The fundamental gap conjecture: a probabilistic approach via the coupling by reflection

Dejun Luo

Institute of Applied Mathematics, AMSS, CAS

Joint work with Fuzhou Gong and Huaiqian Li

Workshop on Markov Processes and Related Topics

BNU & SWJTU, July 6–10, 2013

Outline

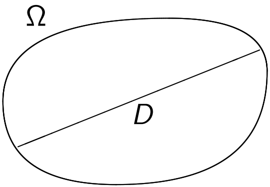
Outline

1. Introduction
2. Log-concavity estimate of ground state
3. Proof of the fundamental gap conjecture

Some notations:

- $\Omega \subset \mathbb{R}^n$: a bounded convex domain of diameter $D = \mathrm{diam}(\Omega)$;
- $V : \Omega \to \mathbb{R}$ a convex potential;
- $L = -\Delta + V$: the Schrödinger operator on Ω with Dirichlet boundary condition;
- Eigenvalues of L: $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, $\lim_{i\to\infty} \lambda_i = +\infty$;
- Eigenfunctions of L: $\phi_0, \phi_1, \phi_2, \ldots$, $\phi_i|_{\partial\Omega} \equiv 0$.

ϕ_0 and λ_0 are called the ground state and ground state energy, respectively. ϕ_0 is strictly positive in Ω.



Fundamental Gap Conjecture (van den Berg, 1983):
The spectral gap of L satisfies

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}. \tag{1}$$

Example 1

Consider the one dimensional case $\Omega = \left(-\frac{D}{2}, \frac{D}{2}\right) \subset \mathbb{R}^1$ and $V \equiv 0$. Then the operator is given by $L = -\frac{d^2}{dt^2}$, and

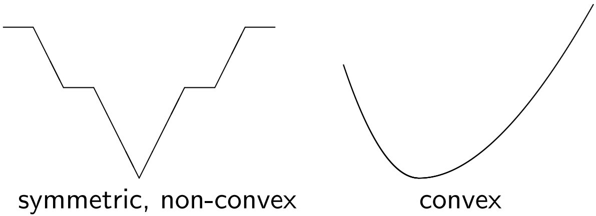
	Eigenvalues λ_i	Eigenfunctions ϕ_i
$i = 0$	$\frac{\pi^2}{D^2}$	$\cos\frac{\pi t}{D}$
$i = 1$	$\frac{4\pi^2}{D^2}$	$\sin\frac{2\pi t}{D}$

Therefore the spectral gap is $\frac{3\pi^2}{D^2}$.

Known results

In one dimension:

- Ashbaugh & Benguria (1989): If V is symmetric and single-well (not necessarily convex), then the conjecture holds;
- Lavine (1994): The conjecture holds if V is convex.



Known results

In higher dimensions:

- Singer, Wong, Yau & Yau (1985): the gap $\lambda_1 - \lambda_0 \geq \frac{\pi^2}{4D^2}$;
- Qi Huang Yu & Jia Qing Zhong (1986): on a compact manifold, the first nontrivial eigenvalue $\geq \frac{\pi^2}{D^2}$;
- Jun Ling (2005): the gap $\geq \frac{\pi^2}{D^2} + \frac{31}{50}\alpha$, where $\alpha = -\sup \nabla^2(\log \phi_0)$;
- Mu-Fa Chen & Feng-Yu Wang (1994, 1997): coupling method yields variational formula for the first nontrivial eigenvalue;

Complete solution

Andrews & Clutterbuck (JAMS, 2011): The gap conjecture holds.

Basic idea: compare the spectral gap with one dimensional case.

Let $\tilde{V} \in C^1\big([-\frac{D}{2}, \frac{D}{2}], \mathbb{R}\big)$ be an even function, such that $\forall\, x, y \in \Omega, x \neq y$,

$$\left\langle \nabla V(x) - \nabla V(y), \frac{x-y}{|x-y|} \right\rangle \geq 2\tilde{V}'\left(\frac{|x-y|}{2}\right). \qquad (2)$$

The function $\tilde{V}$ is called a modulus of convexity of V.

Remark 2

(i) If the sign $\geq$ is replaced by $\leq$, then $\tilde{V}$ is called a modulus of concavity of V.

(ii) If V is convex, then we can choose $\tilde{V} \equiv 0$.

