n-type Markov Branching Processes with Immigration

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Background

- Markov branching process (MBP): 1-type case

- State Space: $\mathbf{Z}_+ = \{0, 1, 2, \cdots\}$

-Transition Rate (q-matrix $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$):

$$q_{ij} = \begin{cases} ib_{j-i+1}, & \text{if } i \geq 1, j \geq i-1 \\ 0, & \text{otherwise.} \end{cases}$$

where $b_k \geq 0 (k \neq 1), \sum_{k \neq 1} b_k = -b_1 < \infty$.

Background

- n-type Markov branching process with immigration (MBPI):

- State Space: $\mathbf{Z}_+^n$

-Transition Rate (q-matrix $Q = (q_{\mathbf{ij}}; \mathbf{i}, \mathbf{j} \in \mathbf{Z}_+^n)$):

$$q_{\mathbf{ij}} = \begin{cases} h_{\mathbf{j}}, & \text{if } |\mathbf{i}| = 0 \\ \sum_{k=1}^n i_k b^{(k)}_{\mathbf{j}-\mathbf{i}+\mathbf{e}_k} + a_{\mathbf{j}-\mathbf{i}}, & \text{if } |\mathbf{i}| > 0 \\ 0, & \text{otherwise} \end{cases} \quad (1.1)$$

where

$$\begin{cases} h_{\mathbf{j}} \geq 0 (\mathbf{j} \neq \mathbf{0}), 0 < \sum_{\mathbf{j} \neq \mathbf{0}} h_{\mathbf{j}} = -h_{\mathbf{0}} < \infty; \\ a_{\mathbf{j}} \geq 0 (\mathbf{j} \neq \mathbf{0}), 0 < \sum_{\mathbf{j} \neq \mathbf{0}} a_{\mathbf{j}} = -a_{\mathbf{0}} < \infty \\ b^{(k)}_{\mathbf{j}} \geq 0 \ (\mathbf{j} \neq \mathbf{e}_k), \ 0 < \sum_{\mathbf{j} \neq \mathbf{e}_k} b^{(k)}_{\mathbf{j}} = -b^{(k)}_{\mathbf{e}_k} < \infty, \ k = 1, \cdots, n. \end{cases} \quad (1.2)$$

Background

- Problems:

(1) Case $h_{\mathbf{0}} = 0$: extinction probability $a_{\mathbf{i0}} = ?$

(2) Case $h_{\mathbf{0}} \neq 0$: recurrence and ergodicity criteria?

- Special cases:

(i) nTBP: $h_{\mathbf{0}} = a_{\mathbf{0}} = 0$

Extinction property: well-known.

Decay property: Li (2009, Sciene in China A).

(ii) Case $n = 1$:

$h_0 \neq 0, a_0 = 0$, Yamazato (1975)

$h_0 \neq 0, a_0 \neq 0$, Li and Chen (2006, Markov Proc. Relat. Fields), Chen and Li (2008, Sciene in China A).

Background

Definition 1. An n-type branching process with immigration (nTBIP) is a continuous-time Markov chain with state space $\mathbf{Z}_+^n$, whose transition function $P(t) = (p_{\mathbf{ij}}(t); \mathbf{i}, \mathbf{j} \in \mathbf{Z}_+^n)$ satisfies Kolmogorov forward equation

$$P'(t) = P(t)Q \tag{1.3}$$

where Q is a nTBI q-matrix as given in (1.1)-(1.2).

Preliminary

We define

$$H(u_1,\cdots,u_n)=\sum_{j_1=0}^{\infty}\cdots\sum_{j_n=0}^{\infty}h_{j_1,\cdots,j_n}u_1^{j_1}\cdots u_n^{j_n};$$

$$A(u_1,\cdots,u_n)=\sum_{j_1=0}^{\infty}\cdots\sum_{j_n=0}^{\infty}a_{j_1,\cdots,j_n}u_1^{j_1}\cdots u_n^{j_n};$$

$$B_i(u_1,\cdots,u_n)=\sum_{j_1=0}^{\infty}\cdots\sum_{j_n=0}^{\infty}b^{(i)}_{j_1,\cdots,j_n}u_1^{j_1}\cdots u_n^{j_n},\quad i=1,\cdots,n.$$

and let

$$H_j=\frac{\partial H}{\partial u_j},\quad A_j=\frac{\partial A}{\partial u_j},\quad j=1,\cdots,n.$$

$$B_{ij}=\frac{\partial B_i}{\partial u_j},\quad g_{ij}=\delta_{ij}-\frac{B_{ij}}{b^{(i)}_{\mathbf{e}_i}},\quad i,j=1,\cdots,n.$$

