A multiparameter Garsia-Rodemich-Rumsey inequality and some applications

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Outline

1. Classical Garsia-Rodemich-Rumsey inequality
2. Some relevant two parameter inequalities
3. Main Result
4. Applications

1 Garsia-Rodemich-Rumsey inequality

Let the function $\Psi : [0, \infty) \to [0, \infty)$ be non decreasing with $\lim_{u \to \infty} \Psi(u) = \infty$ and let the function $p : [0, 1] \to [0, 1]$ be continuous and non decreasing with $p(0) = 0$. Set

$$\Psi^{-1}(u) = \sup_{\Psi(v) \leq u} v \qquad \text{if } \ \Psi(0) \leq u < \infty$$

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Let f be a continuous function on $[0,1]$ and suppose that

$$\int_0^1 \int_0^1 \Psi\left(\frac{|f(x)-f(y)|}{p(x-y)}\right) dxdy \le B < \infty\,.$$

Then for all $x, \ y \in [0, 1]$ we have

$$|f(y) - f(x)| \leq 8 \int_0^{|y-x|} \Psi^{-1} \left(\frac{4B}{u^2} \right) dp(u). \tag{1}$$

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For example if $\Psi(u) = |u|^p$ and $p(u) = |u|^{\alpha+1/p}$, where $p\alpha > 1$, the inequality (1) implies the following Sobolev imbedding inequality

$$|f(s) - f(t)| \le C_{\alpha,p}|t-s|^{\alpha-1/p}\left(\int_0^1\int_0^1 \frac{|f(x)-f(y)|^p}{|x-y|^{\alpha p+1}}dxdy\right)^{1/p}.$$

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But the parameter space is assumed to have a distance (metric space) and the Garsia-Rodemich-Rumsey lemma is with respect to that distance.

This method immediately yields the following result for a fractional Brownian field $W^H(x)$ of Hurst parameter $H = (H_1, \cdots, H_d)$, then for any β_i with $\beta_i < H_i$, $i = 1, \cdots, d$, one has

$$|W(y) - W(x)| \le L \sum_{i=1}^{d} |y_i - x_i|^{\beta_i} \,, \tag{2}$$

where L is an integrable random variable.

Let $W : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables and x and y are two points in $\mathbb{R}^2$. Consider the increment of W along with the rectangle determined by $x = (x_1, x_2)$ and $y = (y_1, y_2)$:

$$\square W := W(y_1, y_2) - W(x_1, y_2) - W(x_2, y_1) + W(x_1, x_2)\,.$$

If $f(x_1, x_2) = x_1 x_2$, then $\square_y^2 f(x) = (x_1 - y_1)(x_2 - y_2)$, which is the area of the rectangle.

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We are seeking

$$|\Box W| \leq L\,|x_1 - y_1|^{\alpha}\,|x_2 - y_2|^{\beta}\ .$$

A known result

If W is a fractional Brownian field with parameters H_1 and H_2, then for any $\beta_1 < H_1$ and $\beta_2 < H_2$, we have

$$|\square W| \leq L_{\beta_1,\beta_2} |y_1 - x_1|^{\beta_1} |y_2 - x_2|^{\beta_2} .$$

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Ral'chenko, K. V.
The two-parameter Garsia-Rodemich-Rumsey inequality and its application to fractional Brownian fields.
Theory Probab. Math. Statist. No. 75 (2007), 167-178.

3. Main Result

Multiparameter Garsia-Rodemich-Rumsey inequality

Let $f(x)$ be a continuous function on $[0,1]^n$ and suppose that

$$\int_{[0,1]^n}\int_{[0,1]^n} \Psi\left(\frac{|\Box_y^n f(x)|}{\prod_{k=1}^n p_k(|x_k - y_k|)}\right) dxdy \le B < \infty\,.$$

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Then for all $s,t\in[0,1]^n$ we have

$$|\Box_s^n f(t)| \leq 8^n \int_0^{|s_1-t_1|}\cdots\int_0^{|s_n-t_n|}\Psi^{-1}\left(\frac{4^n B}{u_1^2\cdots u_n^2}\right)dp_1(u_1)\cdots dp_n(u_n).$$

Lemma

Let $(\Omega, \mathcal{F})$ be a measurable space and let μ be a positive measure on $(\Omega, \mathcal{F})$. Let $g : \Omega \times [0,1] \to \mathbb{R}^m$ be a measurable function such that

$$\int_0^1 \int_0^1 \int_\Omega \Psi\left(\frac{|g(z,t) - g(z,s)|}{p(|t-s|)}\right) \mu(dz)dsdt \leq B < \infty.$$