(iii) Fix any $x \in \Omega$ and $\theta \in \mathbb{R}^n$ with $|\theta| = 1$. For $t > 0$ such that $x + t\theta \in \Omega$, (2) implies

$$\langle \nabla V(x + t\theta) - \nabla V(x), \theta \rangle \geq 2\tilde{V}'\left(\frac{t}{2}\right).$$

Note that $\tilde{V}$ is even, hence $\tilde{V}'(0) = 0$. Dividing both sides by t and letting $t \to 0$ yield

$$\langle [\nabla^2 V(x)]\, \theta, \theta \rangle \geq \tilde{V}''(0).$$

Log-concavity estimate of ground state

Consider the one dimensional Schrödinger operator $\tilde{L} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \tilde{V}$ on $\left[-\frac{D}{2}, \frac{D}{2}\right]$, satisfying the Dirichlet boundary condition.

Denote the corresponding objects by adding a tilde, e.g. $\tilde{\lambda}_i$ and $\tilde{\phi}_i$, $i = 0, 1, 2, \ldots$.

Theorem 3 (Andrews & Clutterbuck, JAMS, 2011, Theorem 1.5)

Assume that $\tilde{V}$ is a modulus of convexity of V, i.e. (2) holds, then $\log \tilde{\phi}_0$ is a modulus of concavity of $\log \phi_0$. More precisely, $\forall x, y \in \Omega, x \neq y$,

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \le 2(\log \tilde{\phi}_0)'\left(\frac{|x-y|}{2}\right). \quad (3)$$

Remarks on Theorem 3

Remark 4

(i) Recall that when V *is convex, then* $\tilde{V} \equiv 0$.

In this case, $\tilde{L} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ *has the ground state* $\tilde{\phi}_0(t) = \cos\frac{\pi t}{D}$, *thus* $(\log\tilde{\phi}_0)'(t) = -\frac{\pi}{D}\tan\frac{\pi t}{D}$, $t \in \big(-\frac{D}{2}, \frac{D}{2}\big)$.

The log-concavity estimate (3) *becomes*

$$\left\langle \nabla\log\phi_0(x) - \nabla\log\phi_0(y), \frac{x-y}{|x-y|}\right\rangle \le -\frac{2\pi}{D}\tan\left(\frac{|x-y|}{2D}\right).$$

(ii) Brascamp & Lieb (JFA, 1976) proved a weaker result: if V *is convex, then the ground state* ϕ_0 *is log-concave.*

Spectral gap comparison theorem

Theorem 5 (Andrews & Clutterbuck, JAMS, 2011, Theorem 1.3)

If $\tilde{V}$ is a modulus of convexity of V, i.e. (2) *holds, then* $\lambda_1 - \lambda_0 \geq \tilde{\lambda}_1 - \tilde{\lambda}_0$.

Ingredients of the proof:

(i) the ground state transform: For $i = 0, 1$, let

$$u_i(t,x) = e^{-\lambda_i t}\phi_i(x) \quad \text{and} \quad v = \frac{u_1}{u_0} = e^{-(\lambda_1-\lambda_0)t}\frac{\phi_1}{\phi_0}.$$

Then $v(t,\cdot) \in C^\infty(\bar{\Omega})$ and

$$\frac{\partial v}{\partial t} = \Delta v + 2\langle \nabla \log \phi_0, \nabla v\rangle;$$

(ii) sharp log-concavity estimate of ground state ϕ_0 (Theorem 3);

(iii) estimate of the modulus of continuity:

$$v(t,x) - v(t,y) \leq C\tilde{v}(t,|x-y|) = Ce^{-(\tilde{\lambda}_1-\tilde{\lambda}_0)t}\frac{\tilde{\phi}_1}{\tilde{\phi}_0}(|x-y|).$$

Recall that $v(t,x) - v(t,y) = e^{-(\lambda_1-\lambda_0)t}\big(\frac{\phi_1}{\phi_0}(x) - \frac{\phi_1}{\phi_0}(y)\big)$,
hence $\forall\, t \geq 0$ and $x, y \in \Omega$,

$$e^{-(\lambda_1-\lambda_0)t}\left(\frac{\phi_1}{\phi_0}(x) - \frac{\phi_1}{\phi_0}(y)\right) \leq Ce^{-(\tilde{\lambda}_1-\tilde{\lambda}_0)t}\frac{\tilde{\phi}_1}{\tilde{\phi}_0}(|x-y|)$$

which implies $\lambda_1 - \lambda_0 \geq \tilde{\lambda}_1 - \tilde{\lambda}_0$. □