Preliminary

Assumptions:

(**A**-1). $\{B_i(u_1, \cdots, u_n); 1 \leq i \leq n\}$ is nonsingular;

(**A**-2). $B_{ij}(1, \cdots, 1) < +\infty, \ i, j = 1, \cdots, n$;

(**A**-3). $G(1, \cdots, 1) = (g_{ij}(1, \cdots, 1))$ is positively regular.

Preliminary

Lemma 1. Suppose $G(1, \cdots, 1)$ is positively regular and $\{B_i(u_1, \cdots, u_n); 1 \leq i \leq n\}$ is nonsingular. Then the equation

$$\begin{cases} B_1(u_1, \cdots, u_n) = 0; \\ B_2(u_1, \cdots, u_n) = 0; \\ \quad \cdots \\ B_n(u_1, \cdots, u_n) = 0. \end{cases} \tag{2.1}$$

has at most two solutions in $[0, 1]^n$. Let $\mathbf{q} = (q_1, \cdots, q_n)$ and $\rho(u_1, \cdots, u_n)$ denote the smallest nonnegative solution to (2.1) and the maximal eigenvalues of $B(u_1, \cdots, u_n)$, respectively. Then,
(i) if $\rho(1, \cdots, 1) \leq 0$, then $\mathbf{q} = \mathbf{1}$; while if $\rho(1, \cdots, 1) > 0$, then $\mathbf{q} < \mathbf{1}$, i.e., $q_1, \cdots, q_n < 1$.
(ii) $\rho(q_1, \cdots, q_n) \leq 0$.

Preliminary

Lemma 2. Let $P(t) = (p_{\mathbf{ij}}(t))$ be the Feller minimal Q-function. Then for any $\mathbf{i} \in \mathbf{Z}_+^n$ and $(u_1, \cdots, u_n) \in [0,1)^n$,

$$\begin{aligned} & \frac{\partial F_{\mathbf{i}}(t, u_1, \cdots, u_n)}{\partial t} \\ = \quad & H(u_1, \cdots, u_n) p_{\mathbf{i0}}(t) + A(u_1, \cdots, u_n) \sum_{\mathbf{j} \in \mathbf{Z}_+^n \setminus \mathbf{0}} p_{\mathbf{ij}}(t) u_1^{j_1} \cdots u_n^{j_n} \\ & + \sum_{k=1}^{n} B_k(u_1, \cdots, u_n) \frac{\partial F_{\mathbf{i}}(t, u_1, \cdots, u_n)}{\partial u_k} \end{aligned}$$

where $F_{\mathbf{i}}(t, u_1, \cdots, u_n) = \sum_{\mathbf{j} \in \mathbf{Z}_+^n} p_{\mathbf{ij}}(t) u_1^{j_1} \cdots u_n^{j_n}$.

Preliminary

Lemma 3. If $\rho(1, \cdots, 1) \leq 0$, then the Q-function is honest.

Theorem 1. Let Q be a nTBI q-matrix defined as (1)–(1). Then there exists exactly one nTBIP, i.e., the Feller minimal process.

Conclusions

- Extinction Property

Theorem 2. Suppose that $G(1,\cdots,1)$ is positively regular, $\{B_i(u_1,\cdots,u_n); 1\le i\le n\}$ is nonsingular. If $B_1(0,\cdots,0)>0$, then the system of equations

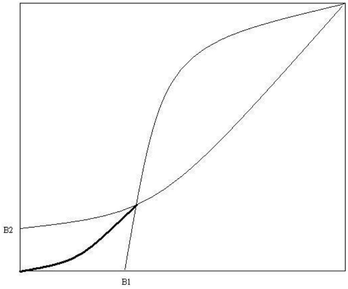
$$\begin{cases} u_k'(u)=\frac{B_k(u,u_2,\cdots,u_n)}{B_1(u,u_2,\cdots,u_n)}, & 2\le k\le n \\ u_k|_{u=0}=0, & 2\le k\le n \end{cases} \tag{3.1}$$

has a unique solution $(u_k(u); 2\le k\le n)$. Furthermore, this solution satisfies

(i) $(u_k(u); 2\le k\le n)$ is well defined on $[0,q_1]$;

(ii) $u_k'(0)\ge 0$ and $u_k'(u)>0$ for all $u\in(0,q_1)$ and $2\le k\le n$;

(iii) $u_k(q_1)=q_k,\quad 2\le k\le n$.


