Strong ergodicity of the regime-switching diffusion processes

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Contents

What is the switching diffusion?

It is a two-component process $(X(t), \Lambda(t))$, where $(X(t))$ describes the continuous dynamics, and $(\Lambda(t))$ describes the random switching device.

- The first component $(X(t))$ satisfies the following SDE

$$dX(t) = \sigma(X(t), \Lambda(t))\mathrm{d}B(t) + b(X(t), \Lambda(t))\mathrm{d}t, \tag{1}$$

  with $X(0) = x \in \mathbb{R}^d$.

- the second component $(\Lambda(t))$ is a continuous time Markov chain with a finite state space $S := \{1, 2, \ldots, m_0\}$, $m_0 > 1$, such that

$$\mathbb{P}\{\Lambda(t+\delta) = l | \Lambda(t) = k\} = \begin{cases} q_{kl}\delta + o(\delta), & \text{if } k \neq l, \\ 1 + q_{kk}\delta + o(\delta), & \text{if } k = l \end{cases} \tag{2}$$

  provided $\delta \downarrow 0$. The Q-matrix (q_{ij}) is irreducible and conservative.

Diffusion process in a fixed environment

For $k \in S$, let $(X^{(k)}(t))$ be a process satisfying the SDE:

$$\mathrm{d}X^{(k)}(t) = \sigma(X^{(k)}(t), k)\mathrm{d}B(t) + b(X^{(k)}(t), k)\mathrm{d}t,$$

with $X^{(k)}(0) = x \in \mathbb{R}^d$. Then $(X^{(k)}(t))$ is called the corresponding diffusion of $(X(t), \Lambda(t))$ in the fixed environment k.

- As $(\Lambda(t))$ is a Q-process in a finite state space with an irreducible Q-matrix, the recurrent property of the process $(X(t), \Lambda(t))$ is the same as that of $(X(t))$, which is obviously connected with the recurrent property of $(X^{(k)}(t))$, $k \in S$.
- Some important phenomena can occur when the environment is random.

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R. Pinsky, M. Scheutzow, 1992

Take $S = \{0, 1\}$. They constructed examples on $[0, \infty) \times S$ with reflection at 0 such that

1. $(X^{(0)}(t))$ and $(X^{(1)}(t))$ are positive recurrent, but $(X(t), \Lambda(t))$ is transient.
2. $(X^{(0)}(t))$ and $(X^{(1)}(t))$ are transient, but $(X(t), \Lambda(t))$ is positive recurrent.

The role of $(\Lambda(t))$ is important.

Consider the geometric Brownian motion in a random environment:

$$\mathrm{d}X(t) = \mu_{\Lambda(t)} X(t)\mathrm{d}t + \sigma_{\Lambda(t)} X(t)\mathrm{d}B(t), \quad X(0) = x > 0,$$

where $(\Lambda(t))$ is a Q-process in the state space $S = \{0, 1\}$, its Q-matrix $\begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}$, where λ_0, λ_1 are two positive constants. μ_0, μ_1, σ_0, σ_1 are constants.

If we use this process to model stock price, the states 0 and 1 will represent respectively the "bull" market and the "bear" market.

Theorem

Set $\Delta_0 = \mu_0 - \frac{1}{2}\sigma_0^2$, $\Delta_1 = \mu_1 - \frac{1}{2}\sigma_1^2$.

(i) *If* $\lambda_0\Delta_1 + \lambda_1\Delta_0 > 0$, *then the process* (X_t, Λ_t) *is transient.*

(ii) *If* $\lambda_0\Delta_1 + \lambda_1\Delta_0 < 0$, *then the process* (X_t, Λ_t) *is positive recurrent.*

Contents

- The semigroup P_t corresponding to the Markov process $(X(t), \Lambda(t))$ is defined by

$$P_t f(x,k) = \mathbb{E}_{x,k} f(X(t), \Lambda(t))$$

for bounded measurable f on $\mathbb{R}^d \times S$.

- Suppose that $\exists !$ stationary distribution π for $(X(t), \Lambda(t))$.
- The process $(X(t), \Lambda(t))$ is called *strongly ergodic*, if there exists an $\varepsilon > 0$ such that

$$\sup_{x\in\mathbb{R}^d,\, k\in S} \|\mathbb{P}(t,(x,k),\cdot) - \pi\|_{\mathrm{var}} = O(e^{-\varepsilon t}), \quad \text{as } t \to \infty. \tag{3}$$

- Define

$$\alpha(\gamma) = \sup\left\{\varepsilon \geq 0;\ \sup_{x\in\mathbb{R}^d, k\in S} \|\mathbb{P}(t,(x,k),\cdot) - \pi\|_{\mathrm{var}} \leq \gamma e^{-\varepsilon t},\ \forall\, t \geq 0\right\}.$$

Set $\alpha = \alpha(\infty) = \lim_{\gamma\to\infty} \alpha(\gamma)$. If $\alpha > 0$, the process $(X(t), \Lambda(t))$ is strongly ergodic.