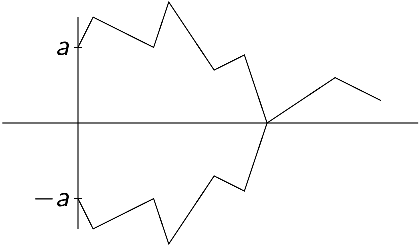
Our purpose: give a probabilistic proof to the gap conjecture by using the coupling by reflection.

Coupling by reflection

The coupling by reflection (also called mirror coupling) was introduced by Lindvall & Rogers (Ann Probab, 1986), see also M.F. Chen &. S.F. Li (Ann Probab, 1989).

Reason for introducing it: to make two multi-dimensional Brownian motions meet in finite time.

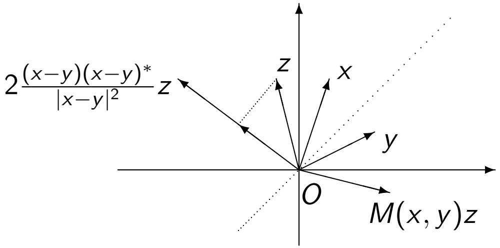
Basic idea:



Introduce the matrix

$$M(x,y) = I_n - 2\frac{(x-y)(x-y)^*}{|x-y|^2}, \quad x, y \in \mathbb{R}^n, x \neq y,$$

which corresponds to the reflection mapping w.r.t. the hyperplane passing through the origin and perpendicular to the vector $x - y$.



Given a smooth vector field $b : \mathbb{R}^n \to \mathbb{R}^n$ and an n-dimensional Brownian motion B_t. Consider the diffusion $(X_t)_{t\geq 0}$ defined by

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t, \quad X_0 = x.$$

The coupling by reflection is given by

$$\mathrm{d}Y_t = \sqrt{2}\,M(X_t, Y_t)\,\mathrm{d}B_t + b(Y_t)\,\mathrm{d}t, \quad Y_0 = y.$$

Define the coupling time $\tau = \inf\{t > 0 : Y_t = X_t\}$.

Lindvall & Rogers (Ann Probab, 1986, Example 5) proved that if

$$\langle x - y, b(x) - b(y)\rangle \leq 0,$$

then $\tau < +\infty$ a.s.

Outline

Equations for $\log \phi_0$

In order to estimate the log-concavity of ϕ_0, we observe that

$$-\Delta\phi_0 + V\phi_0 = \lambda_0\phi_0.$$

Hence

$$\Delta \log \phi_0 + |\nabla \log \phi_0|^2 = V - \lambda_0.$$

Differentiating the equation leads to

$$\Delta(\nabla \log \phi_0) + 2\big\langle \nabla \log \phi_0, \nabla(\nabla \log \phi_0)\big\rangle = \nabla V, \tag{4}$$

or equivalently, in component form,

$$\Delta(\partial_i \log \phi_0) + 2\big\langle \nabla \log \phi_0, \nabla(\partial_i \log \phi_0)\big\rangle = \partial_i V, \quad 1 \le i \le n.$$

Conservative diffusion

The above equations suggest us to consider the following SDE

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t + 2\nabla \log \phi_0(X_t)\,\mathrm{d}t, \quad X_0 = x \in \Omega. \tag{5}$$

where B_t is an n-dimensional standard Brownian motion.

The diffusion $(X_t)_{t\geq 0}$ is conservative, that is, starting from a point $x \in \Omega$, the process X_t will not arrive at the boundary $\partial\Omega$.

- Eric Carlen (Commun. Math. Phys., 1984), P.A. Meyer & W.A. Zheng (Séminaire de probabilités, 1985);
- We can also consider the one dimensional process $\rho_{\partial\Omega}(X_t)$, where $\rho_{\partial\Omega} : \Omega \to \mathbb{R}_+$ is the distance function to the boundary. Using the properties of the drift $2\nabla \log \phi_0 = 2\frac{\nabla\phi_0}{\phi_0}$, we can prove $\rho_{\partial\Omega}(X_t) > 0$ a.s. $\forall\, t \geq 0$.

