On time regularity of Ornstein-Uhlenbeck equation driven by Lévy noise in Hilbert spaces

刘 勇

北 京 大 学

与翟建梁(中国科大)合作

第九届马氏过程及其相关课题研讨班, 2013年7月10日

Outline

1. Background
2. Results
    - Definitions of càdlàg modification
    - General results on càdlàg modification
    - Application
    - Key points of proof
3. Conclusions

Outline

1. Background
2. Results
   - Definitions of càdlàg modification
   - General results on càdlàg modification
   - Application
   - Key points of proof
3. Conclusions

Outline

1 Background

2 Results
- Definitions of càdlàg modification
- General results on càdlàg modification
- Application
- Key points of proof

3 Conclusions

Outline

1 Background

2 Results
- Definitions of càdlàg modification
- General results on càdlàg modification
- Application
- Key points of proof

3 Conclusions

Problems

Ornstein-Uhlenbeck Equation

$$dX(t) = AX(t)dt + dL(t), \ t \geq 0. \quad (1.1)$$

- H, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$;
- A, generator of a C_0-semigroup on H, A^* the adjoint operator of A;
- L, Lévy process,

Problems

Ornstein-Uhlenbeck Equation

$$dX(t) = AX(t)dt + dL(t), \ t \geq 0. \tag{1.1}$$

- H, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$;
- A, generator of a C_0-semigroup on H, A^* the adjoint operator of A;
- L, Lévy process,

Problems

Ornstein-Uhlenbeck Equation

$$dX(t) = AX(t)dt + dL(t), \ t \geq 0. \quad (1.1)$$

- H, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$;
- A, generator of a C_0-semigroup on H, A^* the adjoint operator of A;
- L, Lévy process,

Problems

Problem

If the solution of Eq. (1.1) $(X(t))_{t\geq 0}$ *takes value in* H *for any* t, *is there a* H*-valued (strong, weak,...) càdlàg modification of* X*?*
i.e. $\exists$ *???* *a* H*-valued (strong, weak,...) càdlàg* $(\tilde{X}(t))_{t\geq 0}$ *such that,*

$$\mathbb{P}(X(t) = \tilde{X}(t)) = 1, \quad \text{for any} \quad t. \tag{1.2}$$

Motivations

Motivation?

Why do we consider the problem on càdlàg modification?

Measurability

$\sup_{t\in[a,b]} X(t)$, $\inf_{t\in[a,b]} X(t)$, $\liminf_{t\in[a,b]} X(t)$,... measurable?

Hitting time

Λ: Borel set

Hitting Time:

$$T(\omega) = \inf\{t > 0, X(t,\omega) \in \Lambda\}.$$

- Is $T(\omega)$ the stopping time?
- $X(t,\omega)$: adapted and right continuous.

Measurability

$\sup_{t\in[a,b]} X(t)$, $\inf_{t\in[a,b]} X(t)$, $\liminf_{t\in[a,b]} X(t)$,... measurable?

Hitting time

Λ: Borel set

Hitting Time:

$$T(\omega) \equiv \inf\{t > 0, X(t,\omega) \in \Lambda\}.$$

- Is $T(\omega)$ the stopping time?
- $X(t,\omega)$: adapted and right continuous.

Measurability

$\sup_{t\in[a,b]} X(t)$, $\inf_{t\in[a,b]} X(t)$, $\liminf_{t\in[a,b]} X(t)$,... measurable?

Hitting time

Λ: Borel set

Hitting Time:

$$T(\omega) \equiv \inf\{t > 0, X(t,\omega) \in \Lambda\}.$$

- Is $T(\omega)$ the stopping time?
- $X(t,\omega)$: adapted and right continuous.

Measurability

$\sup_{t\in[a,b]} X(t)$, $\inf_{t\in[a,b]} X(t)$, $\liminf_{t\in[a,b]} X(t)$,... measurable?

Hitting time

Λ: Borel set

Hitting Time:

$$T(\omega) \equiv \inf\{t > 0, X(t,\omega) \in \Lambda\}.$$

- Is $T(\omega)$ the stopping time?
- $X(t,\omega)$: adapted and right continuous.

Doob's stopping time (optional sampling) theorem

$(X(t))_{t\geq 0}$ is right continuous (super)-martingale.