B2
B1

Conclusions

Sketch of the proof. (i) $B_1(u, 0, \cdots, 0) = 0$ has a positive root $u^* \in (0, 1]$. For any $\varepsilon > 0$, $\{\frac{B_k(u,u_2,\cdots,u_n)}{B_1(u,u_2,\cdots,u_n)};\ 2 \le k \le n\}$ satisfy Lipschitz condition on $[0, u^* - \varepsilon] \times [0, 1]^{n-1}$, therefore, (3.1) has a unique solution $(u_k(u); 2 \le k \le n)$ defined on $[0, u^* - \varepsilon]$. Furthermore, (3.1) has a unique solution $(u_k(u); 2 \le k \le n)$ defined on $[0, u^*)$.

(ii) $u_k'(u) \ge 0$ $(2 \le k \le n)$ for all $u \in [0, u^*)$.

(iii) Further, $u_k'(u) > 0$ $(2 \le k \le n)$ for all $u \in (0, u^*]$.

(iv) The solution of (5) can be uniquely extended to $[0, q_1)$.

(v) Finally, $u_k(q_1) = \lim_{u \uparrow q_1} u_k(u) = q_k, \quad k \ge 2$.

Conclusions

Corollary 1. Suppose that $G(1, \cdots, 1)$ is positively regular, $\{B_i(u_1, \cdots, u_n); 1 \leq i \leq n\}$ is nonsingular. If $B_1(0, \cdots, 0) > 0,\ B_2(0, \cdots, 0) > 0$, then the system of equations

$$\begin{cases} u_k'(u) = \frac{B_k(u_1, u, \cdots, u_n)}{B_2(u_1, u, \cdots, u_n)}, & k \neq 2 \\ u_k|_{u=0} = 0, & k \neq 2 \end{cases} \tag{3.2}$$

has the same solution as (3.1).

Conclusions

Theorem 3. For any $\mathbf{i} \neq \mathbf{0}$, $a_{\mathbf{i}\mathbf{0}} = 1$ if and only if $\rho(1, \cdots, 1) \leq 0$ and $J = +\infty$ where

$$J := \int_0^1 \frac{1}{B_1(y, u_2(y), \cdots, u_n(y))} \cdot e^{\int_0^y \frac{A(x,u_2(x),\cdots,u_n(x))}{B_1(x,u_2(x),\cdots,u_n(x))} dx} dy. \quad (3.3)$$

More specifically,

(i) If $\rho(1, \cdots, 1) \leq 0$ and $J = +\infty$, then $a_{\mathbf{i}\mathbf{0}} = 1 (\mathbf{i} \neq \mathbf{0})$.

(ii) If $\rho(1, \cdots, 1) \leq 0$ and $J < +\infty$, then

$$a_{\mathbf{i}\mathbf{0}} = \frac{\int_0^1 \frac{y^{i_1}[u_2(y)]^{i_2}\cdots[u_n(y)]^{i_n}}{B_1(y,u_2(y),\cdots,u_n(y))} \cdot e^{\int_0^y \frac{A(x,u_2(x),\cdots,u_n(x))}{B_1(x,u_2(x),\cdots,u_n(x))} dx} dy}{\int_0^1 \frac{1}{B_1(y,u_2(y),\cdots,u_n(y))} \cdot e^{\int_0^y \frac{A(x,u_2(x),\cdots,u_n(x))}{B_1(x,u_2(x),\cdots,u_n(x))} dx} dy} < 1 \quad (3.4)$$

(iii) If $0 < \rho(1, \cdots, 1) \leq +\infty$,then for $\mathbf{i} \neq \mathbf{0}$,

$$a_{\mathbf{i}\mathbf{0}} = \frac{\int_0^{q_1} \frac{y^{i_1}u_2(y)^{i_2}\cdots u_n(y)^{i_n}}{B_1(y,u_2(y),\cdots,u_n(y))} \cdot e^{\int_0^y \frac{A(x,u_2(x),\cdots,u_n(x))}{B_1(x,u_2(x),\cdots,u_n(x))} dx} dy}{\int_0^{q_1} \frac{1}{B_1(y,u_2(y),\cdots,u_n(y))} \cdot e^{\int_0^y \frac{A(x,u_2(x),\cdots,u_n(x))}{B_1(x,u_2(x),\cdots,u_n(x))} dx} dy} < \prod_{k=1}^n q_k^{i_k} < 1.$$