Some notations

Consider

$$\mathrm{d}Y_t = \sqrt{2}\,M(X_t, Y_t)\,\mathrm{d}B_t + 2\nabla \log \phi_0(Y_t)\,\mathrm{d}t, \quad Y_0 = y \in \Omega. \quad (6)$$

For $\eta, \delta > 0$, define stopping times (by convention: $\inf \emptyset = \infty$)

$$\tau_\eta = \inf\{t > 0 : |X_t - Y_t| = \eta\},$$
$$\sigma_\delta = \inf\{t > 0 : \rho_{\partial\Omega}(X_t) \wedge \rho_{\partial\Omega}(Y_t) = \delta\}.$$

As $\eta \to 0$, $\tau_\eta \uparrow \tau = \inf\{t > 0 : X_t = Y_t\}$: the coupling times.
Set $Y_t = X_t$ for $\tau \le t < +\infty$.

As $\delta \to 0$, a.s. $\sigma_\delta \uparrow +\infty$ since X_t and Y_t do not hit the boundary $\partial\Omega$.

In order to prove the log-concavity estimate

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \leq -\frac{2\pi}{D} \tan\left(\frac{|x-y|}{2D}\right), \quad (7)$$

we consider the processes

$$\begin{aligned} \alpha_t &= \nabla \log \phi_0(X_t) - \nabla \log \phi_0(Y_t), \\ \beta_t &= \frac{X_t - Y_t}{|X_t - Y_t|}, \\ F_t &= \langle \alpha_t, \beta_t \rangle. \end{aligned}$$

Then $\quad \mathrm{d}(X_t - Y_t) = 2\sqrt{2}\, \beta_t \langle \beta_t, \mathrm{d}B_t \rangle + 2\alpha_t \, \mathrm{d}t \quad$ and

$$F_0 = \left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle.$$

Two lemmas

Lemma 6

Assume that the potential $V : \bar{\Omega} \to \mathbb{R}$ is convex. Then for $t \leq \tau_\eta \wedge \sigma_\delta$,

$$\mathrm{d}F_t \geq \langle \beta_t, \mathrm{d}M_t \rangle,$$

where M_t is a vector-valued local martingale.

The proof follows from Itô's formula, the properties of $\log \phi_0$ and of the coupling by reflection: by equations (5) and (6),

$$\mathrm{d}(X_t - Y_t) = 2\sqrt{2}\, \beta_t \langle \beta_t, \mathrm{d}B_t \rangle + 2\alpha_t \, \mathrm{d}t.$$

Hence

$$\mathrm{d}|X_t - Y_t| = \left\langle \frac{X_t - Y_t}{|X_t - Y_t|}, \mathrm{d}(X_t - Y_t) \right\rangle = 2\sqrt{2}\, \langle \beta_t, \mathrm{d}B_t \rangle + 2F_t \, \mathrm{d}t.$$

Two lemmas

Let $\tilde{\phi}_{D,0}(t) = \cos\frac{\pi t}{D},\ t \in \big[-\frac{D}{2}, \frac{D}{2}\big]$ be the first Dirichlet eigenfunction of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ on the interval $\big[-\frac{D}{2}, \frac{D}{2}\big]$.

Define $\psi_D(t) = (\log\tilde{\phi}_{D,0})'(t) = -\frac{\pi}{D}\tan\frac{\pi t}{D},\ t \in \big(-\frac{D}{2}, \frac{D}{2}\big)$.

Since $\psi_D(t)$ explodes at $t = \pm\frac{D}{2}$, we take $D_1 > D$ and consider $\tilde{\phi}_{D_1,0},\ \psi_{D_1} = \big(\log\tilde{\phi}_{D_1,0}\big)'$. Then $\psi_{D_1} \in C_b^\infty\big[0, \frac{D}{2}\big]$ and it satisfies

$$\psi''_{D_1} + 2\psi_{D_1}\psi'_{D_1} = 0.$$

Lemma 7

Set $\xi_t = |X_t - Y_t|/2$. We have for $t \le \tau_\eta \wedge \sigma_\delta$,

$$\mathrm{d}\psi_{D_1}(\xi_t) = \sqrt{2}\,\psi'_{D_1}(\xi_t)\langle\beta_t, \mathrm{d}B_t\rangle + \psi'_{D_1}(\xi_t)\big[F_t - 2\psi_{D_1}(\xi_t)\big]\,\mathrm{d}t.$$

