The Backbone decomposition for spatially dependend supercritical superdiffusions

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Outline

- A brief history of backbones
- The backbone decomposition for continuous-state branching processes
- The backbone decomposition for supercritical superdiffusion.

A brief history of backbones

- Harris and Sevast'yanov (cf Harris' book 1964) for supercritical Galton-Watson process.
- Evans and O'Connell(1994), Engländer and Pinsky (1999) for supercritical superprocess with quadratic branching mechanism (no pathwise construction is offered).
- Salisbury and Verzani (1999); Etheridge and Williams (2003).
- Duquesne and Winkel (2007) for continuous state branching processes with general branching mechanism ψ which satisfies the conditions $0 < -\psi'(0+) < \infty$ and $\int^{\infty} 1/\psi(\xi)\mathrm{d}\xi < \infty$..

A brief history of backbones

- Berestycki, Kyprianou, and Murillo-Salas (2011) for superprocesses with relatively general motion and non-spatial branching mechanism (also allowing for the case that $-\psi'(0+) = \infty$ or $\int^{\infty} 1/\psi(\xi)\mathrm{d}\xi < \infty$).
- Kyprianou and Ren (2012) for a continuous-state branching process with immigration.
- Kyprianou, Pérez and Ren (2013) for a general class of supercritical superdiffusion with spatially dependent branching mechanisms.
- Y.-X. Ren, R. Song and R. Zhang (2013, preprint): Proved central limit theorems for super-OU Processes using the backbone decomposition for super-OU processes.

ψ-CSBP

- $X = \{X_t : t \geq 0\}$ is a continuous-state branching process with branching mechanism ψ: (ψ-CSBP)

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}})\Pi(\mathrm{d}x), \;\; \lambda \geq 0,$$

  where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and Π is a measure concentrated on $(0, \infty)$ which satisfies $\int_{(0,\infty)} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

- Assume that X is conservative and supercritical:

$$\int_{0+} \frac{1}{|\psi(\xi)|}\mathrm{d}\xi = \infty \quad \text{and} \quad -\psi'(0+) > 0.$$

- Suppose that $\mathbb{P}_x$ denotes the law of X on cadlag path space $D[0, \infty)$ when the process is issued from $x \geq 0$.

CSBP conditioned to be extinguished

- Extinguishing probability: for all $x \geq 0$,

$$\mathbb{P}_x(\lim_{t\uparrow\infty} X_t = 0) = e^{-\lambda^* x},$$

where λ^* is the unique root on $(0,\infty)$ of the equation $\psi(\lambda) = 0$. (We assume that $\psi(\infty) = \infty$.)

- $\mathbb{P}_x^*$: the law of $(X, \mathbb{P}_x)$ conditional on $\{\lim_{t\uparrow\infty} X_t = 0\}$.

Under $\mathbb{P}_x^*$ is a ψ^*-CSBP, here

$$\begin{aligned}\psi^*(\lambda) = &\ \psi(\lambda + \lambda^*) \\ = &\ \alpha^*\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}) e^{-\lambda^* x}\Pi(\mathrm{d}x),\end{aligned}$$

where $\alpha^* = \alpha + 2\beta\lambda^* \int_{(0,1)} (1 - e^{-\lambda^* x})x)\Pi(\mathrm{d}x)$.

CSBP conditioned to be extinguished

- Suppose $\mathbb{N}^*$ is the excursion measure on the space $D[0,\infty)$ which satisfies

$$\mathbb{N}^*(1-e^{-\lambda X_t}) = u_t^*(\lambda) = -\frac{1}{x}\log \mathbb{E}_x^*(e^{-\lambda X_t})$$

for $\lambda, t \geq 0$, where $u_t^*(\lambda)$ is the unique solution to the integral equation

$$u_t^*(\lambda) + \int_0^t \psi^*(u_s^*(\lambda)) = \lambda, \tag{0.1}$$

with initial condition $u_0^*(\lambda) = \lambda$. See El Karoui and Roelly (1991), Le Gall (1999), Zenghu Li (2002) and Dynkin and Kuznetsov (2004) for further details.

Backbone decomposition for CSBP

- Dusquene and Winkel (2007) and Berestycki et al. (2011) proved that the law $\mathbb{P}_x$ of process X can be recovered from a supercritical continuous-time Galton-Watson process, issued with a Poisson number of initial ancestors, and dressed in a Poissonian way using $\mathbb{P}^*_x$.