Strong Markov property

$(X(t))_{t\geq 0}$ is a right continuous Feller process.

Stochastic Integral

Predictable, optional process...

$$\int_0^t N(s-)dN(s)$$

Doob's stopping time (optional sampling) theorem

$(X(t))_{t\geq 0}$ is right continuous (super)-martingale.

Strong Markov property

$(X(t))_{t\geq 0}$ is a right continuous Feller process.

Stochastic Integral

Predictable, optional process...

$$\int_0^t N(s-)dN(s)$$

Doob's stopping time (optional sampling) theorem

$(X(t))_{t\geq 0}$ is right continuous (super)-martingale.

Strong Markov property

$(X(t))_{t\geq 0}$ is a right continuous Feller process.

Stochastic Integral

Predictable, optional process...

$$\int_0^t N(s-)dN(s)$$

Some classic results

Brownian Motion

- 1900 Bachilier
- 1905 Einstein

  $p(t,x)$: *the* **distribution density** *of particles at time* t *and position* x.

$$\frac{\partial p(t,x)}{\partial t} = D\frac{\partial^2 p(t,x)}{\partial x^2}.$$

- 1923 Wiener

  Constructed Brownian motion (distribution) on $C[0,1]$*.*

  Let $\xi_0, \xi_1, \xi_2, \cdots$ *i.i.d.* $\sim N(0,1)$

$$\xi_0 t + \sum_{n=1}^{\infty} \sum_{k=2^{n-1}}^{2^n-1} \xi_k \sqrt{2} \frac{\sin(\pi k t)}{\pi k},$$

$$\xi_0 t + \sum_{n=1}^{\infty} \xi_n \sqrt{2} \frac{\sin(\pi n t)}{\pi n}.$$

Wiener process, Wiener measure, Wiener space, $(C[0,1], \mathcal{B}(C[0,1]), \mathbb{P}^W)$.

- 1927 Lévy, Ciesielski
  Haar wavelet's mother function

$$\eta = \begin{cases} 1, & [0, \frac{1}{2}) \\ -1, & [\frac{1}{2}, 1) \\ 0, & \textit{otherwise}. \end{cases}$$

$$\phi_0 = \mathbf{1}_{[0,1)}, \ \ \phi_{2^n+k} = 2^{\frac{n}{2}} \eta(2^n t - k), \ \ t \in [0,1), \ b \in \mathbb{N}, \ \ 0 \le k < 2^n.$$

$$e_n' = \phi_n, \quad \textit{Schauder function},$$

$$B_t = \sum_{n=0}^{\infty} \xi_n e_n.$$

- 1931 Kolmogorov's Forward (Fokker-Planck), Backward Equation.
  1936 Feller

$$\frac{\partial p(t,x,y)}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial y_i \partial y_j}\big(a_{ij}(y)p(t,x,y)\big) - \nabla\cdot\big(b(y)p(t,x,y)\big).$$

$$\frac{\partial p(t,x,y)}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} p(t,x,y) + b(x)\cdot p(t,x,y).$$

- 1942 Itô Stochastic Integral, Stochastic Differential Equation.

W.Feller [4]... no proof was given of the fact that it is possible to introduce by means of this solution a probability measure on some "continuous" function space.

The objective of this article, then, is
1) to formulate the problem precisely, and
2) to give a rigorous proof, à la Doob, for the existence of continuous parameter stochastic processes.

$$dX(t) = \sigma(X(t))dB(t) + b(X(t))dt, \quad \sigma \cdot \sigma^* = (a_{i,j}).$$

- 1942 Itô Stochastic Integral, Stochastic Differential Equation.

W.Feller [4]... no proof was given of the fact that it is possible to introduce by means of this solution a probability measure on some "continuous" function space.

The objective of this article, then, is
1) to formulate the problem precisely, and
2) to give a rigorous proof, à la Doob, for the existence of continuous parameter stochastic processes.

$$dX(t) = \sigma(X(t))dB(t) + b(X(t))dt, \quad \sigma \cdot \sigma^* = (a_{i,j}).$$

- 1942 Itô Stochastic Integral, Stochastic Differential Equation.

W.Feller [4]... no proof was given of the fact that it is possible to introduce by means of this solution a probability measure on some "continuous" function space.

