The deviation matrix, Poisson's equation, and Quasi-birth-death processes (QBDs)

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Institute of Probability and Statistics, School of Mathematics and Statistics, Central South University

the Ninth Workshop on Markov Processes and Related Topics
SWJTU and BNU, July 6-13, 2013

Joint work with Sarah Dendievel and Guy Latouche

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Poisson's equation

Let $\{\Phi_0, \Phi_1, \Phi_2, \dots\}$ be a discrete-time Markov chain.

Poisson's equation:

$$(I - P)\underline{x} = \underline{g}$$

where

- P is the transition matrix $P \geq 0$, $P\underline{1} = \underline{1}$,
- $\underline{g}$ is a given vector indexed by the state space

Assume

- denumerable state space, irreducible, positive recurrent
- $\underline{\pi}^{\mathrm{t}}\underline{g} = 0$. w.l.g.

Link with Central limit theorem

Define $S_n = \frac{1}{n} \sum_{0 \le t \le n-1} g_{\Phi_t}$

Strong Law of Large Number:

$$S_n \to \underline{\pi}^{\mathrm{t}} \underline{g} \qquad \text{a.s. for } n \to \infty$$

Central limit theorem:

$$\sqrt{n}(S_n - \underline{\pi}^{\mathrm{t}} \underline{g}) \Rightarrow N(0, \sigma_g^2) \qquad \text{for } n \to \infty$$

where $\sigma_g^2 = \sum_i \pi_i (2 h_i \bar{g}_i - \bar{g}_i^2)$

$$(I - P)\underline{h} = \underline{\bar{g}} \qquad \underline{\bar{g}} = \underline{g} - (\underline{\pi}^{\mathrm{t}} \underline{g})\underline{1}$$

Link with perturbation

We need P is aperiodic.
Take perturbation E: matrix s.t. $Q = P + E$ is stochastic, irreducible, positive recurrent, ...

Define $\underline{\alpha}$ to be the stationary probability vector of Q

If E is sufficiently small,

$$\underline{\alpha}^{\mathrm{t}} = \underline{\pi}^{\mathrm{t}} \sum_{n \geq 0} (E\mathcal{D})^n$$

where $\mathcal{D} := \sum_{n \geq 0}(P^n - \underline{1}\underline{\pi}^{\mathrm{t}})$ is the deviation matrix such that

$$(I - P)\mathcal{D} = I - \underline{1}\underline{\pi}^{\mathrm{t}} \qquad \underline{\pi}^{\mathrm{t}}\mathcal{D} = \underline{0}$$

Outline

Finite state space is easy

Poisson's equation

$$(I - P)\underline{x} = \underline{g}$$

P stochastic, irreducible, finite size: $\underline{x}$ is unique, up to an additive constant

$$\underline{x} = (I - P)^{\#}\underline{g} + c\underline{1}$$

- $(I - P)^{\#}$ is the group inverse of $(I - P)$
- c is an arbitrary constant
- actually, $\underline{\pi}^{t}\underline{g} = 0$ otherwise system doesn't make sense

Groupe inverses and deviation matrix

Groupe inverse of A (unique when it exists):

(1) $AA^{\#}A = A, \;\; A^{\#}AA^{\#} = A^{\#}$

(2) $AA^{\#} = A^{\#}A$

- For properties and computation:
Campbell and Meyer, *Generalized Inverses of Linear Transformations*, 1979

- Irreducible finite MC: $(I - P)^{\#}$ exists and is unique solution to

$$(I - P)(I - P)^{\#} = I - \underline{1}\underline{\pi}^{t}, \qquad \underline{\pi}^{t}(I - P)^{\#} = \underline{0}$$

Deviation matrix: (in addition, P is non-periodic)

$$\mathcal{D} := \sum_{n \geq 0} (P^n - \underline{1}\underline{\pi}^{t}) = (I - P)^{\#}.$$

Constructive solution

GLynn and Meyn (1996): Assume $\underline{\pi}^{\mathrm{t}}|\underline{g}| < \infty$

(i) Take j to be an arbitrary state and T to be its first return time

One solution of the Poisson equation $(I - P)\underline{x} = \underline{g}$ is given by

$$x_i = \mathrm{E}[\sum_{0 \le n < T} g_{\Phi_n} | \Phi_0 = i]$$

$$x_j = 0.$$

(ii) (uniqueness:) solutions are given up to an arbitrary constant.