Log-concavity estimate of the ground state

Theorem 8 (Modulus of log-concavity)

Assume that the potential function $V : \Omega \to \mathbb{R}$ is convex. Then for all $x, y \in \Omega$ with $x \neq y$, it holds

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \leq -\frac{2\pi}{D} \tan\left(\frac{\pi|x-y|}{2D}\right).$$

Sketch of proof. Fix $\eta > 0$, $\delta > 0$ and $D_1 > D$. Lemmas 6 and 7 lead to

$$\mathrm{d}\big[F_t - 2\psi_{D_1}(\xi_t)\big] \geq \mathrm{d}\tilde{M}_t - 2\psi'_{D_1}(\xi_t)\big[F_t - 2\psi_{D_1}(\xi_t)\big]\,\mathrm{d}t,$$

in which $\mathrm{d}\tilde{M}_t$ is the martingale part.

Sketch of proof

The above inequality is equivalent to

$$\mathrm{d}\left\{\big[F_t - 2\psi_{D_1}(\xi_t)\big]\exp\left[\int_0^t 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right]\right\} \geq \exp\left[\int_0^t 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right]\mathrm{d}\tilde{M}_t.$$

Integrating from 0 to $t \wedge \tau_\eta \wedge \sigma_\delta$ and taking expectation on both sides give us

$$\begin{aligned} F_0 - 2\psi_{D_1}(\xi_0) \leq \mathbb{E}\bigg\{ & \big[F_{t\wedge\tau_\eta\wedge\sigma_\delta} - 2\psi_{D_1}(\xi_{t\wedge\tau_\eta\wedge\sigma_\delta})\big] \\ & \times \exp\left[\int_0^{t\wedge\tau_\eta\wedge\sigma_\delta} 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right]\bigg\}. \qquad (8) \end{aligned}$$

Brascamp & Lieb (JFA, 1976): if V is convex, then the ground state ϕ_0 is log-concave. Hence $F_{t\wedge\tau_\eta\wedge\sigma_\delta} \leq 0$ a.s.

Sketch of proof

$$F_0-2\psi_{D_1}(\xi_0) \le -2\,\mathbb{E}\left\{\psi_{D_1}(\xi_{t\wedge\tau_\eta\wedge\sigma_\delta})\exp\left[\int_0^{t\wedge\tau_\eta\wedge\sigma_\delta} 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right]\right\}.$$

Notice the following facts:

- Lindvall & Rogers (Ann. Probab., 1986): the log-concavity of the drift $\nabla\log\phi_0$ implies the coupling (X_t, Y_t) is successful, i.e., $\tau_\eta\uparrow\tau<+\infty$ a.s.
- ψ_{D_1} is a bounded function on $[0, D/2]$.
- $\psi'_{D_1}(z) = -\frac{\pi^2}{D_1^2}\sec^2(\frac{\pi z}{D_1}) \le 0$ for $z\in[0, D/2]$, thus $\exp\left[\int_0^{t\wedge\tau_\eta\wedge\sigma_\delta} 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right] \le 1$ for all $t>0$.

Letting $t\uparrow\infty$ and $\delta,\eta\downarrow 0$, the dominated convergence theorem yields

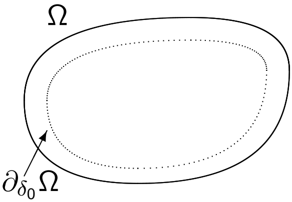
$$F_0-2\psi_{D_1}(\xi_0) \le -2\,\mathbb{E}\left\{\psi_{D_1}(\xi_\tau)\exp\left[\int_0^{\tau} 2\psi'_{D_1}(\xi_s)\,\mathrm{d}s\right]\right\} = 0.$$

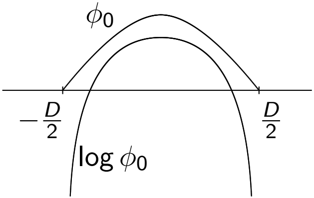
If we do not use the results of Brascamp & Lieb (JFA, 1976) and Lindvall & Rogers (Ann. Probab., 1986), then we need some estimates on the ground state ϕ_0.