The objective of this article, then, is
1) to formulate the problem precisely, and
2) to give a rigorous proof, à la Doob, for the existence of continuous parameter stochastic processes.

$$dX(t) = \sigma(X(t))dB(t) + b(X(t))dt, \quad \sigma \cdot \sigma^* = (a_{i,j}).$$

- 1969-1982 Stroock, Varadhan, Solution of Martingale Problem

Theory of Stochastic Analysis without Stochastic Integral.

Constructing $\mathbb{P}$ on $C[0,1]$ or $D[0,1]$ such that the transition probability of the canonical process satisfies Kolmogorov's Equation.

- Dynkin-Kinney' criterion for Markov processes

$$\psi_\varepsilon(\delta) = \sup_{x \in \mathcal{S}, |t| < \delta} P(t; x, B(x, \varepsilon)^c)$$

$$C[0,1]: \quad \psi_\varepsilon(\delta) = o(\delta);$$

$$D[0,1]: \quad \psi_\varepsilon(\delta) = o(1).$$

- Kolmogorov-Censov's moment criterion for general processes

$C[0,1]$: for any $t, s \in [0,1]$,

$$E\|X(t) - X(s)\|^\alpha \le C|t-s|^{1+\beta};$$

$D[0,1]$: $\exists p, q \ge 0, r > 0, \forall, 0 \le t_1 < t_2 < t_3 \le 1,$

$$E(\|X(t_2) - X(t_1)\|^p \|X(t_3) - X(t_2)\|^q) \le C|t_3 - t_1|^{1+r}.$$

- Dynkin-Kinney' criterion for Markov processes

$$\psi_\varepsilon(\delta) = \sup_{x\in\mathcal{S},|t|<\delta} P(t; x, B(x,\varepsilon)^c)$$

$$C[0,1]: \quad \psi_\varepsilon(\delta) = o(\delta);$$

$$D[0,1]: \quad \psi_\varepsilon(\delta) = o(1).$$

- Kolmogorov-Censov's moment criterion for general processes
  $C[0,1]$: for any $t, s \in [0,1]$,

$$E\|X(t) - X(s)\|^\alpha \leq C|t-s|^{1+\beta};$$

$D[0,1]$: $\exists p, q \geq 0, r > 0$, $\forall, 0 \leq t_1 < t_2 < t_3 \leq 1$,

$$E(\|X(t_2) - X(t_1)\|^p \|X(t_3) - X(t_2)\|^q) \leq C|t_3 - t_1|^{1+r}.$$

Interests in Researching (Generalized) O-U Equations

(Generalized) O-U Equations: linear, solvable, Markov, stationary, Gaussian or stable ...

- *"Test" stochastic equation for some abstract theories.*
- $dX(t) = AX(t)dt + dL(t) +$ "purtubation".

2010 Brzeźniak, Goldys, Imkeller, Peszat, Priola, Zabczyk

The time regularity of the process X is of prime interest in the study of non-linear Stochastic PDEs.

Interests in Researching (Generalized) O-U Equations

(Generalized) O-U Equations: linear, solvable, Markov, stationary, Gaussian or stable ...

- *"Test" stochastic equation for some abstract theories.*
- $dX(t) = AX(t)dt + dL(t) +$ "purtubation".

2010 Brzeźniak, Goldys, Imkeller, Peszat, Priola, Zabczyk

The time regularity of the process X is of prime interest in the study of non-linear Stochastic PDEs.

Interests in Researching (Generalized) O-U Equations

(Generalized) O-U Equations: linear, solvable, Markov, stationary, Gaussian or stable ...

- *"Test" stochastic equation for some abstract theories.*
- $dX(t) = AX(t)dt + dL(t) +$ "purtubation".

2010 Brzeźniak, Goldys, Imkeller, Peszat, Priola, Zabczyk

The time regularity of the process X is of prime interest in the study of non-linear Stochastic PDEs.