  The aim is to construct a process $(\Lambda_t : t \geq 0; \mathbf{P}_x)$ such that

$$(X_t : t \geq 0; \mathbb{P}_x) = (\Lambda_t : t \geq 0; \mathbf{P}_x) \quad \text{(in distribution)}$$

Backbone: F-GW process

- Suppose $Z = \{Z_t : t \geq 0\}$ is a continuous-time GW process with branching generator F and Z_0 has a Poisson distribution with parameter $\lambda^* x$, where

$$F(r) = q\left(\sum_{n\geq 0} p_n r^n - r\right) = \frac{1}{\lambda^*}\psi(\lambda^*(1-r)), \qquad (0.2)$$

$q = \psi'(\lambda^*)$, and $\{p_n : n \geq 0\}$ is the offspring distribution: $p_0 = p_1 = 0$, and for $n \geq 2$, $p_n := p_n[0,\infty)$ where for $y \geq 0$,

$$p_n(\mathrm{d}y) = \frac{1}{\lambda^*\psi'(\lambda^*)}\left\{\beta(\lambda^*)^2\delta_0(\mathrm{d}y)\mathbf{1}_{\{n=2\}} + (\lambda^*)^n\frac{y^n}{n!}e^{-\lambda^* y}\Pi(\mathrm{d}y)\right\}.$$

Dressing

(i) Along the life length of each individual alive in the process Z, there is Poissonian dressing with rate

$$2\beta \mathrm{d}\mathbb{N}^* + \int_0^\infty y e^{-\lambda^* y} \Pi(\mathrm{d}y) \mathrm{d}\mathbb{P}_y^*. \tag{0.3}$$

(ii) At the branch points of Z, on the event that there are n offspring, an additional copy of a ψ^*-CSBP with initial mass $y \geq 0$ is issued with probability $p_n(\mathrm{d}y)$.

Define

$$\Lambda_t := X_t^* + I_t^{\mathbb{N}^*} + I_t^{\mathbb{P}^*} + I_t^{bp},$$

where X_t^* is the mass at time t of an independent ψ^*-CSBP issued at time zero with initial mass x.

Backbone decomposition for CSBP

Theorem 1 *(i) Fix $x > 0$. The law of $(X, \mathbb{P}_x)$ agrees with that of $(\Lambda, \mathbf{P}_x)$.*
(ii) Moreover, for all $t \geq 0$, the law of Z_t given Λ_t is that of a Poisson random variable with law $\lambda^* \Lambda_t$.

Superdiffusion

- $\xi = \{\xi_t : t \geq 0; \Pi_x\}$: a diffusion on E (a domain of $\mathbb{R}^d$) with infinitesimal generator $L = \sum_{i,j} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}$, where the coefficients $a_{i,j}$ and b_j satisfy:
  - **(Uniform Elliptically)** There exists a constant $\gamma > 0$ such that
$$\sum_{i,l} a_{i,j} u_i u_j \geq \gamma \sum_i u_i^2$$
    for all $x \in E$ and $u_1, \cdots u_d \in \mathbb{R}$.
  - **(Hölder continuity)** The coefficients $a_{i,j}$ and b_i are uniformly bounded and Hölder continuous.
  - The semigroup of ξ will be denoted by $\mathcal{P} = \{\mathcal{P}_t : t \geq 0\}$.

Superdiffusion

- Branching mechanism:

$$\psi(x,\lambda) = -\alpha(x)\lambda + \beta(x)\lambda^2 + \int_{(0,\infty)} (e^{-\lambda z} - 1 + \lambda z)\pi(x, \mathrm{d}z), \tag{0.4}$$

  where α and $\beta \geq 0$ are bounded measurable mappings from E to $\mathbb{R}$ and $[0,\infty)$ respectively and for each $x \in E$, $\pi(x, \mathrm{d}z)$ is a measure concentrated on $(0,\infty)$ such that $x \longrightarrow \int_{(0,\infty)} (z \wedge z^2)\pi(x, \mathrm{d}z)$ is bounded and measurable.

