Heisenberg Inequality in Infinite Dimensions

Yuh-Jia Lee[1]
National University of Kaohsiung
Kaohsiung, TAIWAN 811

July 6, 2013

[1]Talk given in The 9th Workshop on Markov Processes and Related Topics, July 6-14, 2013, Ermei Campus of Southwest Jiaotong University

Portrait of Werner Heisenberg

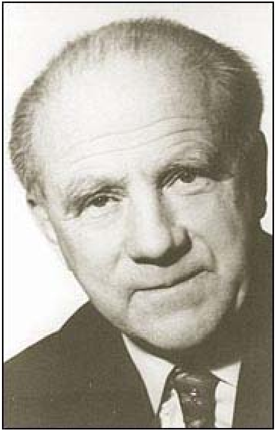

Figure : Werner Heisenberg

Heisenberg as a founder of quantum mechanics

WERNER HEISENBERG (1901 - 1976) was one of the greatest physicists of the twentieth century. He is best known as a founder of quantum mechanics, the new physics of the atomic world, and especially for the uncertainty principle in quantum theory. He is also known for his controversial role as a leader of Germany's nuclear fission research during World War II. After the war he was active in elementary particle physics and West German science policy.

The uncertainty principle

A quantum mechanical principle due to Werner Heisenberg (1927) states that it is not possible to simultaneously determine the position and momentum of a particle. It shows that the more precisely the POSITION is determined, the less precisely the MOMENTUM is known (and vice versa). The principle is sometimes known as the Heisenberg uncertainty principle, and can be stated exactly as

$$\Delta x \Delta p \geq \frac{1}{2}\hbar,$$

where x is the uncertain position and p is the uncertain momentum and $h = 2\pi\hbar$ is Planck's constant.

Basic Notations(1)

$$\mu : mass$$

$$\omega : \text{ frequence}$$

$$\hbar : \text{ reduced Planck's constant}$$

$$h = 2\pi\hbar \text{ Planck's constant}$$

$$P = \frac{\hbar}{i}\frac{\partial}{\partial x}, \text{ the observable momentum (operator)}$$

$$Q = x \cdot \quad (Q\varphi = x\varphi), \text{ observable position (operator)}$$

$$A = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\mu\omega}{\hbar}}Q + \frac{i}{\sqrt{\mu\omega\hbar}}P\right)$$

$$A^* = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\mu\omega}{\hbar}}Q - \frac{i}{\sqrt{\mu\omega\hbar}}P\right)$$

The number operator and the observable energy operator

$$H = \frac{1}{2\mu} P^2 + \frac{\mu\omega^2}{2} Q^2, \qquad \text{the energy operator}$$

$$N = A\,A^* = \frac{1}{\omega\hbar} H + \frac{1}{2} I, \quad \text{the number operator}$$

CCR relations

- $[P, Q] = PQ - QP = \frac{\hbar}{i} I$
- $[A, A^*] = I$

A as an annihilation operator and A^* as a creation operator

Let φ_λ be the eigenvector of N with eigenvalue λ. Then we have

$$\begin{aligned} N(A\varphi_\lambda) &= (A^*A)A\varphi_\lambda \\ &= (AA^* - I)A\varphi_\lambda \\ &= A(A^*A - I)\varphi_\lambda \\ &= A(N - I)\varphi_\lambda \\ &= (\lambda - 1)A\varphi_\lambda. \end{aligned}$$

Thus either $A\varphi_\lambda = 0$ or $A\varphi_\lambda$ is an eigenvector of N with eigenvalue $\lambda - 1$.

Suppose the latter is the case. We define

$$\varphi_{\lambda-1} := A\varphi_\lambda.$$

Then we find that

$$\varphi_{\lambda-2} := A^2\varphi_\lambda$$

is either 0 or an eigenvector with eigenvalue $\lambda - 2$. Continue in this way we obtain a sequence of vectors

$$\varphi_{\lambda-m} := A^m\varphi_\lambda \quad (m = 0, 1, 2, \dots),$$

which are eigenvectors with eigenvalue $(\lambda - m)$ as long as $\varphi_{\lambda-m} \neq 0$.