Cases of Brownian Motion

- Da Prato, G., Kwapień, S., Zabczyk, J. 1987
- Iscoe, I., Marcus, M.B., McDonald, D., Talagrand, M., Zinn, J. 1990
- Da Prato, G., Zabczyk, J. 1992
- Millet, A., Smoleński, W. 1992
- Brzeźniak, Z., Peszat, S., Zabczyk, J. 2001

For example, in 1990, Iscoe, et.al. gave a simple but quite sharp criterion for continuity of X_t in l^2,

$$dx_k(t) = -\lambda_k x_k(t)dt + \sqrt{2a_k}dB_k, \quad k = 1, 2, \cdots .$$

Iscoe, I., Marcus, M.B., McDonald, D., Talagrand, M., Zinn, J. 1990

$f(x)$ positive function on $[0,\infty)$ such that $\frac{f(x)}{x}$ nondecreasing for $x \geq x_1 > 0$ and

$$\int_{x_1}^{\infty} \frac{dx}{f(x)} < \infty, \quad \sum_k \frac{a_k}{\lambda_k} < \infty, \quad \sup_k \frac{f(a_k) \vee x_1}{\lambda_k \vee 1} < \infty. \tag{1.3}$$

Then, x_t is continuous in l^2 a.s. Moreover, this result is best possible in the sense that it is false for any function $f(x)$, which satisfies all the above hypotheses with the exception that $\int_{x_1}^{\infty} \frac{dx}{f(x)} = \infty$.

Outline

1 Background

2 Results
- Definitions of càdlàg modification
- General results on càdlàg modification
- Application
- Key points of proof

3 Conclusions

OU-Equation

$$dX(t) = AX(t)dt + dL(t), \ t \geq 0. \tag{2.1}$$

- H, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$;
- A, generator of a C_0-semigroup on H, A^* adjoint operator of A;
- L, pure jump Lévy process, ν, Lévy measure;
  - $L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, L^n, i.i.d., càdlàg Lévy processes on $\mathbb{R}$;
  - $\{e_n\}_{n \in \mathbb{N}}$, fixed reference orthonormal basis in H;
  - $\beta_n \geq 0$.

OU-Equation

$$dX(t) = AX(t)dt + dL(t),\ \ t \geq 0. \tag{2.1}$$

- H, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$;
- A, generator of a C_0-semigroup on H, A^* adjoint operator of A;
- L, pure jump Lévy process, ν, Lévy measure;
  - $L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, L^n, i.i.d., càdlàg Lévy processes on $\mathbb{R}$;
  - $\{e_n\}_{n \in \mathbb{N}}$, fixed reference orthonormal basis in H;
  - $\beta_n \geq 0$.

OU-Equation

$$dX(t) = AX(t)dt + dL(t), \ t \geq 0. \tag{2.1}$$

- H, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$;
- A, generator of a C_0-semigroup on H, A^* adjoint operator of A;
- L, pure jump Lévy process, ν, Lévy measure;
  - $L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, L^n, i.i.d., càdlàg Lévy processes on $\mathbb{R}$;
  - $\{e_n\}_{n \in \mathbb{N}}$, fixed reference orthonormal basis in H;
  - $\beta_n \geq 0$.

OU-Equation

$$dX(t) = AX(t)dt + dL(t), \ t \geq 0. \tag{2.1}$$

- H, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$;
- A, generator of a C_0-semigroup on H, A^* adjoint operator of A;
- L, pure jump Lévy process, ν, Lévy measure;
  - $L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, L^n, i.i.d., càdlàg Lévy processes on $\mathbb{R}$;
  - $\{e_n\}_{n \in \mathbb{N}}$, fixed reference orthonormal basis in H;
  - $\beta_n \geq 0$.

OU-Equation

$$dX(t) = AX(t)dt + dL(t),\ t \geq 0. \tag{2.1}$$

- H, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$;
- A, generator of a C_0-semigroup on H, A^* adjoint operator of A;
- L, pure jump Lévy process, ν, Lévy measure;
  - $L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, L^n, i.i.d., càdlàg Lévy processes on $\mathbb{R}$;
  - $\{e_n\}_{n \in \mathbb{N}}$, fixed reference orthonormal basis in H;
  - $\beta_n \geq 0$.

OU-Equation

$$dX(t) = AX(t)dt + dL(t),\ t \geq 0. \tag{2.1}$$

- H, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$;
- A, generator of a C_0-semigroup on H, A^* adjoint operator of A;
- L, pure jump Lévy process, ν, Lévy measure;
  - $L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, L^n, i.i.d., càdlàg Lévy processes on $\mathbb{R}$;
  - $\{e_n\}_{n \in \mathbb{N}}$, fixed reference orthonormal basis in H;
  - $\beta_n \geq 0$.

