1) RW

$$
\begin{aligned}
&P(X_{n+1} = x-l | X_n = x) = q_l, \ l = 1, 2; \\
&P(X_{n+1} = x+1 | X_n = x) = p, \ p + q_1 + q_2 = 1.
\end{aligned}
$$

3. (L,R) RW

$$P(X_{n+1} = X_n + l | X_n, ..., X_0) = p_l, \ l \in \{-L, R\}/\{0\},$$

where $\sum_l p_l = 1$.

Question

For the above random walk, define

$$T = \inf\{n \geq 0 : X_n > 0\},$$
$$\zeta(s) = E(s^T), \ |s| < 1.$$

Question: $\zeta(s) =?$ $P(T = n) =?$

Hint: Using the branching structure hidden in the path of random walk.

For nearest simple RW, define for $i \leq 0$,

$$U_i = \#\{0 \leq n < T : X_n = i + 1, X_{n+1} = i\}.$$

$\{U_i\}$ is a G-W process and

$$T = 1 + 2 \sum_{i \leq 0} U_i.$$

Basic idea

T is a linear functional of the total progeny W of a Galton-Watson Process.

To study the distribution of T, it is enough to study W.

Nearest simple RW $\leftrightarrow$ **single type G-W process.**

RW with bounded jumps$\leftrightarrow$**multitype G-W process**.

Our goal:

(a) Study the p.g.f. of the total progeny of multitype G-W process

(b) Give the p.g.f. of T for some non nearest RW and estimate the tail probability $P(T > n)$.

Notations

$\mathbb{Z}_+ = \{0, 1, 2, ...\}$.

$p_i(\cdot),\ i = 1, ..., L$ are probability measures on $\mathbb{Z}_+^L$.

$\{Z_n\}$ is a multitype G-W process with offspring distribution

$P(Z_{n+1} = \left(n^{(1)}, ..., n^{(L)}\right) | Z_n = \mathbf{e}_i) = p_i \left(n^{(1)}, ..., n^{(L)}\right),\ i = 1, ..., L.$

$$\phi^{(i)}(s^{(1)}, ..., s^{(L)}) = E\left(\left(s^{(1)}\right)^{Z_1^{(1)}} \cdots \left(s^{(L)}\right)^{Z_1^{(L)}} \Big| Z_0 = \mathbf{e}_i\right), i = 1, ..., L.$$

$Y_n = \sum_{i=0}^{n} Z_i$: the total progeny of the first n generation.

$$G_n^{(i)}(s^{(1)}, ..., s^{(L)}) = E\left(\left(s^{(1)}\right)^{Y_n^{(1)}} \cdots \left(s^{(L)}\right)^{Y_n^{(L)}} \Big| Z_0 = \mathbf{e}_i\right),$$

$|s^{(l)}| < 1, 1 \leq l \leq L.$

Notations

Let $\pi^{(i)} = P(Z_n = \mathbf{0} \text{ for some } n | Z_0 = \mathbf{e}_i)$, being the extinction probabilities.

Introduce

$$\begin{aligned}
&\mathbf{s} = (s^{(1)}, ..., s^{(L)}), \\
&\boldsymbol{\phi}(\mathbf{s}) = (\phi^{(1)}(\mathbf{s}), ..., \phi^{(L)}(\mathbf{s})), \\
&\mathbf{G}_n(\mathbf{s}) = (G_n^{(1)}(\mathbf{s}), ..., G_n^{(L)}(\mathbf{s})), \\
&\boldsymbol{\pi} = (\pi^{(1)}, ..., \pi^{(L)}), \\
&\mathbf{1} = (1, ..., 1).
\end{aligned}$$

The main results—p.g.f. for the total progeny

The following theorem gives the p.g.f. of the total progeny of a critical or subcritical multitype G-W process. For the single type case, see Feller (1968) and Dwass (1969).

Theorem 1 (p.g.f. of the total progeny)

Suppose that the branching process $\{Z_n\}$ is extinct, that is

$$\boldsymbol{\pi} = (\pi^{(1)}, ..., \pi^{(L)}) = (1, ..., 1).$$

Then the limit

$$\boldsymbol{\rho}(\mathbf{s}) := \lim_{n\to\infty} \mathbf{G}_n(\mathbf{s})$$

exists and for fixed $\mathbf{s}$, $\boldsymbol{\rho}(\mathbf{s})$ is the **unique** solution of equation

$$\mathbf{u} = \mathbf{s}\boldsymbol{\phi}(\mathbf{u}).$$

Moreover, $\boldsymbol{\rho}(\mathbf{s})$ is an **honest probability generating function**.

