Dirichlet Heat Kernels for Rotationally symmetric Lévy Processes

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Outline

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Brownian motion has been widely used in various fields. However, in some applications, Brownian motion is obviously not appropriate.

Because the sample paths of BM are continuous, it is not useful to model jump behavior. Because BM has exponential tails, it is not useful to model heavy tailed phenomenon. Yet, jumps and heavy tailed phenomena occur a lot in applications.

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However, general Lévy processes are not very tractable. We would like to find a sub class of Lévy processes which is large enough, and yet more tractable than general Lévy processes.

Let $W = (W_t : t \geq 0)$ be a d-dimensional Brownian motion, and let $S = (S_t : t \geq 0)$ be an independent subordinator (that is, an increasing Lévy process starting from 0). The process $X = (X_t : t \geq 0)$ defined by $X_t := W_{S_t}$, $t \geq 0$ is called a subordinate Brownian motion. A subordinate BM is a Lévy process.

Subordinate Brownian motions form a large subclass of Lévy processes, yet they are much more tractable than general Lévy processes. Subordinate Brownian motions are discontinuous in general and can have heavy tails. Subordinate Brownian motions are widely used in various applications. For instance, subordinate Brownian motions are used in mathematical finance, as the subordintaor can be thought of as the "operational time" or "intrinsic time".

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A subordinator $S = (S_t : t \geq 0)$ is usually characterized by its Laplace transform

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \qquad \forall\, t, \lambda > 0.$$

The function ϕ is called the Laplace exponent of the subordinator.

The Laplace exponent of a subordinator can be written in the form

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)$$

where $b \geq 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$. b is called the drift coefficient and μ the Lévy measure of the subordinator.

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Here are some examples of Laplace exponents of subordinators.:
$\phi(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0, 2]$;
$\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$, $\alpha \in (0, 2)$ and $m > 0$;
$\phi(\lambda) = \lambda + \lambda^{\alpha/2}$, $\alpha \in (0, 2)$;
$\phi(\lambda) = \lambda^{\alpha/2} + \lambda^{\beta/2}$, $0 < \beta < \alpha < 2$.

If the Laplace exponent of S is ϕ, then the Lévy exponent of the subordinate Brownian motion X is given by $\Phi(\theta) = \phi(|\theta|^2)$. The infinitesimal generator can be written as $-\phi(-\Delta)$.

When $\phi(\lambda) = \lambda^{\alpha/2}$, the resulting subordinate Brownian motion turns out to be a symmetric α-stable process. The infinitesimal generator of this process can be written as $-(-\Delta)^{\alpha/2}$.

When $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$, the resulting subordinate Brownian motion turns out to be a relativistic α-stable process with mass m. The infinitesimal generator of this process can be written as $m - (-\Delta + m^{2/\alpha})^{\alpha/2}$.

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When $\phi(\lambda) = \lambda + \lambda^{\alpha/2}$ for some $\alpha \in (0, 2)$, the resulting subordinate Brownian motion turns out to be the independent sum of a Brownian motion and an α-stable process. The infinitesimal generator of this process can be written as $\Delta - (-\Delta)^{\alpha/2}$.

When $\phi(\lambda) = \lambda^{\beta/2} + \lambda^{\alpha/2}$ the resulting subordinate Brownian motion turns out to be the independent sum of β and α-stable processes. The infinitesimal generator of this process can be written as $-(-\Delta)^{\beta/2} - (-\Delta)^{\alpha/2}$.

We are mainly interested in subordinators which are not compound Poisson processes. That is, we will assume that $\phi(\infty) = \infty$. We will also assume some technical condition on ϕ which I am not going to mention here.

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A subordinate Brownian motion has a transition density given by

$$p(t,x,y) = \int_{[0,\infty)} q(s,x,y)\mathbb{P}(S_t \in ds)$$

where

$$q(s,x,y) = (4\pi t)^{-d/2} \exp(-\frac{|x-y|^2}{4t}).$$

$p(t,x,y)$ is the fundamental solution of the equation

$$\partial_t u = -\phi(-\Delta)u.$$

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The Lévy exponent of any Lévy process admits a Lévy-Khintchine decomposition. The Lévy measure of the process X has a density J, called the Lévy density, given by

$$J(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} \mu(t) dt, \qquad x \in \mathbb{R}^d.$$

Thus $J(x) = j(|x|)$ with

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t) dt, \qquad r > 0.$$

Note that the function $r \mapsto j(r)$ is continuous and decreasing on $(0, \infty)$. We will sometimes use notation $J(x, y)$ for $J(x - y)$.

For any open subset $D \subset \mathbb{R}^d$, we use τ_D to denote the first time the process X exits D. We define the process X^D by $X_t^D = X_t$ for $t < \tau_D$ and $X_t^D = \partial$ for $t \geq \tau_D$, where ∂ is a cemetery point. X^D is called the subprocess of X killed upon exiting D. The infinitesimal generator of X^D is $-\phi(-\Delta)|_D$.