In a similar way we get

$$N(A^*\varphi) = (\lambda + 1)(A^*\varphi).$$

Hence $\varphi_{\lambda+1} := A^*\varphi_\lambda$ is either 0 or an eigenvector of N with eigenvalue $\lambda + 1$.
It can be shown that $\varphi_{\lambda+1} \neq 0$. In fact if $\|A^*\varphi_\lambda\| = 0$, we have

$$\begin{aligned}
\|A^*\varphi_\lambda\|^2 &= \langle A^*\varphi_\lambda, A^*\varphi_\lambda\rangle \\
&= \langle \varphi_\lambda, AA^*\varphi_\lambda\rangle \\
&= \langle \varphi_\lambda, A^*A\varphi_\lambda\rangle + \langle \varphi_\lambda, \varphi_\lambda\rangle \\
&= \|A\varphi_\lambda\|^2 + \|\varphi_\lambda\|^2 \\
&\neq 0,
\end{aligned}$$

since $\varphi_\lambda \neq 0$.

Thus $\varphi_{\lambda+1}$ is always an eigenvector of N with eigenvalue $\lambda + 1$.
Again repeating the above process one obtains a sequence of vectors $\varphi_{\lambda+n} \quad (n = 0, 1, 2, \dots)$ which are eigenvectors of N with eigenvalue $\lambda + n$.
Next we determine when $\varphi_{\lambda-m}$ can be zero. We calculate

$$\langle \varphi_{\lambda-m}, N\varphi_{\lambda-m} \rangle = (\lambda - m)\|\varphi_{\lambda-m}\|^2$$

and

$$\langle \varphi_{\lambda-m}, N\varphi_{\lambda-m} \rangle = \|A\varphi_{\lambda-m}\|^2.$$

It follows that

$$\lambda - m = \frac{\|A\varphi_{\lambda-m}\|^2}{\|\varphi_{\lambda-m}\|^2} \geq 0.$$

The sequence of eigenvectors $\varphi_{\lambda-m}$ must terminate after finite number of steps, and therefore there must exist one vector φ_0 such that

$$A\varphi_0 = 0.$$

φ_0 is an eigenvector of N with eigenvalue zero since

$$N\varphi_0 = A^*A\varphi_0 = A^*0 = 0.$$

Now we define the normalized vectors

$$\begin{aligned}\phi_0 &= \frac{\varphi_0}{\|\varphi_0\|},\\ \phi_1 &= C_1 A^* \phi_0,\\ &\vdots\\ \phi_n &= C_n (A^*)^n \phi_0,\\ &\vdots\end{aligned}$$

where C_n are chosen such that $\|\phi_n\| = 1$.

Thus we have

$$N\phi_n = N(C_n(A^*)^n\phi_0 = n\phi_n, \ \ \|\phi_n\| = 1.$$

The C_n can be calculated as follows: By the definition of ϕ_n, we have

$$\begin{aligned} 1 = \|\phi_n\|^2 &= \langle (A^*)^n\phi_0, (A^*)^n\phi_0\rangle |C_n|^2, \\ &= \frac{|C_n|^2}{|C_{n-1}|^2}\langle \phi_{n-1}, AA^*\phi_{n-1}\rangle \\ &= \frac{|C_n|^2}{|C_{n-1}|^2}\langle \phi_{n-1}, (A^*A+1)\phi_{n-1}\rangle \\ &= n\,\frac{|C_n|^2}{|C_{n-1}|^2}\langle \phi_{n-1}, \phi_{n-1}\rangle \\ &= n\,\frac{|C_n|^2}{|C_{n-1}|^2}; \end{aligned}$$

hence C_n must be chosen so that

$$n\,|C_n|^2 = |C_{n-1}|^2\,,$$

or

$$C_n = \sqrt{\frac{1}{n!}}\,.$$

Summary

Start with a normalized eigenvector having the property:

$$A\phi_0 = 0.$$

We obtain a system of orthonormal system $\{\phi_n\}$ consisting of eigenvectors of N defined by

$$\phi_n = \frac{1}{n!}(A^*)^n \phi_0$$

which satisfy

$$A\phi_n = \sqrt{n}\phi_{n-1}$$
$$A^*\phi_n = \sqrt{n+1}\phi_{n+1}.$$

Concept of quantum probability

Let $\{\omega_n\}$ be a sequence of numbers such that

$$\sum_n \omega_n = 1, \quad 0 < \omega_n < 1.$$

and W a s. a.positive operator with $Tr[W] = 1$ such that

$$W = \sum_n \omega_n \Lambda_n,$$

where Λ_n is a projection onto the eigenspace spanned by ϕ_n. W is referred as the "probability" in the Quantum Probability Theory. Given an operator A, the expectation or average of A w.r.t. W is defined by