Conclusions

Sketch of the proof. (a) By Theorem 2 and Lemma 2 (with $h_{\mathbf{0}} = 0$), we have for any $u \in [0, 1)$ and $\mathbf{i} \neq \mathbf{0}$,

$$\begin{aligned} & a_{\mathbf{i0}} - u^{i_1} u_2(u)^{i_2} \cdots u_n(u)^{i_n} \\ = \; & B_1(u, u_2(u), \cdots, u_n(u)) \cdot G'_{\mathbf{i}}(u) \\ & + A(u, u_2(u), \cdots, u_n(u)) \cdot G_{\mathbf{i}}(u) \end{aligned} \tag{3.5}$$

where
$G_{\mathbf{i}}(u) = \sum_{\mathbf{k} \neq \mathbf{0}} (\int_0^\infty p_{\mathbf{ik}}(t)dt) \cdot u^{k_1} [u_2(u)]^{k_2} \cdots [u_n(u)]^{k_n} < +\infty$.

Conclusions

(b) First consider the case $\rho(1,\cdots,1)\le 0$. Solving (3.5) for $u\in[0,1)$ immediately yields

$$G_{\mathbf{i}}(u)\cdot e^{\int_0^u \frac{A(x,u_2(x),\cdots,u_n(x))}{B_1(x,u_2(x),\cdots,u_n(x))}dx} = \int_0^u \frac{a_{\mathbf{i}\mathbf{0}}-y^{i_1}u_2(y)^{i_2}\cdots u_n(y)^{i_n}}{B_1(y,u_2(y),\cdots,u_n(y))}\cdot e^{\int_0^y \frac{A(x,u_2(x),\cdots,u_n(x))}{B_1(x,u_2(x),\cdots,u_n(x))}dx}dy$$

which implies that if $J=+\infty$, then $a_{\mathbf{i}\mathbf{0}}=1$.

Conclusions

(c) For (ii), we can prove that (3.5) is the minimal solution of the equation

$$\sum_{\mathbf{j}\neq\mathbf{0}} q_{\mathbf{ij}}x_{\mathbf{j}}^{*} + q_{\mathbf{i0}} = 0, \quad 0 \leq x_{\mathbf{j}}^{*} \leq 1, \mathbf{i} \neq \mathbf{0}$$

and hence (3.5) is the extinction probability.

(d) (iii) is similar as (ii). □

Conclusions

Theorem 4. Suppose that $\rho(1, \cdots, 1) \leq 0$ and $J = \infty$ where J is given in (3.3) and thus the extinction probability $a_{\mathbf{i}\mathbf{0}} = 1(\mathbf{i} \neq \mathbf{0})$. Then for any $\mathbf{i} \neq \mathbf{0}$, $E_{\mathbf{i}}[\tau_0] < \infty$ if and only if

$$\int_0^1 \frac{1 - yu_2(y)\cdots u_n(y) - A(y, u_2(y), \cdots, u_n(y))}{B_1(y, u_2(y), \cdots, u_n(y))} dy < \infty \quad (3.6)$$

and in which case, $E_{\mathbf{i}}[\tau_0]$ is given by

$$E_{\mathbf{i}}[\tau_0] = \int_0^1 \frac{1 - y^{i_1} u_2(y)^{i_2} \cdots u_n(y)^{i_n}}{B_1(y, u_2(y), \cdots, u_n(y))} \cdot e^{-\int_y^1 \frac{A(x, u_2(x), \cdots, u_n(x))}{B_1(x, u_2(x), \cdots, u_n(x))} dx} dy. \quad (3.7)$$

Conclusions

- Recurrence Property

Theorem 5. The nTBIP is recurrent if and only if $\rho(1, \cdots, 1) \leq 0$ and $J = +\infty$, where J is given in (3.3).