X^D has a continuous transition density $p_D(t, x, y)$ with respect to the Lebesgue measure. $p_D(t, x, y)$ contains all the statistical information about the process X^D, yet it is very rarely known explicitly.

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Due to the complication of the boundary behavior, sharp two-sided estimates on the Dirichlet heat kernels in smooth domains are more difficult. The complete solution was achieved in 2002.

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The recent study of subordinate BMs started with the special case of symmetric α-stable processes. We first assume that X is a symmetric α-stable process, $\alpha \in (0, 2)$.

It is a classical result that the heat kernel $p(t, x, y)$ of a symmetric α-stable process admits the following two-sided estimates

$$p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}$$

The Dirichlet heat kernel estimates is much more difficult. The first break-through is the following two-sided estimates on the Dirichlet heat kernel $p_D(t, x, y)$ of the fractional Laplacian $-(\Delta)^{\alpha/2}|_D$ when D is a $C^{1,1}$ open set.

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Theorem [CKS, EJMS 10]

Let D be a $C^{1,1}$ open subset of $\mathbb{R}^d$ and $\delta_D(x)$ the distance between x and D^c.
(i) For every $T > 0$, on $(0, T] \times D \times D$

$$p_D(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

(ii) Suppose in addition that D is bounded. For every $T > 0$, there are positive constants $c_1 < c_2$ so that on $[T, \infty) \times D \times D$,

$$c_1\, e^{-\lambda_1 t}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2} \le p_D(t,x,y) \le c_2\, e^{-\lambda_1 t}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2},$$

where $\lambda_1 > 0$ is the smallest eigenvalue of the Dirichlet fractional Laplacian $(-\Delta)^{\alpha/2}|_D$.

[CKS, EJMS 10] provides a road-map for obtaining sharp two-sided heat kernel estimates for other Lévy processes. The techniques of [1] have been adapted for many other processes.

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In [CKS, JLMS 11], sharp two-sided Dirichlet heat kernel estimates for mixtures of Brownian motion and stable processes (that is, subordinate Brownian motions via the Bernstein function $\phi(\lambda) = \lambda + \lambda^{\alpha/2}$) were established.

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Now I am going to present recent results for general subordinate Brownian motions. X is now a subordinate Brownian motion via a subordinator with Laplace exponent ϕ.

We will assume:
(A): There exist constants $\delta_1, \delta_2 \in (0,1)$, $a_1 \in (0,1)$, $a_2 \in (1,\infty)$ and $R_0 > 0$ such that

$$a_1 \lambda^{\delta_1} \phi(r) \le \phi(\lambda r) \le a_2 \lambda^{\delta_2} \phi(r) \qquad \text{for } \lambda \ge 1 \text{ and } r \ge R_0.$$

Define

$$\Phi(r) = \frac{1}{\phi(r^{-2})}.$$

and let Φ^{-1} be the inverse function of Φ.

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$$\Phi(r) = \frac{1}{\phi(r^{-2})}.$$

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We may assume the following two conditions later.

(B): There exist constants $C_1 > 0$ and $C_2 \in (0, 1]$ such that

$$p(t, u) \le C_1 p(t, C_2 r) \qquad \text{for } t \in (0, 1] \text{ and } u \ge r > 0.$$

(C): There exist constants $C_3 > 0$ and $C_4 \in (0, 1]$ such that

$$p(t, r) \le C_3 t j(C_4 r) \qquad \text{for } t \in (0, 1] \text{ and } r > 0.$$

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Theorem [CKS, 13a]

Suppose that (**A**) holds and D is a $C^{1,1}$ open set in $\mathbb{R}^d$.

(1) For every $T > 0$, there exists $c_1 > 0$ such that for all $(t, x, y) \in (0, T] \times D \times D$,

$$p_D(t,x,y) \geq c_1 \left(1 \wedge \frac{\Phi(\delta_D(x))}{t}\right)^{1/2} \left(1 \wedge \frac{\Phi(\delta_D(y))}{t}\right)^{1/2} \left(\Phi^{-1}(t)^{-d} \wedge tJ(x,y)\right).$$

(2) For every $T > 0$, there exists $c_2 > 0$ such that for all $(t, x, y) \in (0, T] \times D \times D$,

$$p_D(t,x,y) \leq c_2 \left(1 \wedge \frac{\Phi(\delta_D(x))}{t}\right)^{1/2} \left(1 \wedge \frac{\Phi(\delta_D(y))}{t}\right)^{1/2} p(t, |x-y|/4).$$

Corollary [CKS, 13a]