OU-Equation

$$dX(t) = AX(t)dt + dL(t), \ t \geq 0. \tag{2.1}$$

- H, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$;
- A, generator of a C_0-semigroup on H, A^* adjoint operator of A;
- L, pure jump Lévy process, ν, Lévy measure;
  - ☐ $L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, L^n, i.i.d., càdlàg Lévy processes on $\mathbb{R}$;
  - ☐ $\{e_n\}_{n\in\mathbb{N}}$, fixed reference orthonormal basis in H;
  - ☐ $\beta_n \geq 0$.

Definitions of càdlàg modification of processes in Hilbert spaces

Peszat, S., Zabczyk, J. SPA 2013

H-strong càdlàg modification

X, H-valued, if exists a $\tilde{X}$ H-càdlàg trajectories, such that $P(X_t = \tilde{X}_t) = 1, \ \forall t > 0$.

H-weak càdlàg modification

X, H-valued, if exists a $\tilde{X}$ such that for any $z \in H$, the real-valued process $\langle \widetilde{X}(\cdot), z \rangle_H$ has càdlàg trajectories.

H-cylindrical modification

X, H-valued, if for any $z \in H$, the real-valued process $\langle X(\cdot), z \rangle$ has a càdlàg modification.

Definitions of càdlàg modification of processes in Hilbert spaces

Peszat, S., Zabczyk, J. SPA 2013

H-strong càdlàg modification

X, H-valued, if exists a $\tilde{X}$ H-càdlàg trajectories, such that $P(X_t = \tilde{X}_t) = 1, \ \forall t > 0$.

H-weak càdlàg modification

X, H-valued, if exists a $\tilde{X}$ such that for any $z \in H$, the real-valued process $\langle \widetilde{X}(\cdot), z \rangle_H$ has càdlàg trajectories.

H-cylindrical modification

X, H-valued, if for any $z \in H$, the real-valued process $\langle X(\cdot), z \rangle$ has a càdlàg modification.

Definitions of càdlàg modification of processes in Hilbert spaces

Peszat, S., Zabczyk, J. SPA 2013

H-strong càdlàg modification

X, H-valued, if exists a $\tilde{X}$ H-càdlàg trajectories, such that $P(X_t = \tilde{X}_t) = 1, \ \forall t > 0$.

H-weak càdlàg modification

X, H-valued, if exists a $\tilde{X}$ such that for any $z \in H$, the real-valued process $\langle \widetilde{X}(\cdot), z \rangle_H$ has càdlàg trajectories.

H-cylindrical modification

X, H-valued, if for any $z \in H$, the real-valued process $\langle X(\cdot), z \rangle$ has a càdlàg modification.

Definitions of càdlàg modification of processes in Hilbert spaces

V-cylindrical modification

$V \hookrightarrow H$, V-valued process H, if for any $v \in V^*$, the real-valued process $\langle v^*, X\rangle_{V,V^*}$ has a càdlàg modification.

Remark

H-strong $\Rightarrow$ H-weak $\Rightarrow$ H-cylindrical $\Leftarrow$ V-cylindrical

Definitions of càdlàg modification of processes in Hilbert spaces

V-cylindrical modification

$V \hookrightarrow H$, V-valued process H, if for any $v \in V^*$, the real-valued process $\langle v^*, X\rangle_{V,V^*}$ has a càdlàg modification.

Remark

H-strong $\Rightarrow$ H-weak $\Rightarrow$ H-cylindrical $\Leftarrow$ V-cylindrical

Strong càdlàg modification

Brzeźniak, Goldys, Imkeller, Peszat, Priola, Zabczyk, *C. R. Acad. Sci. Paris, Ser. I* 2010

$$dX(t) = AX(t)dt + dL(t), \ t \geq 0, \tag{2.2}$$

$L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, L^n, i.i.d., càdlàg Lévy processes on $\mathbb{R}$.

Theorem (Time irregularity)

X, H-valued process, $(e_n) \in \mathcal{D}(A^*)$, $\beta_n \nrightarrow 0$, then X has no H-càdlàg modification with probability 1.