- Let $(X(t), \Lambda(t), Y(t), \Lambda'(t))$ be a coupling process, and let $T = \inf\{t \geq 0;\ (X(t), \Lambda(t)) = (Y(t), \Lambda'(t))\}$ be the coupling time.

Theorem (Yong-Hua, Mao, '02,'06)

If $\exists\ \lambda > 0$ *such that*

$$M := \sup\left\{\mathbb{E}_{x,k,y,l}[e^{\lambda T}];\ x,\, y \in \mathbb{R}^d,\ k,\, l \in S\right\} < \infty,$$

then $\alpha \geq \alpha(2M) \geq \lambda > 0.$

One dimensional case

Let $(X(t), \Lambda(t))$ be a regime-switching diffusion process on $[0, \infty) \times S$ with reflection at 0, where $S = \{1, 2, \dots, m_0\}$ for some fixed $m_0 > 1$. Its corresponding diffusion in each fixed environment $k \in S$ is also denoted by $(X^{(k)}(t))$. Recall that Q-matrix of $(\Lambda(t))$ is assumed to be irreducible.

Theorem

If for each $k \in S$, the process $(X^{(k)}(t))$ is strongly ergodic, then $(X(t), \Lambda(t))$ is strongly ergodic.

- Is the converse of this theorem true ?

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If for each $k \in S$, the process $(X^{(k)}(t))$ is strongly ergodic, then $(X(t), \Lambda(t))$ is strongly ergodic.

- Is the converse of this theorem true ? ...... No

One dimensional case: Example

$(X(t))$ satisfies the following SDE:

$$\mathrm{d}X(t) = b(X(t), \Lambda(t))\mathrm{d}t + \mathrm{d}B(t),$$

where $b(x, 0) = b_0(x) = \frac{1}{2}$, and $b(x, 1) = b_1(x) = -\frac{x^2}{2}$.
The Q-matrix of $(\Lambda(t))$ is $\begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}$.

$$\mathrm{d}X^{(k)}(t) = b_k(X^{(k)}(t))\mathrm{d}t + \mathrm{d}B(t).$$

Then $(X^{(0)}(t))$ is not ergodic, and $(X^{(1)}(t))$ is strongly ergodic.

- We can find q_1, $q_2 > 0$ such that $(X(t), \Lambda(t))$ is strongly ergodic.

Contents

- Suppose that there is another switching diffusion $(Y(t), \Lambda'(t))$ on $\mathbb{R}^d \times S$ such that $(X(t), \Lambda(t))$ and $(Y(t), \Lambda'(t))$ admit the same transition probability, then $(X(t), \Lambda(t), Y(t), \Lambda'(t))$ constitutes a coupling process.
- Its infinitesimal generator

$$\mathscr{A} f(x,k,y,l) = L_{k,l} f(x,k,y,l) + Q f(x,k,y,l), \;\; x,y \in \mathbb{R}^d, \;\; k,l \in S,$$

for $f \in C^2(\mathbb{R}^d \times S \times \mathbb{R}^d \times S)$.

Here

$$L_{k,l}f(x,y) = \frac{1}{2}\sum_{i,j=1}^{2d} a_{ij}(x,k,y,l)\frac{\partial^2 f}{\partial z_i \partial z_j}(x,y) + \sum_{i=1}^{2d} b_i(x,k,y,l)\frac{\partial f}{\partial z_i}(x,y),,$$

for $f \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ where $z_i = x_i$ for $1 \le i \le d$; $z_i = y_{i-d}$ for $d < i \le 2d$, and

$$\big(a_{ij}(x,k,y,l)\big) = \begin{pmatrix} a(x,k) & c(x,k,y,l) \\ c(x,k,y,l)^* & a(y,l) \end{pmatrix}; \quad \big(b_i(x,k,y,l)\big) = \begin{pmatrix} b(x,k) \\ b(y,l) \end{pmatrix}.$$

C^* denotes the transpose of matrix C. The matrix $c(x,k,y,l)$ is a $d \times d$-matrix such that $(a_{ij}(x,k,y,l))$ is positive definite for each x, $y \in \mathbb{R}^d$, k, $l \in S$.