Theorem 6. The nTBIP is positive recurrent (i.e., ergodic) if and only if $\rho(1, \cdots, 1) \leq 0$ and

$$\int_0^1 \frac{-A(y, u_2(y), \cdots, u_n(y)) - H(y, u_2(y), \cdots, u_n(y))}{B_1(y, u_2(y), \cdots, u_n(y))} dy < \infty. \quad (3.8)$$

Moreover, if $\rho(1, \cdots, 1) < 0$ and $\sum_{j=1}^{n} [A_j(1, \cdots, 1) + H_j(1, \cdots, 1)] < \infty$, then the process is exponentially ergodic.

Conclusions

Theorem 7. Suppose that the nTBIP is positive recurrent. Then its equilibrium distribution $(\pi_{\mathbf{j}}; \mathbf{j} \in \mathbf{Z}_+^n)$ is given by

$$\pi(s) = \pi_{\mathbf{0}}[1 + \int_0^s \frac{-H(y, u_2(y), \cdots, u_n(y))}{B_1(y, u_2(y), \cdots, u_n(y))} \cdot e^{-\int_y^s \frac{A(x, u_2(x), \cdots, u_n(x))}{B_1(x, u_2(x), \cdots, u_n(x))} dx} dy]$$

where $\pi(s) = \sum_{\mathbf{j} \in \mathbf{Z}_+^n} \pi_{\mathbf{j}} s^{j_1} u_2(s)^{j_2} \cdots u_n(s)^{j_n}$.

By Theorem 4, we have

Theorem 8. The nTBIP is never strongly ergodic.

Conclusions

- Branching Property ($h_{\mathbf{j}} = a_{\mathbf{j}}$)

Theorem 9. Let $P(t) = (p_{\mathbf{ij}}(t); \mathbf{i}, \mathbf{j} \in \mathbf{Z}_+^n)$ be a transition function. Then the following statements are equivalent.
(i) $P(t)$ is the Feller minimal Q-function, where Q takes the form of (1.1)-(1.2) with $h_{\mathbf{j}} = a_{\mathbf{j}}$.
(ii) For any $\mathbf{i} \in \mathbf{Z}_+^n$, $t \geq 0$, $\mathbf{s} \in [-1, 1]^n$, we have

$$F_{\mathbf{i}}(t, \mathbf{s}) = F_{\mathbf{0}}(t, \mathbf{s}) \cdot \prod_{k=1}^{n} \Big(\sum_{\mathbf{j} \in \mathbf{Z}_+^n} \tilde{p}_{\mathbf{e}_k \mathbf{j}}(t) \mathbf{s}^{\mathbf{j}} \Big)^{i_k} \qquad (3.10)$$

where $F_{\mathbf{i}}(t, \mathbf{s}) = \sum_{\mathbf{j} \in \mathbf{Z}_+^n} p_{\mathbf{ij}}(t) \mathbf{s}^{\mathbf{j}}$ $(\mathbf{i} \in \mathbf{Z}_+^n, \mathbf{s} \in [-1, 1]^n)$ and $(\tilde{p}_{\mathbf{e}_i \mathbf{j}}(t); \mathbf{j} \in \mathbf{Z}_+^n)$ is the Feller minimal $\tilde{Q}$-function and $\tilde{Q}$ is an n-type ordinary branching q-matrix (but may not be conservative).

Conclusions

(iii) For any $\mathbf{i} \in \mathbf{Z}_+^n$, $t \geq 0$, $\mathbf{s} \in [-1,1]^n$, we have

$$F_{\mathbf{i}}(t,\mathbf{s}) = F_{\mathbf{0}}(t,\mathbf{s}) \cdot \prod_{k=1}^{n} (F_{\mathbf{e}_k}(t,\mathbf{s})/F_{\mathbf{0}}(t,\mathbf{s}))^{i_k}. \tag{3.11}$$

In particular,

$$p_{\mathbf{i}\mathbf{0}}(t) = p_{\mathbf{0}\mathbf{0}}(t) \cdot \prod_{k=1}^{n} (p_{\mathbf{e}_k\mathbf{0}}(t)/p_{\mathbf{0}\mathbf{0}}(t))^{i_k}, \quad |\mathbf{i}| \geq 1 \tag{3.12}$$

Conclusions

- Decay Property $(h_{\mathbf{j}} = a_{\mathbf{j}})$

Theorem 10.

$$\lambda_Z = -A(q_1, \cdots, q_n),$$

where $(q_1, \cdots, q_n)$ is the minimal nonnegative solution of (2.1).

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