4 Questions ????

Strong càdlàg modification

Theorem (L. and J.L.Zhai 2012)

$(X(t), t \geq 0)$ *has* H*-càdlàg (resp. continuous) modification*

$$\Leftrightarrow$$

$\forall n \in \mathbb{N}$, $(\langle X(t), e_n \rangle, t \geq 0)$ *càdlàg (resp. continuous)*

$$\mathbb{P}\Big(\lim_{N\to\infty} \sup_{t\in[0,T]} \sum_{i=N}^{\infty} \langle X(t), e_n \rangle^2 = 0 \Big) = 1. \quad (2.3)$$

Weak càdlàg modification

Theorem (Peszat-Zabczyk 2013, L. and J.L.Zhai 2013)

The following four assertions are equivalent:

(1) *The process X has a weakly càdlàg modification;*

(2) $\mathbb{P}\Big(\sup_{t\in[0,T]}\sum_{n=1}^{\infty}\langle X(t),e_n\rangle^2<\infty\Big)=1$;

(3) $\mathbb{P}\Big(\sup_{t\in[0,T]}\sup_{\|z\|_H\le 1}\langle X(t),z\rangle_H<\infty\Big)=1$;

(4) $\mathbb{P}\Big(\sup_{\|z\|_H\le 1}\sup_{t\in[0,T]}|\langle X(t),z\rangle_H|<\infty\Big)=1$;

H-cylindrical càdlàg modification

Theorem (L. and J.L.Zhai 2013)

If for every $y = \sum_{n=1}^{\infty} y_n e_n \in H$,

$$\mathbb{P}\Big(\lim_{i\to\infty} \sup_{t\in[0,T]} |\langle X(t), \sum_{n=i}^{\infty} y_n e_n\rangle_H| = 0\Big) = 1, \qquad (2.4)$$

then process X *has* H*-cylindrical càdlàg modification*

V-cylindrical càdlàg modification

Theorem (L. and J.L.Zhai 2013)

If for every $y^* = \sum_{n=1}^{\infty} y_n^* e_n \in V^*$,

$$\mathbb{P}\Big(\lim_{i\to\infty} \sup_{t\in[0,T]} |\langle X(t), \sum_{n=i}^{\infty} y_n^* e_n \rangle_{V,V^*}| = 0 \Big) = 1, \qquad (2.5)$$

then the process X *has V-cylindrical càdlàg modification*

$$dX(t) = AX(t)dt + dL(t), \ t \geq 0.$$

Corollary (L.and J.L.Zhai 2012, 2013)

Assume that $\nu(V \setminus H)$*=0*

(1) X *has* H*-strong càdlàg modification, then* $\forall\ \epsilon > 0$, $\nu(\|y\|_H \geq \epsilon) < \infty$;

(2) X *has* H*-weak càdlàg modification, then* $\exists\ \epsilon_0 > 0$, $\nu(\|y\|_H \geq \epsilon_0) < \infty$;

Remark

- *This corollary implies the main result in* Brzeźniak *et.al.* 2010

$$\beta_n \nrightarrow 0 \Rightarrow \text{no càdlàg}$$

- *An example solution to OU-Eq. admits H-weak càdlàg but not H-strong càdlàg. Answer a question in* Peszat, Zabczyk 2013

Generalized OU-equation with cylindrical α-stable processes

$$dX(t) = AX(t)dt + dL(t), \;\; t \geq 0, \tag{2.6}$$

$L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n, \quad L^n$, i.i.d., α-stable processes

Theorem (L. and J.L.Zhai 2012,2013)

The following assertions are equivalent:

(a) *The process* L *is a Lévy process on* H*;*

(b) *The process* $(X(t), t \geq 0)$ *has* H*-càdlàg modification;*

(c) *The process* $(X(t), t \geq 0)$ *has weakly càdlàg modification*

(d) $\sum_{n=1}^{\infty} |\beta_n|^{\alpha} < \infty$*;*

(e) *For any* $\epsilon > 0$, $\sum_{n=1}^{\infty} \mu_n\big(|y| \geq \epsilon\big) < \infty$*;*

(f) *There exists* $\epsilon_0 > 0$*, such that* $\sum_{n=1}^{\infty} \mu_n\big(|y| \geq \epsilon_0\big) < \infty$*;*

Remark

The above theorem answer 2 of 4 questions and partly answer another one addressed by Brzeźniak et.al. 2010

- *Does $\beta_n \to 0$ imply existence of a càdlàg modification of X? NO!*
- *Is S.H.E. H_δ-càdlàg for $\delta \in [-\frac{1}{\alpha}, 0)$? YES!*
- *Is $L \in H$ necessary for X having H-strong càdlàg? YES! for OU-with stable noise.*

Remark

The above theorem denies the conjecture in Priola, Zabczyk 2011.