- Notation: $\mathcal{M}_F(E)$ is the sapce of finite measures on E.

$$\langle f, \mu \rangle = \int_E f(x)\mu(dx).$$

Superdiffusion

- $(\mathcal{P}, \psi)$-superdiffusion: $X = \{X_t, t \geq 0\}$ is a strong Markov process taking values in $\mathcal{M}_F(E)$ such that for each $\mu \in \mathcal{M}_F(E)$, $\mathbb{P}_\mu(X_0 = \mu) = 1$ and all $f \in \mathrm{bp}(E)$,

$$\mathbb{E}_\mu(e^{-\langle f, X_t\rangle}) = \exp\left\{-\int_E u_f(x,t)\mu(\mathrm{d}x)\right\} \qquad t \geq 0, \quad (0.5)$$

where $u_f(x,t)$ is the unique non-negative solution to the equation

$$u_f(x,t) = \mathcal{P}_t[f](x) - \int_0^t \mathrm{d}s \cdot \mathcal{P}_s[\psi(\cdot, u_f(\cdot, t-s))](x) \qquad x \in E, t \geq 0. \quad (0.6)$$

The event of extinction

- We define the event of *extinction*:

$$\mathcal{E} = \{\langle 1, X_t\rangle = 0 \text{ for some } t > 0\}.$$

For each $x \in E$ write

$$w(x) = -\log \mathbb{P}_{\delta_x}(\mathcal{E}). \qquad (0.7)$$

We see that

$$\mathbb{P}_\mu(\mathcal{E}) = \exp\left\{-\int_E w(x)\mu(\mathrm{d}x)\right\}. \qquad (0.8)$$

- Assume that w is locally bounded away from 0 and ∞.

Remark: For the special case that ψ does not depend on x and $\mathcal{P}$ is conservative, $\langle 1, X_t \rangle$ is a CSBP. If $\psi(\lambda)$ satisfy the following condition:

$$\int^{\infty} \frac{1}{\psi(\lambda)} \mathrm{d}\lambda < \infty,$$

then $\mathbb{P}_\mu$-a.s. we have

$$\mathcal{E} = \{ \lim_{t \longrightarrow \infty} \langle 1, X_t \rangle = 0 \},$$

that is to say the event of *extinction* is equivalent to the event of *extinguishing*.

Localization

Suppose D is a bounded domain such that $D \subset\subset E$.

- $X^D = \{X_t^D : t \geq 0\}$ is a superprocess with branching mechanism $\psi(x,\lambda)\mathbf{1}_D(x)$, but whose associated semi-group is replaced by that of the process ξ killed upon leaving D.
- It can be proved that

$$w(\xi_{t\wedge\tau^D})\exp\left\{-\int_0^{t\wedge\tau^D}\frac{\psi(\xi_s, w(\xi_s))}{w(\xi_s)}\mathrm{d}s\right\}, \qquad t\geq 0, \quad (0.9)$$

  is a martingale.

Conditioning on extinction

- Define $\mathbb{P}^*_\mu(\cdot) = \mathbb{P}_\mu(\cdot|\mathcal{E})$.
- Then for any $\mu \in \mathcal{M}_F(D)$ satisfying $\langle w, \mu\rangle < \infty$,

$$-\log \mathbb{E}^*_\mu\left(e^{-\langle f, X^D_t\rangle}\right) = \int_D u^{D,*}_f(x,t)\mu(\mathrm{d}x),$$

where $u^{D,*}_f(x,t)$ is the unique solution of

$$u^{D,*}_f(x,t) = \Pi_x[f(\xi_t); t < \tau^D] - \Pi_x\left[\int_0^{t\wedge\tau_D} \psi^*(\xi_s, u^{D,*}_f(\xi_s, t-s))\mathrm{d}s\right], \quad x \tag{0.10}$$

where ψ^* is defined by

$$\psi^*(x,\lambda) = -\alpha^*(x)\lambda + \beta(x)\lambda^2 + \int_{(0,\infty)} (e^{-\lambda z} - 1 + \lambda z)\pi^*(x, \mathrm{d}z), \tag{0.11}$$

Conditioned on extinction

where

$$\alpha^*(x) = \alpha(x) - 2\beta(x)w(x) - \int_{(0,\infty)} (1 - e^{-w(x)z})z\pi(x, \mathrm{d}z)$$

and

$$\pi^*(x, \mathrm{d}z) = e^{-w(x)z}\pi(x, \mathrm{d}z) \text{ on } E \times (0, \infty).$$

- Let $\mathbb{N}^*_x$ be the excursion measure corresponding to $\mathbb{P}^*_{\delta_x}$.

Backbone: $(\mathcal{P}^w, F)$-branching diffusion

The backbone $Z = \{Z_t : t \geq 0\}$ is a $(\mathcal{P}^w, F)$-Markov branching process.

- Spacial motion: For $t \geq 0$, $x \in E$, the formula

$$\left.\frac{\mathrm{d}\Pi_x^w}{\mathrm{d}\Pi_x}\right|_{\mathcal{F}_t^\xi} = \frac{w(\xi_t)}{w(x)} \exp\left\{-\int_0^t \frac{\psi(\xi_s, w(\xi_s))}{w(\xi_s)}\mathrm{d}s\right\} \qquad \text{on } \{t < \zeta\} \tag{0.12}$$

  uniquely determines a family of (sub-)probability measures $\{\Pi_x^w : x \in E\}$.