Comments of (i):

It is not convenient to consider single state j for matrix-analytic models.

Censoring — a.k.a. Schur complementation

Take subset of states A, T first return time to A,

$$P = \begin{bmatrix} P_{AA} & P_{AB} \\ P_{BA} & P_{BB} \end{bmatrix} \qquad N_B = \sum_{n \geq 0} P_{BB}^n$$

$$\beta_i = \mathrm{E}[\sum_{0 \leq n < T} g_{\Phi_n} | \Phi_0 = i]$$

Dendievel, Latouche and Liu (2013):

Theorem 1: One solution is given by

$$\underline{x}_A = \underline{\beta}_A + (P_{AA} + P_{AB} N_B P_{BA}) \underline{x}_A$$

$$\underline{x}_B = \underline{\beta}_B + N_B P_{BA} \underline{x}_A$$

Schur complementation, same as censoring for stationary distribution.

Deviation matrix $\mathcal{D} = \sum_{n \geq 0}(P^n - \underline{1}\underline{\pi}^{\mathrm{t}})$

The deviation matrix exits (i.e. the series converges) if and only if $\mathrm{E}[T^2(j)|\Phi_0 = j] < \infty$

Bhulai and Spieksma (2003)

Assume P is geometrically ergodic.

(i) $\mathcal{D}$ is the unique solution of

$$(I - P)\mathcal{D} = I - \underline{1}\underline{\pi}^{\mathrm{t}}, \qquad \underline{\pi}^{\mathrm{t}}\mathcal{D} = \underline{0}$$

(ii) A solution of $(I - P)\underline{x} = \underline{g}$, with $\underline{\pi}^{\mathrm{t}}\underline{g} = 0$, $\underline{\pi}^{\mathrm{t}}|\underline{g}| < \infty$, is $\underline{x} = \mathcal{D}\underline{g} + c\underline{1}$, where c is arbitrary.

In this way, $\mathcal{D}$ is just like $(I - P)^{\#}$.

QBDs

QBDs are Markov chains on a two-dimensional state space

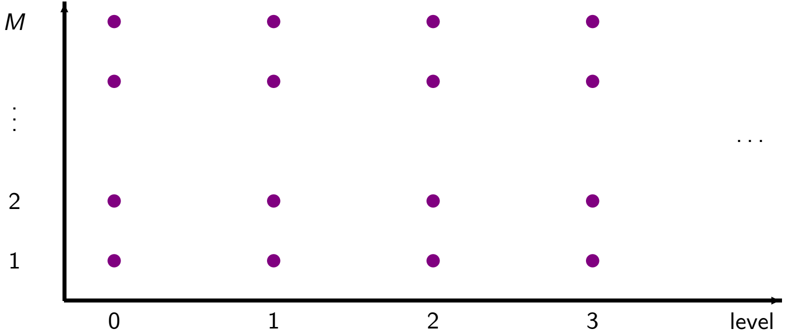
$$(n, \varphi) : \quad n = 0, 1, 2, \ldots; \quad \varphi = 1, 2, \ldots, M$$

here $M < \infty$.

Often,

- n is length of a queue, named the level,
- φ may be many different things, named the phase.