Examples

Example 1

Let $\{Z_n\}_{n\geq 0}$ be a 2-type branching process with offspring distributions

$$P\left(Z_{n+1}=(a,b)|Z_n=\mathbf{e}_1\right)=\frac{(a+b)!}{a!b!}q^a r^b p,\ \ a,b\geq 0,$$
$$P\left(Z_{n+1}=\mathbf{0}|Z_n=\mathbf{e}_2\right)=1,$$

with $p,q,r>0,\ p+q+r=1$. Let $Y_n=\sum_{i=0}^n Z_i$ and $Y=\lim_{n\to\infty} Y_n$. If $q\leq p$, then $P(Y<\infty)=1$ and the p.g.f. of Y

$$\boldsymbol{\rho}(\mathbf{s})=(\rho^{(1)}(\mathbf{s}),\rho^{(2)}(\mathbf{s}))=\left(\frac{1-rs^{(2)}-\sqrt{\left(1-rs^{(2)}\right)^2-4pqs^{(1)}}}{2q},s^{(2)}\right).$$

Moreover, if $p=q=\frac{1-r}{2}$(**Critical**) and $P(Z_0=\mathbf{e}_1)=1$ then

$$\lim_{n\to\infty}\sqrt{n}P(|Y|>n)=\frac{1}{\sqrt{\pi}}\sqrt{\frac{1+r}{1-r}}.$$

Example 2

If we replace the offspring distribution of Example 1 by

$$P\left(Z_{n+1}=(a,b)|Z_n=\mathbf{e}_1\right)=\frac{(a+b)!}{a!b!}q^a r^b p,$$
$$P\left(Z_{n+1}=(a+1,b)|Z_n=\mathbf{e}_2\right)=\frac{(a+b)!}{a!b!}q^a r^b p,\ \ a,b\geq 0,$$

then $\rho^{(1)}(\mathbf{u})$ is the smallest real solution of

$$r\frac{u^{(2)}}{u^{(1)}}\left(\rho^{(1)}\right)^3-q\left(\rho^{(1)}\right)^2-\rho^{(1)}+pu^{(1)}=0$$

and $\rho^{(2)}(\mathbf{u})=\left(\rho^{(1)}\right)^2\frac{u^{(2)}}{u^{(1)}}$.

A remark

Remark 1

The 2-type G-W process in Example 1 could be got by decomposing the path of RW with stay, while the one in Example 2 describes the branching structure with in (2,1) RW.
Therefore, by using the conclusions in Example 1 and 2, one could derive the probability generating functions of the first passage time T of the corresponding random walk.

Results for random walk with stay

The following theorem gives the p.g.f. and the tail probability estimate of the first passage time T of random walk with stay.

Theorem 2 (Random walk with stay)

Suppose that $\{X_n\}$ is a random walk with stay and that $q \leq p$. Let T be its first passage time of position 1. Then

$$E(u^T) = \frac{1 - ru - \sqrt{(1-ru)^2 - 4pqu^2}}{2qu}, \ |u| < 1.$$

Moreover if $p = q = \frac{1-r}{2}$ (**Implying the walk is recurrent**) then

$$\lim_{n\to\infty} \sqrt{n} P(T \geq n) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1-r}}.$$

Results for (2,1) random walk

For (2,1) RW which transient to the right or recurrent, we have

Theorem 3 ((2,1) random walk)

Suppose that $\{X_n\}$ is a (2-1) random walk and that $p - q_1 - 2q_2 \geq 0$, (**implying** $\limsup_{n\to\infty} X_n = \infty$). Let $h(u)$ be the probability generating function of the first passage time T. Then $h(s)$ is the smallest real solution of equation

$$q_2 s h^3 + q_1 s h^2 - h + ps = 0, \ |s| < 1. \tag{1}$$

Notes:

The p.g.f. of T could be calculated by path decomposition up to the first step of the walk.

In Theorem 2 and 3, we illustrate how to solve the problems of random walk with bounded jumps by means of branching processes.

Proofs

Theorem 1 (p.g.f. of the total progeny)

Suppose that the branching process $\{Z_n\}$ is extinct, that is

$$\boldsymbol{\pi} = (\pi^{(1)}, ..., \pi^{(L)}) = (1, ..., 1).$$

Then the limit

$$\boldsymbol{\rho}(\mathbf{s}) := \lim_{n \to \infty} \mathbf{G}_n(\mathbf{s})$$

exists and for fixed $\mathbf{s}$, $\boldsymbol{\rho}(\mathbf{s})$ is the **unique** solution of equation

$$\mathbf{u} = \mathbf{s}\boldsymbol{\phi}(\mathbf{u}).$$

Moreover, $\boldsymbol{\rho}(\mathbf{s})$ is an **honest probability generating function**.