$$\langle A \rangle := trace[AW].$$

From the definition of A and A^*, we have

$$P = \frac{i\sqrt{\mu\omega\hbar}(A^* - A)}{\sqrt{2}}$$

$$Q = \sqrt{\frac{\hbar}{\mu\omega}}\left(\frac{A + A^*}{\sqrt{2}}\right)$$

The Pure sate case: $W = \Lambda_n$

$$\langle P \rangle = 0$$
$$\langle Q \rangle = 0$$
$$\langle P^2 \rangle = \langle \phi_n, P^2 \phi_n \rangle = \mu\omega\hbar(n + \frac{1}{2})$$
$$\langle Q^2 \rangle = \frac{\hbar}{2\mu\omega}(n + \frac{1}{2})$$

Define

$$dispA := \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$$

and define

$$\Delta A := \sqrt{dispA}$$

Then we have

$$\Delta P \cdot \Delta Q = \sqrt{\langle P^2 \rangle}\sqrt{\langle Q^2 \rangle} = \hbar(n + \frac{1}{2}) > \frac{\hbar}{2}.$$

General case: W any state

$$\langle P\rangle = \sum_n \omega_n \langle P\phi_n, \phi_n\rangle = 0$$

$$\langle Q\rangle = 0$$

$$\langle P^2\rangle = \sum_n \mu\omega_n \hbar (n + \frac{1}{2})$$

$$\langle Q^2\rangle = \sum_n \frac{\hbar}{2\mu\omega_n}(n + \frac{1}{2})$$

Then we have

$$\Delta P \cdot \Delta Q > \frac{\hbar}{2}.$$

This proves the HUP in the classical setting.

Let $A \in L(H,H)$ and A^* the adjoint of A. Suppose that the CCR relation holds:

$$[A, A^*] = I$$

Then we have

$$\begin{aligned}
&\langle (A - A^*)(A + A^*)\rangle\varphi, \varphi\rangle \\
&= \langle (A + A^*))\varphi, (A - A^*)\varphi\rangle \\
&= \langle (AA^* - A^*A)\varphi, \varphi\rangle \\
&= \langle [A, A^*]\varphi, \varphi\rangle \\
&= \|\varphi\|^2
\end{aligned}$$

It follows that we have

$$\|(A^* - A)\varphi\| \|(A + A^*)\varphi\| \geq \|\varphi\|^2.$$

Example 1

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dx)$. Let

$$A = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\mu\omega}{\hbar}}Q + \frac{i}{\sqrt{\mu\omega\hbar}}P\right)$$
$$A^* = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\mu\omega}{\hbar}}Q - \frac{i}{\sqrt{\mu\omega\hbar}}P\right)$$

as given before. Then $A + A^*$ and $A - A^*$ can be represented by

$$A + A^* = \sqrt{\frac{2\mu\omega}{\hbar}}Q$$
$$A - A^* = i\sqrt{\frac{2}{\mu\omega\hbar}}P$$

Then we have

$$\begin{aligned}\|\varphi\|^4 &\leq \frac{4}{\hbar^2}\|P\varphi\|^2\|Q\varphi\|^2 \\ &\leq \hbar^2 \left(\int_{\mathbb{R}} \varphi'(x)^2\,dx\right)\left(\int_{\mathbb{R}} (x\varphi(x))^2\,dx\right) \\ &= 4\left(\int_{\mathbb{R}} (x\hat{\varphi}(x))^2\,dx\right)\left(\int_{\mathbb{R}} (x\varphi(x))^2\,dx\right).\end{aligned}$$

Finally the HUP becomes

$$\left(\int_{\mathbb{R}} (x\hat{\varphi}(x))^2\,dx\right)\left(\int_{\mathbb{R}} (x\varphi(x))^2\,dx\right) \geq \frac{1}{4}\|\varphi\|^4.$$