Transition graph (such as it is)



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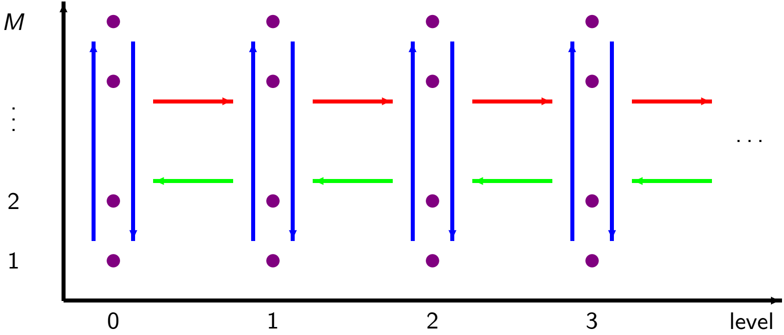

Transition matrix

Block-structured transition matrix:

$$P = \begin{bmatrix} A_* & A_1 & 0 & 0 & \cdots \\ A_{-1} & A_0 & A_1 & 0 & \\ 0 & A_{-1} & A_0 & A_1 & \ddots \\ 0 & 0 & A_{-1} & A_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

Transition probabilities:

$(A_1)_{ij}$ probability to go up from (n, i) to $(n+1, j)$

Transition matrix

Block-structured transition matrix:

$$P = \begin{bmatrix} A_* & A_1 & 0 & 0 & \cdots \\ A_{-1} & A_0 & A_1 & 0 & \\ 0 & A_{-1} & A_0 & A_1 & \ddots \\ 0 & 0 & A_{-1} & A_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

Transition probabilities:

$(A_{-1})_{ij}$ probability to go down from (n, i) to $(n-1, j)$

Transition matrix

Block-structured transition matrix:

$$P = \begin{bmatrix} A_* & A_1 & 0 & 0 & \cdots \\ A_{-1} & A_0 & A_1 & 0 & \\ 0 & A_{-1} & A_0 & A_1 & \ddots \\ 0 & 0 & A_{-1} & A_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

Transition probabilities:

$(A_0)_{ij}$ probability to stay in level n, (n, i) to (n, j), $n \neq 0$

Transition matrix

Block-structured transition matrix:

$$P = \begin{bmatrix} A_* & A_1 & 0 & 0 & \cdots \\ A_{-1} & A_0 & A_1 & 0 & \\ 0 & A_{-1} & A_0 & A_1 & \ddots \\ 0 & 0 & A_{-1} & A_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

Transition probabilities:

$(A_*)_{ij}$ probability to remain in level 0, $(0, i)$ to $(0, j)$

matrices for QBDs

Analysis makes extensive use of matrices

$$G_{ij} = \mathrm{P}[T < \infty, \Phi_T = (0, j) | \Phi_0 = (1, i)],$$

$$R_{ij} = \mathrm{E}[\sum_{0 \le t < T} \mathbb{1}[\Phi_t = (1, j)] | \Phi_0 = (0, i)],$$

$$U = A_0 + A_1 G$$

T is the first return time to level 0.

Computing matrices G and R.

Latouche-Ramaswami's Algorithm (1993) is powerful.

QBDs

Our focus:

Look for "constructive" solution to

$$(I - P)\underline{x} = \underline{g},$$

where P is the transition matrix of an irreducible, aperiodic and positive recurrent QBD.

Solution 1: through the first return time

Let $A = \ell(0)$ denote the level 0 and $B = \mathbb{E}\backslash A$. Using Theorem 1, we have:

Dendievel, Latouche and Liu (2013)