Suppose that (**A**) holds and D is a $C^{1,1}$ open set in $\mathbb{R}^d$. If D is unbounded, we assume in addition that condition **(C)** holds. For every $T > 0$, there exist $c_1 > 0$ and $c_2 > 0$ such that for $0 < t \le T$, $x, y \in D$,

$$c_1 \left(1 \wedge \frac{\Phi(\delta_D(x))}{t}\right)^{1/2} \left(1 \wedge \frac{\Phi(\delta_D(y))}{t}\right)^{1/2} \left(\Phi^{-1}(t)^{-d} \wedge tj(|x-y|)\right)$$

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Outline

In this section we will assume that X is a purely discontinuous rotationally symmetric Lévy process with Lévy exponent $\Psi(\xi)$. Because of rotational symmetry, the function Ψ depends on $|\xi|$ only, and thus we write $\Psi(\xi) = \Psi(|\xi|)$. We will use $p(t,x,y) = p(t,y-x)$ to denote the transition density of X. For any open set D, we will use $p_D(t,x,y)$ to denote the transition density of X^D.

We further assume that the Lévy measure of X has a density with respect to the Lebesgue measure on $\mathbb{R}^d$, which is denoted by $J_X(x,y) = J_X(x-y) = j_X(|y-x|)$. We assume that $j_X(r)$ is continuous on $(0,\infty)$ and that there is a constant $\gamma > 1$ such that

$$\gamma^{-1} j(r) \le J_X(r) \le \gamma j(r) \quad \text{for all } r > 0.$$

This implies that

$$\gamma^{-1}\phi(|\xi|^2) \le \Psi(|\xi|) \le \gamma\phi(|\xi|^2) \quad \text{for all } \xi \in \mathbb{R}^d.$$

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This implies that

$$\gamma^{-1}\phi(|\xi|^2) \le \Psi(|\xi|) \le \gamma\phi(|\xi|^2) \quad \text{for all } \xi \in \mathbb{R}^d.$$

In this section we will assume that X is a purely discontinuous rotationally symmetric Lévy process with Lévy exponent $\Psi(\xi)$. Because of rotational symmetry, the function Ψ depends on $|\xi|$ only, and thus we write $\Psi(\xi) = \Psi(|\xi|)$. We will use $p(t, x, y) = p(t, y - x)$ to denote the transition density of X. For any open set D, we will use $p_D(t, x, y)$ to denote the transition density of X^D.

We further assume that the Lévy measure of X has a density with respect to the Lebesgue measure on $\mathbb{R}^d$, which is denoted by $J_X(x, y) = J_X(x - y) = j_X(|y - x|)$. We assume that $j_X(r)$ is continuous on $(0, \infty)$ and that there is a constant $\gamma > 1$ such that

$$\gamma^{-1} j(r) \le J_X(r) \le \gamma j(r) \quad \text{for all } r > 0.$$

This implies that

$$\gamma^{-1} \phi(|\xi|^2) \le \Psi(|\xi|) \le \gamma \phi(|\xi|^2) \quad \text{for all } \xi \in \mathbb{R}^d .$$

Definition

Let $0 < \kappa \leq 1$. We say that a open set D is κ-fat if there is $R_1 > 0$ such that for all $x \in \overline{D}$ and all $r \in (0, R_1]$, there is a ball $B(A_r(x), \kappa r) \subset D \cap B(x, r)$.

Theorem [CKS, 13a]

Suppose that (**A**) and the comparability conditions above hold and that D is a κ-fat open set.

(1) For every $T > 0$, there exists $c_1 > 0$ such that for $0 < t \leq T$, $x, y \in D$,

$$p_D(t,x,y) \geq c_1 \mathbb{P}_x(\tau_D > t)\mathbb{P}_y(\tau_D > t)\left(\Phi^{-1}(t)^{-d} \wedge tJ(x,y)\right).$$

(2) If D is unbounded, we assume in addition that condition **(B)** holds. For every $T > 0$, there exists $c_2 > 0$ such that for $0 < t \leq T$, $x, y \in D$,

$$p_D(t,x,y) \leq c_2 \mathbb{P}_x(\tau_D > t)\mathbb{P}_y(\tau_D > t) p(t, C_5 x, C_5 y),$$

where $C_5 = C_2^2/4$.

Corollary [CKS, 13a]

Suppose that (**A**) and the comparability conditions above hold and that D is a κ-fat open set. If D is unbounded, we assume in addition that condition **(C)** holds. For every $T > 0$, there exist $c_1 > 0$ and $c_2 > 0$ such that for $0 < t \leq T$, $x, y \in D$,

$$c_1 \mathbb{P}_x(\tau_D > t)\mathbb{P}_y(\tau_D > t)\left(\Phi^{-1}(t)^{-d} \wedge tj(|x-y|)\right)$$

$$\leq p_D(t,x,y) \leq c_2 \mathbb{P}_x(\tau_D > t)\mathbb{P}_y(\tau_D > t)\left(\Phi^{-1}(t)^{-d} \wedge tj(C_6|x-y|)\right),$$

where $C_6 = C_4^3/4$.

Outline

References

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Thank you!