Example 2: Gaussian case

Let $H = L^2(\mathbb{R}, \mu)$, where μ is the standard Gaussian measure. Define $A\varphi = \varphi'$ for sufficient smooth φ. Then we have

$$\langle x\,\varphi, \psi\rangle = \langle A\varphi, \psi\rangle + \langle \varphi, A\psi\rangle$$

so that

$$\langle x\,\varphi, \psi\rangle = \langle (A + A^*)\varphi, \psi\rangle,$$

where

$$A^*\varphi = x\,\varphi - \varphi'.$$

It is easy to see that $[A, A^*] = I$. Then the HUP read

$$\left(\int_{\mathbb{R}} |x\,\varphi(x) - 2\varphi'(x)|^2 \mu(dx)\right)\left(\int_{\mathbb{R}} |x\,\varphi(x)|^2 \mu(dx)\right) \geq \|\varphi\|^4.$$

Next we define the Fourier Wiener transform by

$$\mathcal{F}\varphi(y) = \int_{\mathbb{R}} \varphi(\sqrt{2}x + iy)\mu(dy).$$

Then we have

$$\mathcal{F}[x\varphi - 2\varphi'](y) = iy \cdot \mathcal{F}(y)$$

and

$$\begin{aligned} \int_{\mathbb{R}} |x\,\varphi(x) - 2\varphi'(x)|^2 \mu(dx) &= \int_{\mathbb{R}} |\mathcal{F}[x\,\varphi - 2\varphi'](y)|^2 \mu(dy) \\ &= \int_{\mathbb{R}} |y\,\mathcal{F}\varphi(y)|^2 \mu(dy). \end{aligned}$$

The HUP now becomes

$$\|\varphi\|^4 \leq \left(\int_{\mathbb{R}} |x\,\mathcal{F}\varphi(x)|^2 \mu(dx)\right)\left(\int_{\mathbb{R}} |x\,\varphi(x)|^2 \mu(dx)\right)$$

It is not hard to verify that on $L^(\mathbb{R}^d, \mu)$ the HUP read

$$\|\varphi\|^4 \leq \left(\int_{\mathbb{R}^d} |\langle x\,,\mathcal{F}\varphi(x)\rangle|^2 \mu(dx)\right)\left(\int_{\mathbb{R}^d} |\langle x\,,\varphi(x)\rangle|^2 \mu(dx)\right)$$

It can be show that the equality holds iff φ is of the form

$$\varphi(x) = e^{\frac{\alpha}{2}} \langle x, u_\eta \rangle^2 \varphi(P_\eta^\perp x)$$

for any real number α such that $|\alpha| < 1$, where η is a non-zero vector and u_η is the normalized vector of η and P_η is the projection P_η and $P_\eta^\perp = I - P_\eta$.

HUP on infinite dimensional spaces

Let (H, B) be an abstract Wiener pair and p_1 the abstract Wiener measure with variance parameter 1. Then the HUP read

$$\|\varphi\|^4 \leq \left(\int_B |\langle x\,, \mathcal{F}\varphi(x)\rangle|^2\, p_1(dx)\right)\left(\int_B |\langle x\,, \varphi(x)\rangle|^2\, p_1(dx)\right)$$

It can be show that the equality holds iff φ is of the form

$$\varphi(x) = e^{\frac{\alpha}{2}\langle x, u_\eta\rangle^2}\varphi(P_\eta^\perp x)$$

for any real number α such that $|\alpha| < 1$, where η is a non-zero vector and u_η is the normalized vector of η and P_η is the projection P_η and $P_\eta^\perp = I - P_\eta$ dimensional space remain true, we refer the reader to [2].

General Form of HUP

Let (H, B) be an abstract Wiener space with abstract Wiener measure p_1. For $\varphi \in L^2(p_1)$ and $T \in \mathcal{L}(B, H)$, we have

$$\left[\int_B |Tx|_H^2 |\varphi(x)|^2 p_1(dx)\right] \left[\int_B |Tx|_H^2 |\mathcal{F}\varphi(x)|^2 p_1(dx)\right] \geq \| T|_H \|_{HS}^4 \|\varphi\|_2^4.$$