Theorem 2. A solution of the Poisson's equation for a QBD is given by

$$\mathbf{h_0} = (I - P^*)^{\#}\left[\mathbf{y_0} - (\boldsymbol{\pi}\mathbf{g})\boldsymbol{\tau_0}\right] + \mathbf{1},$$
$$\mathbf{h_n} = \mathbf{y_n} - (\boldsymbol{\pi}\mathbf{g})\,\boldsymbol{\tau_n} + G^n\mathbf{h_0}, \quad n \geq 1,$$

where $P^* = B + A_1 G$, and $\mathbf{y_n}$, $\boldsymbol{\tau_n}$ are obtained by

$$\boldsymbol{\tau_0} = \mathbf{1} + A_1\boldsymbol{\tau_1},$$
$$\boldsymbol{\tau_n} = \left[(I - G^n)(I - G)^{\#} + n\mathbf{1}\boldsymbol{\nu}^\top\right](I - U)^{-1}(I - R)^{-1}\mathbf{1}, \quad n \geq 1,$$
$$\mathbf{y_0} = g_0 + A_1\mathbf{y_1},$$
$$\mathbf{y_n} = \sum_{i=0}^{n-1} G^i (I - U)^{-1}\left[\sum_{l\geq 0} R^l g_{n-i+l}\right], \quad n \geq 1,$$

Solution 2: through the deviation matrix

Let $C = I - \underline{1}\,\underline{\pi}^{t}$. Look for the deviation matrix by solving

$$(I - P)X = C$$

i.e. $X = \mathcal{D}$.

Details are a bit messy, focus on the structure.

$$\left[\begin{array}{c|ccc} I - A_* & -A_1 & 0 & \dots \\ \hline -A_{-1} & I - A_0 & -A_1 & \\ 0 & -A_{-1} & I - A_0 & \\ \vdots & & & \ddots \end{array}\right] \left[\begin{array}{c} X_0 \\ \hline X_1 \\ X_2 \\ \vdots \end{array}\right] = \left[\begin{array}{c} C_0 \\ \hline C_1 \\ C_2 \\ \vdots \end{array}\right]$$

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Schur complementation

$$\left[\begin{array}{c|ccc} I-A_* & -A_1 & 0 & \dots \\ \hline -A_{-1} & I-A_0 & -A_1 & \\ 0 & -A_{-1} & I-A_0 & \\ \vdots & & & \ddots \end{array}\right] \left[\begin{array}{c} X_0 \\ \hline X_1 \\ X_2 \\ \vdots \end{array}\right] = \left[\begin{array}{c} C_0 \\ \hline C_1 \\ C_2 \\ \vdots \end{array}\right]$$

Gaussian elimination:

$$\begin{bmatrix} I-A_0 & -A_1 & \\ -A_{-1} & I-A_0 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \end{bmatrix} + \begin{bmatrix} A_{-1} \\ 0 \\ \vdots \end{bmatrix} X_0$$

Isolate $\begin{bmatrix} X_1 & X_2 & \dots \end{bmatrix}$ and inject in first equation

Schur complementation (contd)

$$\begin{bmatrix} I - A_0 & -A_1 & \\ -A_{-1} & I - A_0 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \end{bmatrix} + \begin{bmatrix} A_{-1} \\ 0 \\ \vdots \end{bmatrix} X_0$$

Need W such that

$$W \begin{bmatrix} I - A_0 & -A_1 & \\ -A_{-1} & I - A_0 & \\ & & \ddots \end{bmatrix} = I$$

Or $\quad W = \sum_{n \geq 0} H^n,$

$$H = \begin{bmatrix} A_0 & A_1 & \\ A_{-1} & A_0 & \\ & & \ddots \end{bmatrix}$$

W: the matrix of the expected sojourn times

$W_{(n,i)(k,j)}$ is expected number of visits to (k, j) before level zero, starting from (n, i). A bit calculation yields

$$W_{nk} = \begin{cases} G^{n-k} W_{kk}, & n > k, \\ W_{nn} R^{k-n}, & n < k, \\ \sum_{\nu=0}^{n-1} G^{\nu} (I - U)^{-1} R^{\nu}, & n = k. \end{cases}$$

Probability explanation of the first case: $n > k$

$$n \underset{G}{\rightarrow} n-1 \underset{G}{\rightarrow} \cdots \underset{G}{\rightarrow} k$$

(a) go down $n - k$ levels from n to k and

(b) start counting

$$W_{nk} = G^{n-k} W_{kk}$$

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Schur complementation (end)

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First equation becomes

$$(I - P_*)X_0 = \sum_{i \geq 0} R^i C_i,$$

where $P_* = B + A_1 G$ is the transition matrix of the censored Markov chain.