- *much weaker than $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty \Rightarrow X$, H-càdlàg.*

Remark

The above theorem answer 2 of 4 questions and partly answer another one addressed by Brzeźniak et.al. 2010

- *Does $\beta_n \to 0$ imply existence of a càdlàg modification of X? NO!*
- *Is S.H.E. H_δ-càdlàg for $\delta \in [-\frac{1}{\alpha}, 0)$? YES!*
- *Is $L \in H$ necessary for X having H-strong càdlàg? YES! for OU-with stable noise.*

Remark

The above theorem denies the conjecture in Priola, Zabczyk 2011.

- *much weaker than $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty \Rightarrow X$, H-càdlàg.*

Remark

The above theorem answer 2 of 4 questions and partly answer another one addressed by Brzeźniak et.al. 2010

- *Does* $\beta_n \to 0$ *imply existence of a càdlàg modification of* X*? NO!*
- *Is S.H.E.* H_δ*-càdlàg for* $\delta \in [-\frac{1}{\alpha}, 0)$*? YES!*
- *Is* $L \in H$ *necessary for* X *having* H*-strong càdlàg? YES! for OU-with stable noise.*

Remark

The above theorem denies the conjecture in Priola, Zabczyk 2011.

- *much weaker than* $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty \Rightarrow X$*,* H*-càdlàg.*

Remark

The above theorem answer 2 of 4 questions and partly answer another one addressed by Brzeźniak et.al. 2010

- *Does $\beta_n \to 0$ imply existence of a càdlàg modification of X? NO!*
- *Is S.H.E. H_δ-càdlàg for $\delta \in [-\frac{1}{\alpha}, 0)$? YES!*
- *Is $L \in H$ necessary for X having H-strong càdlàg? YES! for OU-with stable noise.*

Remark

The above theorem denies the conjecture in Priola, Zabczyk 2011.

- *much weaker than $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty \Rightarrow X$, H-càdlàg.*

Remark

The above theorem answer 2 of 4 questions and partly answer another one addressed by Brzeźniak et.al. 2010

- *Does $\beta_n \to 0$ imply existence of a càdlàg modification of X? NO!*
- *Is S.H.E. H_δ-càdlàg for $\delta \in [-\frac{1}{\alpha}, 0)$? YES!*
- *Is $L \in H$ necessary for X having H-strong càdlàg? YES! for OU-with stable noise.*

Remark

The above theorem denies the conjecture in Priola, Zabczyk 2011.

- *much weaker than $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty \Rightarrow X,\ H$-càdlàg.*

Remark

The above theorem answer 2 of 4 questions and partly answer another one addressed by Brzeźniak et.al. 2010

- *Does* $\beta_n \to 0$ *imply existence of a càdlàg modification of* X*? NO!*
- *Is S.H.E.* H_δ*-càdlàg for* $\delta \in [-\frac{1}{\alpha}, 0)$*? YES!*
- *Is* $L \in H$ *necessary for* X *having* H*-strong càdlàg? YES! for OU-with stable noise.*

Remark

The above theorem denies the conjecture in Priola, Zabczyk 2011.

□ *much weaker than* $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty \Rightarrow X$*,* H*-càdlàg.*

Theorem (Peszat, Zabczyk 2013)

Let X be the solution to (1.1), where A is the generator of an exponential stable analytic semigroup S on a Hilbert space H. Let L be a Lévy process taking value in a Hilbert space $U = H_{-\rho}$ for a certain $\rho < \frac{1}{2}$. Assume that the Lévy measure ν of L satisfies $\nu(H_{-\rho} \setminus H) = 0$ and that

$$\int_H \left(|z|^2_{-\rho} + |z|^4_{\varepsilon}\right)\nu(dz) < \infty$$

for certain $\varepsilon > 0$.
Then X has a càdlàg modification in H and

$$\mathbb{E} \sup_{0 \geq t \geq T} |X(t)|^q_H < \infty, \ \forall T < \infty, \forall q \in [1, 4).$$

For $0 < v_1 \le v_2 \le \cdots \to +\infty$,
$V \equiv \{y = \sum_{n=1}^{\infty} y_n e_n \in H,\ \sum_{n=1}^{\infty} y_n^2 v_n^2 < \infty\}$,
$V^* = \{y = \sum_{n=1}^{\infty} y_n^* e_n,\ \sum_{n=1}^{\infty} {y^*}_n^2 / v_n^2 < \infty\}$.