Lemma 9

There exists $\delta_0 > 0$ and $C_0 > 0$, such that $\forall\, x \in \partial_{\delta_0}\Omega$,

$$\nabla^2 \log \phi_0(x) \leq -\frac{C_0}{\rho_{\partial\Omega}(x)}.$$





Using the above lemma, we can prove

Lemma 10 (see also J. Wolfson, arXiv:1212.1669)

Fix any $\varepsilon > 0$.
(i) Near-diagonal estimate. There is $\eta_1 > 0$ such that for all $x, y \in \Omega$ with $|x-y| \le \eta_1$, it holds

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \le \varepsilon.$$

(ii) Near-boundary estimate. Let $\eta_1 > 0$ be given as above. There is $\delta_1 > 0$ small enough such that if $\delta < \delta_1$ and $x \in \partial_\delta \Omega$, $y \in \Omega$ with $|x-y| > \eta_1$, it holds

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \le -C_1 \log \frac{\delta_1}{\delta} + C_2$$

for some constants $C_1, C_2 > 0$.

Alternative proof of Theorem 8

Let $\varepsilon > 0$ and $\eta_1 > 0$ be given as in Lemma 10. Take sufficiently small $\delta_2 < \delta_1$. Applying (8) with η_1 and δ_2 gives

$$F_0 - 2\psi_{D_1}(\xi_0) \le \mathbb{E}\Bigg(\left[F_{t\wedge\tau_{\eta_1}\wedge\sigma_{\delta_2}} - 2\psi_{D_1}(\xi_{t\wedge\tau_{\eta_1}\wedge\sigma_{\delta_2}}) \right] \times \exp\left[\int_0^{t\wedge\tau_{\eta_1}\wedge\sigma_{\delta_2}} 2\psi'_{D_1}(\xi_s)\,ds \right] \Bigg).$$

Letting $t \to \infty$,

$$F_0 - 2\psi_{D_1}(\xi_0) \le \mathbb{E}\Bigg(\left[F_{\tau_{\eta_1}\wedge\sigma_{\delta_2}} - 2\psi_{D_1}(\xi_{\tau_{\eta_1}\wedge\sigma_{\delta_2}}) \right] \times \exp\left[\int_0^{\tau_{\eta_1}\wedge\sigma_{\delta_2}} 2\psi'_{D_1}(\xi_s)\,ds \right] \Bigg).$$

By Lemma 10 we can prove

$$F_{\tau_{\eta_1}\wedge\sigma_{\delta_2}} - 2\psi_{D_1}(\xi_{\tau_{\eta_1}\wedge\sigma_{\delta_2}}) \le 2\varepsilon.$$

Outline

1. Introduction
2. Log-concavity estimate of ground state
3. Proof of the fundamental gap conjecture

Simple preparation

Recall the coupling processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t + 2\nabla \log \phi_0(X_t)\,\mathrm{d}t, \quad X_0 = x \in \Omega.$$
$$\mathrm{d}Y_t = \sqrt{2}\,M(X_t, Y_t)\,\mathrm{d}B_t + 2\nabla \log \phi_0(Y_t)\,\mathrm{d}t, \quad Y_0 = y \in \Omega.$$

We still denote by $\xi_t = |X_t - Y_t|/2$ which satisfies

$$\begin{aligned} \mathrm{d}\xi_t &= \sqrt{2}\,\langle \beta_t, \mathrm{d}B_t\rangle + F_t\,\mathrm{d}t \\ &\leq \sqrt{2}\,\langle \beta_t, \mathrm{d}B_t\rangle - \frac{2\pi}{D}\tan\left(\frac{\pi\xi_t}{D}\right)\mathrm{d}t. \end{aligned} \tag{9}$$

Lemma 11

We have for all $t \geq 0$,

$$\mathbb{E}\sin\left(\frac{\pi\xi_t}{D}\right) \leq \exp\left(-\frac{3\pi^2 t}{D^2}\right)\sin\left(\frac{\pi|x-y|}{2D}\right) \leq \exp\left(-\frac{3\pi^2 t}{D^2}\right).$$