Proofs

Sketch proof of Theorem 1: By induction,

$$G_{n+1}^{(i)}(\mathbf{s}) = s^{(i)}\phi^{(i)}(\mathbf{G}_n(\mathbf{s})).$$

For $\mathbf{0} \ll \mathbf{s} \ll \mathbf{1}$, $\mathbf{G}_1(\mathbf{s}) = \mathbf{s}\boldsymbol{\phi}(\mathbf{s}) \ll \mathbf{s} = \mathbf{G}_0(\mathbf{s})$.

Again by induction, $\{\mathbf{G}_n(s)\}$ is monotone decreasing in n.

The limit $\boldsymbol{\rho}(\mathbf{s}) := \lim_{n\to\infty} \mathbf{G}_n(\mathbf{s})$ exists and $\boldsymbol{\rho}(\mathbf{s})$ satisfies equation

$$\boldsymbol{\rho}(\mathbf{s}) = \mathbf{s}\boldsymbol{\phi}(\boldsymbol{\rho}(\mathbf{s})). \tag{2}$$

Fixing $\mathbf{0} \ll \mathbf{s} \ll \mathbf{1}$, define $F : [0,1]^L \mapsto [0,1]^L$ by

$$\mathbf{F}(\mathbf{u}) = \mathbf{s}\boldsymbol{\phi}(\mathbf{u}).$$

A version of fixed point theorem (See Smith and Stuart [8].) yields that $\mathbf{F}$ has a unique fixed point in $[0,1]^L$.

Proofs

Therefore there exists a unique $\mathbf{u} \in [0,1]^L$ such that

$$\mathbf{u} = \mathbf{s}\boldsymbol{\phi}(\mathbf{u}).$$

Then $\boldsymbol{\rho}(\mathbf{s})$ is the unique solution of

$$\mathbf{u} = \mathbf{s}\boldsymbol{\phi}(\mathbf{u}).$$

$$\boldsymbol{\rho}(\mathbf{1}) = \mathbf{1}\boldsymbol{\phi}(\mathbf{1}) = \mathbf{1}.$$

$\boldsymbol{\rho}(\mathbf{s})$ is an honest p.g.f. function. Indeed, it is the p.f.g. of

$$Y = \sum_{i=0}^{\infty} Y_n.$$

□

Results for random walk with stay

Theorem 2 (Random walk with stay)

Suppose that $\{X_n\}$ is a random walk with stay and that $q \leq p$. Let T be its first passage time of position 1. Then

$$E(u^T) = \frac{1 - ru - \sqrt{(1-ru)^2 - 4pqu^2}}{2qu}, \ |u| < 1.$$

Moreover if $p = q = \frac{1-r}{2}$ (**Implying the walk is recurrent**) then

$$\lim_{n \to \infty} \sqrt{n} P(T \geq n) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1-r}}.$$

Branching structure for RW

For RW with stay, let

$$\left(U_1^{(1)}, U_1^{(2)}\right) = (1, 0), \tag{3}$$

and for $i \leq 0$ define

$$\begin{aligned} U_i^{(1)} &= \#\{0 \leq n < T_1 : X_n = i, X_{n+1} = i - 1\}, \\ U_i^{(2)} &= \#\{0 \leq n < T_1 : X_n = i, X_{n+1} = i\} \end{aligned} \tag{4}$$

counting the number of steps by the walk from i to $i-1$ and the steps from i to i itself respectively.

The branching structure for random walk with stay could be found in Wang [10] and Zeitouni [11].

Branching structure for RW

Theorem A:Branching structure for RW with stay

Let $\{X_n\}$ be a random walk with stay. If $q \leq p$, then $\left\{\left(U_n^{(1)}, U_n^{(2)}\right)\right\}_{n\leq 1}$ defined in (3) and (4) forms a 2-type branching process. Its offspring distributions are

$$P\left(\left(U_i^{(1)}, U_i^{(2)}\right) = (a,b) \middle| \left(U_{i+1}^{(1)}, U_{i+1}^{(2)}\right) = (1,0)\right) = \frac{(a+b)!}{a!b!} q^a r^b p,$$
$$P\left(\left(U_i^{(1)}, U_i^{(2)}\right) = (0,0) \middle| \left(U_{i+1}^{(1)}, U_{i+1}^{(2)}\right) = (0,1)\right) = 1.$$