  - We will denote by $\mathcal{P}^w$ the semi-group of the $E \cup \{\dagger\}$-valued process ξ.
  - Particles move with associated semi-group $\mathcal{P}^w$.

Backbone: $(\mathcal{P}^w, F)$-branching diffusion

- The branching generator is given by

$$F(x,s) = q(x)\sum_{n\geq 0} p_n(x)(s^n - s), \qquad (0.13)$$

where

$$q(x) = \psi'(x, w(x)) - \frac{\psi(x, w(x))}{w(x)}, \qquad (0.14)$$

$p_0(x) = p_1(x) = 0$ and for $n \geq 2$,

$$p_n(x) = \frac{1}{w(x)q(x)}\left\{\beta(x)w^2(x)1_{\{n=2\}} + w^n(x)\int_{(0,\infty)} \frac{y^n}{n!}e^{-w(x)y}\pi(x, \mathrm{d}y)\right\}$$

- Z_0 is a Poisson random measure on E with intensity $w(x)\mu(\mathrm{d}x)$.

Dressing

- **(Continuum immigration:)** The process $I^{\mathbb{N}^*}$ is measure-valued on E such that

$$I_t^{\mathbb{N}^*} = \sum_{u \in \mathcal{T}} \sum_{b_u < r \le t \wedge d_u} X_{t-r}^{(1,u,r)},$$

where, given Z, independently for each $u \in \mathcal{T}$ such that $b_u < t$, the processes $X^{(1,u,r)}$ are independent copies of the canonical process X, immigrated along the space-time trajectory $\{(z_u(r), r) : r \in (b_u, t \wedge d_u]\}$ with rate

$$2\beta(z_u(r))\mathrm{d}r \times \mathrm{d}\mathbb{N}^*_{z_u(r)}.$$

Dressing

- (**Discontinuous immigration:**) The process $I^{\mathbb{P}^*}$ is measure-valued on E such that

$$I_t^{\mathbb{P}^*} = \sum_{u \in \mathcal{T}} \sum_{b_u < r \leq t \wedge d_u} X^{(2,u,r)},$$

where, given Z, independently for each $u \in \mathcal{T}$ such that $b_u < t$, the processes $X^{(2,u,r)}$ are independent copies of the canonical process X, immigrated along the space-time trajectory $\{(z_u(r), r) : r \in (b_u, t \wedge d_u]\}$ with rate

$$\mathrm{d}r \times \int_{y \in (0,\infty)} y e^{-w(z_u(r))y} \pi(z_u(r), \mathrm{d}y) \mathrm{d}\mathbb{P}^*_{y\delta_{z_u(r)}}.$$

Dressing

- **(Branch point biased immigration:)**

$$I_t^\eta = \sum_{u \in \mathcal{T}^D} \mathbf{1}_{\{d_u \le t\}} X^{(3,u)}_{t-d_u},$$

where, given Z, independently for each $u \in \mathcal{T}$ such that $d_u < t$, the processes $X^{(3,u)}$ are independent copies of the canonical process X issued at time d_u with law $\mathbb{P}^*_{Y_u \delta_{z_u(d_u)}}$ such that, given u has $n \ge 2$ offspring, the independent random variable Y_u has distribution $\eta_n(z_u(r), \mathrm{d}y)$, where

$$\begin{aligned} \eta_n(x, \mathrm{d}y) = \; & \frac{1}{q(x)w(x)p_n(x)} \Big\{ \beta(x) w^2(x) \delta_0(\mathrm{d}y) \mathbf{1}_{\{n=2\}} \\ & + w(x)^n \frac{y^n}{n!} e^{-w(x)y} \pi(x, \mathrm{d}y) \Big\} \end{aligned} \quad . \tag{0.15}$$

Backbone decomposition for SD

Theorem 2 *Let* X^* *be an independent copy of* $(X, \mathbb{P}^*_\mu)$*.*
Then define the measure valued stochastic process $\Delta = \{\Delta_t : t \geq 0\}$ *such that, for* $t \geq 0$*,*

$$\Delta_t = X^*_t + I^{\mathbb{N}^*}_t + I^{\mathbb{P}^*}_t + I^\eta_t. \qquad (0.16)$$

Then $(\Delta, \mathbf{P}_\mu)$ *is Markovian and has the same law as* $(X, \mathbb{P}_\mu)$*.*

END

Thank you!

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