Theorem (L. and J.L. Zhai 2013)

(1) X *has H-cylindrical càdlàg modification if and only if for every* $z = \sum_{n=1}^{\infty} z_n e_n \in H$, $\sum_{n=1}^{\infty} |z_n|^{\alpha} |\beta_n|^{\alpha} < \infty$;

(2) X *has V-cylindrical càdlàg modification if and only if for every* $z = \sum_{n=1}^{\infty} z_n e_n \in V^*$, $\sum_{n=1}^{\infty} |v_n z_n|^{\alpha} |\beta_n|^{\alpha} < \infty$.

Theorem

(1) *If* $\sum_{n=1}^{\infty} |\beta_n|^{\frac{2\alpha}{2-\alpha}} < \infty$, *then* X *has* H*-cylindrical càdlàg modification;*

(2) *If* $\sum_{n=1}^{\infty} \left(v_n|\beta_n|\right)^{\frac{2\alpha}{2-\alpha}} < \infty$, *then* X *has* V*-cylindrical càdlàg modification.*

Remark

The above theorem answer a question in Peszat, Zabczyk 2013.

Remark

The above theorems imply that H-cylindrical modification

$$\not\Rightarrow$$

H-weak càdlàg modification, V-cylindrical modification.

Remark

We have an example that
H-weak càdlàg modification

$$\not\Rightarrow$$

H-strong càdlàg modification.

Remark

The above theorem answer a question in Peszat, Zabczyk 2013.

Remark

The above theorems imply that H-cylindrical modification

$$\not\Rightarrow$$

H-weak càdlàg modification, V-cylindrical modification.

Remark

We have an example that
H-weak càdlàg modification

$$\not\Rightarrow$$

H-strong càdlàg modification.

Remark

The above theorem answer a question in Peszat, Zabczyk 2013.

Remark

The above theorems imply that H-cylindrical modification

$$\nRightarrow$$

H-weak càdlàg modification, V-cylindrical modification.

Remark

We have an example that
H-weak càdlàg modification

$$\nRightarrow$$

H-strong càdlàg modification.

Key points of proof

(1) The proper characteristics of compact set in $D([0,t],H)$ in different topologies.

(2) The property of power law (scale law) of Lévy measure of α-stable process.

Outline

1 Background

2 Results
- Definitions of càdlàg modification
- General results on càdlàg modification
- Application
- Key points of proof

3 Conclusions

Conclusions

- Some (necessary and) sufficient conditions are given for a process X has strong (weak, cylindrical...) càdlàg modification;
- A complete answer on four kinks of càdlàg modification of generalized OU-equation with cylindrical α-stable noise is given;
- Clarify some relationships of four kinks of càdlàg modification in Hilbert space;
- Some questions and conjecture are answered.

Conclusions

- Some (necessary and) sufficient conditions are given for a process X has strong (weak, cylindrical...) càdlàg modification;
- A complete answer on four kinks of càdlàg modification of generalized OU-equation with cylindrical α-stable noise is given;
- Clarify some relationships of four kinks of càdlàg modification in Hilbert space;
- Some questions and conjecture are answered.

Conclusions

- Some (necessary and) sufficient conditions are given for a process X has strong (weak, cylindrical...) càdlàg modification;
- A complete answer on four kinks of càdlàg modification of generalized OU-equation with cylindrical α-stable noise is given;
- Clarify some relationships of four kinks of càdlàg modification in Hilbert space;
- Some questions and conjecture are answered.

Conclusions

- Some (necessary and) sufficient conditions are given for a process X has strong (weak, cylindrical...) càdlàg modification;
- A complete answer on four kinks of càdlàg modification of generalized OU-equation with cylindrical α-stable noise is given;
- Clarify some relationships of four kinks of càdlàg modification in Hilbert space;
- Some questions and conjecture are answered.

Thanks!