Moreover the hitting time T could be expressed by the branching process as

$$T = 1 + \sum_{i\leq 0} 2U_i^{(1)} + U_i^{(2)} = \sum_{i\leq 0} (U_i^{(1)}, U_i^{(2)}) \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Sketch proof of Theorem 2

Proof of Theorem 2:

Let $\{Z_n\}$ be the 2-type branching process in Example 1, $Y = \sum_{n=0}^{\infty} Z_n$. Then the p.g.f. of Y

$$\boldsymbol{\rho}(\mathbf{s}) = (\rho^{(1)}(\mathbf{s}), \rho^{(2)}(\mathbf{s})) = \left(\frac{1 - rs^{(2)} - \sqrt{\left(1 - rs^{(2)}\right)^2 - 4pqs^{(1)}}}{2q}, s^{(2)} \right).$$

Comparing the branching process $\{Z_n\}$ in Theorem 1 and $\{(U_n^{(1)}, U_n^{(2)})\}_{n \leq 1}$ in Theorem A, if $P(Z_0 = \mathbf{e}_1) = 1$ then T has the same distribution with

$$1 + \sum_{n=1}^{\infty} 2Z_n^{(1)} + Z_n^{(2)} = \sum_{n=0}^{\infty} 2Z_n^{(1)} + Z_n^{(2)} - 1 = 2Y^{(1)} + Y^{(2)} - 1.$$

Proof of Theorem 2

Letting $\eta(u) := E(u^{T+1})$, then

$$\eta(u) = E(u^{2Y^{(1)}} u^{Y^{(2)}}) = \rho^{(1)}(u^2, u) = \frac{1 - ru - \sqrt{(1-ru)^2 - (1-r)^2 u^2}}{1-r}.$$

Define $\alpha_n = P(T \geq n)$ and let $\alpha(u) = \sum_{n=0}^{\infty} \alpha_n u^n$. Some calculation yields

$$\alpha(u) = \frac{1-\eta(u)}{1-u} = \frac{1}{1-r}\sqrt{\frac{1-(2r-1)u}{1-u}} - \frac{r}{1-r}.$$

By some subtle estimation, we could prove that

$$\lim_{n\to\infty} \sqrt{n}\alpha_n = \frac{1}{1-r}\Big(\frac{2}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}\left(2 - \sqrt{1-(2r-1)}\right)\Big) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{1-r}}.$$

□

Branching structure of (2,1) RW

Next, consider (2-1) random walk. Let $T = \inf[n > 0 : X_n = 1]$. Define, for $-\infty < i \leq 0$,

$$\begin{aligned} U_i^{(1)} &= \#\{0 < k < T_1 : X_{k-1} > i, X_k = i\} \\ U_i^{(2)} &= \#\{0 < k < T_1 : X_{k-1} > i, X_k = i - 1\} \end{aligned} \tag{5}$$

and set

$$\left(U_1^{(1)}, U_1^{(2)}\right) = (1, 0). \tag{6}$$

Branching structure of (2,1) RW

Then we have the following theorem.

Theorem B (Hong and Wang [4])

Let $\{X_n\}$ is a (2-1) random walk. Suppose that $E(X_1) = p - q_1 - 2q_2 \geq 0$. Then $\left\{\left(U_i^{(1)}, U_i^{(2)}\right)\right\}_{i\leq 1}$ defined in (5) and (6) forms a 2-type branching process with offspring distributions

$$P\left((U_{i-1}^{(1)}, U_{i-1}^{(2)}) = (a,b) \middle| (U_i^{(1)}, U_i^{(2)}) = (1,0)\right) = \frac{(a+b)!}{a!b!} q_1^a q_2^b p,$$

$$P\left((U_{i-1}^{(1)}, U_{i-1}^{(2)}) = (a+1,b) \middle| (U_i^{(1)}, U_i^{(2)}) = (0,1)\right) = \frac{(a+b)!}{a!b!} q_1^a q_2^b p,$$

and that

$$T = 1 + \sum_{i\leq 0} 2U_i^{(1)} + U_i^{(2)}.$$

A remark on (L, R) RW

With the branching structure for (2,1) RW in hand, using the some argument as in the proof of Theorem 2, we can easily carry out the proof of Theorem 3.

Remark 2: about (L,R) RW

The branching structure of (L,R) RW was revealed in Hong and Wang [5]. However, it evolved a $(1 + ... + L) \times (1 + ...R)$-type branching process. By a similar method one could use the branching structure to give the probability generating function of T. Of course, in this general case, the root for equation $\mathbf{u} = \mathbf{s}\boldsymbol{\phi}(\mathbf{u})$ will be very complicated.

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Thanks a lot

非常感谢

hmking@mail.ahnu.edu.cn