$Q = \big(q_{(k,l)(m,n)}\big)$ is a coupling operator of the q-matrix (q_{km}).

Assumptions:

1. The coupling process is non-explosive.
2. $q_{(k,k)(m,n)} = 0$ for $m \neq n,\ k \in S$ and $\big(q_{(k,l)(m,n)}\big)$ is irreducible.

Necessary notations

$$A_k(x,y) = a(x,k) + a(y,k) - 2c(x,k,y,k),$$

$$B_k(x,y) = \sum_{i=1}^{d} (b_i(x,k) - b_i(y,k))(x_i - y_i),$$

$$\tilde{A}_k(x,y) = \Big(\sum_{i,j=1}^{d} \big(A_k(x,y)\big)_{ij} (x_i - y_i)(x_j - y_j) \Big) / |x-y|^2, \quad x \neq y.$$

Let $\tilde{\gamma}(r,k)$, $\underline{\gamma}(r,k)$, $\alpha_*(r,k)$ and $\alpha^*(r,k)$ be continuous on $(0,\infty)\times S$ satisfying

$$\tilde{\gamma}(r,k) \geq \sup_{|x-y|=r} \frac{\sum_{i=1}^d \big(A_k(x,y)\big)_{ii} - \tilde{A}_k(x,y) + 2B_k(x,y)}{\tilde{A}_k(x,y)},\ \ r>0,$$

$$\underline{\gamma}(r,k) \leq \inf_{|x-y|=r} \frac{\sum_{i=1}^d \big(A_k(x,y)\big)_{ii} - \tilde{A}_k(x,y) + 2B_k(x,y)}{\tilde{A}_k(x,y)},\ \ r>0,$$

$$\alpha_*(r,k) \leq \inf_{|x-y|=r}\{\tilde{A}_k(x,y)\} \leq \sup_{|x-y|=r}\{\tilde{A}_k(x,y)\} \leq \alpha^*(r,k),\ \ r>0,\ \ k\in S.$$

Set

$$I(s,k)=\int_1^s \frac{\tilde{\gamma}(u,k)}{u}\mathrm{d}u,\ \ s>0,\quad \underline{I}(s,k)=\int_1^s \frac{\underline{\gamma}(u,k)}{u}\mathrm{d}u,\ \ s>0,\ \ k\in S.$$

Criterion on success of coupling

Theorem

If for each $k \in S$

$$\delta_k := \int_0^\infty e^{-\bar{I}(s,k)}\Big(\int_s^\infty \frac{e^{\bar{I}(u,k)}}{\alpha_*(u,k)}\mathrm{d}u\Big)\mathrm{d}s < \infty,$$

then the coupling $\big(X(t), \Lambda(t), Y(t), \Lambda'(t)\big)$ starting at every (x, k, y, l) in $\mathbb{R}^d \times S \times \mathbb{R}^d \times S$ is successful.

Proposition

If

$$\delta_k := \int_0^\infty e^{-\bar{I}(s,k)}\Big(\int_s^\infty \frac{e^{\bar{I}(u,k)}}{\underline{\alpha}(u,k)}\mathrm{d}u\Big)\mathrm{d}s < \infty,$$

then for every $0 < \lambda < \tilde{\theta} e^{-2\tilde{\Theta}\delta}$, it holds

$$\sup_{x,k,y,k} \mathbb{E}_{x,k,y,k}\big[e^{\lambda T}\big] \leq \big(1 - \lambda\,\tilde{\theta}^{-1} e^{2\tilde{\Theta}\delta}\big)^{-1}.$$

Here $\tilde{\Theta} = \max_{k\in S} q_{(k,k)} = \max_{k\in S} \sum_{j\neq k} q_{(k,k)(j,j)}$, $\tilde{\theta} = \min_{k\in S} q_{(k,k)}$, $\delta = \max_{k\in S} \delta_k$.

Strong ergodicity in multidimensional case

Theorem

If there exists a coupling process $(X(t), \Lambda(t), Y(t), \Lambda'(t))$ in the previous form. If for each $k \in S$

$$\delta_k := \int_0^\infty e^{-\bar{I}(s,k)} \Big(\int_s^\infty \frac{2\lambda e^{\bar{I}(u,k)}}{\alpha_*(u,k)} \mathrm{d}u \Big) \mathrm{d}s < \infty,$$

then the process $(X(t), \Lambda(t))$ is strongly ergodic.

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THE END

Thank you for your attention!