The expression of the deviation matrix

For deviation matrix, let $C = I - \underline{1}\,\underline{\pi}^{\mathrm{t}}$ (in $(I-P)X = C$).

Dendievel, Latouche and Liu (2013)

Theorem 3. The deviation matrix $\mathcal{D}$ is given by $\mathcal{D} = (I - \underline{1}\,\underline{\pi}^{\mathrm{t}})K$, where

$$K_{0k} = (I - P_*)^{\#} \quad (I - \underline{\tau}_0 \underline{\pi}_0^{\mathrm{t}}) R^k \qquad k \geq 0$$

$$K_{n0} = -\underline{\tau}_n \underline{\pi}_0^{\mathrm{t}} + G^n K_{00} \qquad n \geq 1$$

$$K_{nk} = W_{nk} - \underline{\tau}_n \underline{\pi}_k^{\mathrm{t}} + G^n K_{0k} \qquad n, k \geq 1,$$

- One solution of the Poisson equation is given by $\underline{x} = \mathcal{D}\underline{g} + c\underline{1}$.

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$$
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K_{nk} &= W_{nk} - \underline{\tau}_n \underline{\pi}_k^{\mathrm{t}} + G^n K_{0k} & & n, k \geq 1,
\end{aligned}
$$

- One solution of the Poisson equation is given by $\underline{x} = \mathcal{D}\underline{g} + c\underline{1}$.

Illustration

Assume QBD is a queueing system, level = number of customers.

Define

$$m_{\ell,j} = \frac{1}{L} \sum_{n\geq 0} (\mathrm{E}[Y_n | Y_0 = \ell, \varphi_0 = j] - L)$$

where L is stationary expected number of customers.

One has

$$\underline{m} = \frac{1}{L} \mathcal{D} \underline{\gamma} \quad \text{with} \underline{\gamma}_n = n\underline{1}.$$

Special form of $\underline{\gamma}$ allows for further simplification and makes it possible to compute $\underline{m}$ with finite computations.

PH/M/1 queue

Example: PH/M/1 queue, services are exponential, interarrival times are PH($\underline{\tau}$, T) with

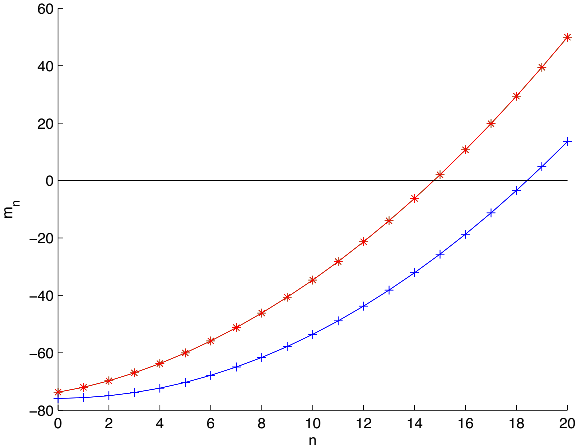
$$\underline{\tau} = \begin{bmatrix} 0.1127 & 0.8873 \end{bmatrix},$$

$$T = \begin{bmatrix} -0.2254 & 0 \\ 0 & -1.7746 \end{bmatrix}.$$

Service rate $= 1.2$.

Traffic coefficient $= 0.8333$.

PH/M/1 queue (Contd)



$L = 11.1$. Plot of $m_{n,i}$: blue line for phase $i = 1$; red line for phase $i = 2$.


That's all Folks!