Conditions on the HUP equality

Let $T \in \mathcal{L}(B, H)$ and write $T^*Tx = \widetilde{T}^*Tx = \sum_{j=1}^{r} {\lambda_j}^2 (x, e_j)e_j$, where r is the rank of T and $\{e_j : j = 1, \ldots, r\} \subset B^*$ is the orthonormal set consisting of eigenvalues of T^*T. Denote by $P_{_T}^{\perp}$ the projection of B onto the closure of the subspace of H spanned by $\{e_j : j = 1, \ldots, r\}$ and $P_{_T}^{\perp} = I - P_{_T}$. Then the equality holds iff $r < \infty$ and f is of the form

$$f(x) \;=\; f(P_{_T}^{\perp} x) \exp\Big\{ \frac{\alpha}{2} |P_{_T} x|^2 \Big\}, \tag{3.1}$$

where $|\alpha| < \frac{1}{2}$.

Non-Gaussian Cases

Let f be an analytic functional.

(1) In the Gaussian white noise case,

$$\partial_t f(x) = Df(x)\,\delta_t.$$

(2) In the Poisson white noise case,

$$\partial_t f(x) = f(x+\delta_t) - f(x).$$

(3) In the Gamma white noise case,

$$\partial_t f(x) = \int_0^\infty \left(f(x+u\,\delta_t) - f(x)\right) e^{-u}\,du.$$

Classical Probability	Algebraic Probability	Remark
Probability measure P	ρ (state)	$\rho(\varphi) = \int_\Omega \varphi(\omega)P(d\omega)$
Sample space $(\Omega, \mathcal{F}, P)$	$\mathcal{A} = L^\infty(\Omega, P)$	$(\mathcal{A}, \rho)$ (a $*$-algebra)
Event $E \in \mathcal{F}$	$p = \mathbf{1}_E \in \mathcal{A}$	$p = p^* = p^2$
Random variable $X : \Omega \to \mathbb{R}$	$a = X \in L^\infty(\Omega, P)$	$a = a^* \in \mathcal{A}$
Expectation $\mathbb{E}[X]$	$\rho(a)$	
Probability $P(E)$	$\rho(p) = \mathbb{E}[\mathbf{1}_E]$	
$\mathbb{E}[X^m]$	$\rho(a^m)$	
Distribution μ_X	μ_a	$\mu_a(a^m) = \rho(a^m)$

Let$\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$, $\mathcal{O}(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ denote the spaces of bounded operator $\mathcal{H}$, self-adjoint operator and orthogonal operators, respectively. $\mathcal{L}(\mathcal{H})$, $\mathcal{O}(\mathcal{H})$, $\mathcal{P}(\mathcal{H})$ are all $*$-algebra.
Let ρ be a positive operator such that $trace(\rho) = 1$, we call ρ a state. An one dimensional state is also called a pure state. Then $(\mathcal{H}, \mathcal{P}(\mathcal{H}), \rho)$ is called a quantum probability space.

Classical Probability	Quantum Probability
Probability measure(P)	Positive operator ρ on $\mathcal{H}$
Event(E)	$E \in \mathcal{P}(\mathcal{H})$ $(0 \leq E \leq 1)$
Impossible event(ϕ)	$0 \in \mathcal{P}(\mathcal{H})$
Certain event(Ω)	$1 \in \mathcal{P}(\mathcal{H})$
$E_1 \subset E_2$	$E_1 \leq E_2$
E^c	$1 - E$
$\bigcup E_i$	$\bigvee E_i$
$\bigcap E_i$	$\bigwedge E_i$
$E_1 \cap E_2 = \phi$	$E_1 E_2 = 0$
$P(E)$	trace$[\rho E]$
Random Variable X	$X \in \mathcal{O}(\mathcal{H})$ (observables)
$X = \sum_j x_j \mathbf{1}_{E_j}$	$X = \sum_j x_j E_j^X$ (x_j distinct, $E_j^X \in \mathcal{O}(\mathcal{H})$ and $E_j^X(h) = x_j h$)
$f(X) = \sum_j f(x_j) \mathbf{1}_{E_j}$	$f(X) = \sum_j f(x_j) E_j^X$

References

1. Arno BÖhm, *Quantum Mechanics*, Springer-Verlag (1979)

2. Yuh-Jia Lee and Aurel Stan, An Infinite-Dimensional Heisenberg Uncertainty Principle, Taiwanese J. Math. 3(1999), 529-538.

3. J. Von Neumann, *Mathmatical Foundation of Quantum Mechanics*, Princeton University Press, 1955