Sketch of proof. By Itô's formula and (9),

$$\begin{aligned} \mathrm{d}\sin\left(\frac{\pi\xi_t}{D}\right) &= \frac{\pi}{D}\cos\left(\frac{\pi\xi_t}{D}\right)\mathrm{d}\xi_t - \frac{\pi^2}{2D^2}\sin\left(\frac{\pi\xi_t}{D}\right)\mathrm{d}\xi_t\cdot\mathrm{d}\xi_t \\ &\leq \sqrt{2}\,\frac{\pi}{D}\cos\left(\frac{\pi\xi_t}{D}\right)\langle\beta_t,\mathrm{d}B_t\rangle - \frac{3\pi^2}{D^2}\sin\left(\frac{\pi\xi_t}{D}\right)\mathrm{d}t. \end{aligned}$$

Denote by $\hat{M}_t$ the martingale part. Then

$$\mathrm{d}\left[\exp\left(\frac{3\pi^2 t}{D^2}\right)\sin\left(\frac{\pi\xi_t}{D}\right)\right] \leq \exp\left(\frac{3\pi^2 t}{D^2}\right)\mathrm{d}\hat{M}_t.$$

Integrating from 0 to $t \wedge \tau_\eta \wedge \sigma_\delta$ and taking expectation yield

$$\mathbb{E}\left[\exp\left(\frac{3\pi^2(t \wedge \tau_\eta \wedge \sigma_\delta)}{D^2}\right)\sin\left(\frac{\pi \xi_{t\wedge\tau_\eta\wedge\sigma_\delta}}{D}\right)\right] \le \sin\left(\frac{\pi|x-y|}{2D}\right).$$

Letting δ and η tend to 0 gives us

$$\mathbb{E}\left[\exp\left(\frac{3\pi^2(t \wedge \tau)}{D^2}\right)\sin\left(\frac{\pi \xi_{t\wedge\tau}}{D}\right)\right] \le \sin\left(\frac{\pi|x-y|}{2D}\right).$$

Recall that $\xi_t = 0$ almost surely for $t \ge \tau$; thus we have

$$\mathbb{E}\left[\exp\left(\frac{3\pi^2(t \wedge \tau)}{D^2}\right)\sin\left(\frac{\pi \xi_{t\wedge\tau}}{D}\right)\right] = \mathbb{E}\left[\exp\left(\frac{3\pi^2 t}{D^2}\right)\sin\left(\frac{\pi \xi_t}{D}\right)\right],$$

which leads to the desired result. □

Theorem 12 (Fundamental gap conjecture)

If the potential V of the Schrödinger operator $L = -\Delta + V$ is convex, then the spectral gap $\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}$.

Proof. Recall that the ground state transform
$v = \frac{e^{-\lambda_1 t}\phi_1}{e^{-\lambda_0 t}\phi_0} =: e^{-(\lambda_1-\lambda_0)t} v_0$ solves

$$\frac{\partial v}{\partial t} = \Delta v + 2\langle \nabla \log \phi_0, \nabla v\rangle.$$

Hence by (5) and (6),

$$v(t,x) = \mathbb{E} v_0(X_t), \quad v(t,y) = \mathbb{E} v_0(Y_t).$$

Since $v_0 = \frac{\phi_1}{\phi_0}$ is Lipschitz continuous on $\bar{\Omega}$ with a constant $K > 0$,

$$|v(t,x) - v(t,y)| \leq \mathbb{E}|v_0(X_t) - v_0(Y_t)| \leq K\mathbb{E}|X_t - Y_t| = 2K\mathbb{E}\xi_t.$$

Next $\sin\frac{\pi z}{D} \geq \frac{2z}{D}$ for $z \in \left[0, \frac{D}{2}\right]$, hence

$$|v(t,x) - v(t,y)| \leq KD\mathbb{E}\sin\left(\frac{\pi\xi_t}{D}\right) \leq KD\exp\left(-\frac{3\pi^2 t}{D^2}\right),$$

where the last inequality is due to Lemma 11.

Noting that $v(t,x) - v(t,y) = e^{-(\lambda_1-\lambda_0)t}(v_0(x) - v_0(y))$, we obtain

$$e^{-(\lambda_1-\lambda_0)t}|v_0(x) - v_0(y)| \leq KD\exp\left(-\frac{3\pi^2 t}{D^2}\right)$$

for all $t \geq 0$ and $x, y, \in \Omega$. Since $v_0 = \frac{\phi_1}{\phi_0}$ is not constant, we conclude that

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}.$$

Thank you for your attention!