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Then there exist two decreasing sequences $\{t_k\,,k=0,1,\cdots\}$ and $\{d_k\,,k=0,1,\cdots\}$ with

$$t_k \le d_{k-1} = p^{-1}\left(\frac{1}{2}p(t_{k-1})\right)\,,\quad k=1,2,\cdots$$

such that the following inequality holds

$$\int_\Omega \Psi\left(\frac{|g(z,t_k)-g(z,t_{k-1})|}{p(|s-t|)}\right)\mu(dz) \le \frac{4B}{d_{k-1}^2}\,.$$

4. Applications

Multiparameter Kolmogorov lemma

Let W be a random field on $\mathbb{R}^n$. Suppose there exist positive constants α, β_k $(1 \leq k \leq n)$ and K such that for every x, y in $[0, 1]^n$,

$$\mathbb{E}\left[\left|\Box_y^n W(x)\right|^\alpha\right] \leq K \prod_{k=1}^n |x_k - y_k|^{1+\beta_k}.$$

Then, for every $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $0 < \epsilon_k \alpha < \beta_k$ $(1 \leq k \leq n)$, there exist a random variable η with $\mathbb{E}\eta^\alpha \leq K$, such that the following inequality holds almost surely

$$|\Box_t^n W(s)| \leq C\eta(\omega) \prod_{k=1}^n |t_k - s_k|^{\beta_k \alpha^{-1} - \epsilon_k}$$

for all s, t in $[0, 1]^n$, where C is a constant defined by

$$C = 8^n 4^{n/\alpha} \prod_{k=1}^n \left(1 + \frac{2}{\beta_k - \alpha\epsilon_k}\right).$$

Fractional Brownian field

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Then, there exist an integrable random constant $C_{n,H_1,\cdots,H_n}$ such that for every x,y in $[0,1]^n$

$$|\Box_y^n W^H(x)| \leq C_{n,H_1,\cdots,H_n} \sqrt{\left|\log\left(\prod_{k=1}^n |x_k-y_k|\right)\right|} \prod_{k=1}^n |x_k-y_k|^{H_k}.$$

Comparison with a work of Ayache and Xiao

Ayache, A. and Xiao, Y. proved
For fractional Brownian field W^H on $\mathbb{R}^n$, there exist a random variable $A_1 = A_1(\omega) > 0$ of finite moments of any order and an event Ω_1^* of probability 1 such that for any $\omega \in \Omega_1^*$,

$$\sup_{s,t\in[0,1]^n} \frac{|W^H(s,\omega) - W^H(t,\omega)|}{\sum_{j=1}^n |s_j - t_j|^{H_j}\sqrt{\log\left(3 + |s_j - t_j|^{-1}\right)}} \leq A_1(\omega)\,.$$

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Asymptotic properties and Hausdorff dimensions of fractional Brownian sheets.
J. Fourier Anal. Appl. 11 (2005), no. 4, 407-439.

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Indeed, our estimate gives the increment along an edge of the n-dimensional rectangle $[s_1, t_1] \times \cdots \times [s_n, t_n]$

$$\begin{aligned} &\left| W^H(s_1, \cdots, s_n) - W^H(s_1, \cdots, s_{n-1}, t_n) \right| \\ &\quad \le A_1(\omega) \left(\prod_{k=1}^{n-1} s_k^{H_k} \right) \left| \log \prod_{k=1}^{n-1} s_k \right|^{1/2} |s_n - t_n|^{H_n} \left| \log |s_n - t_n| \right|^{1/2}. \end{aligned}$$

Similarly, we can obtain analogue estimates along any edge of the n-dimensional rectangle $[s_1, t_1] \times \cdots \times [s_n, t_n]$. The increment along the diagonal is majorized by the total increments along all the edges connecting s and t. Hence, this argument yields the following estimate

$$\left| W^H(s) - W^H(t) \right| \le A_1(\omega) \sum_{k=1}^{n} \left(\prod_{j \neq k} s_j^{H_j} \right) \left| \log \prod_{j \neq k} s_j \right|^{1/2} |s_k - t_k|^{H_k} | \log |s_k - t_k||^{1/2}.$$

stochastic partial differential equation

Consider the following one dimensional stochastic partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \dot{W} & 0 < t \leq T\,,\ y \in \mathbb{R} \\ u(0, y) = 0 & y \in \mathbb{R}\,, \end{cases} \tag{3}$$

where $\Delta u = \frac{\partial^2}{\partial y^2} u$, W is space time standard Brownian sheet, and $\dot{W} = \frac{\partial^2}{\partial t \partial y} W$.

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where $\Delta u = \frac{\partial^2}{\partial y^2} u$, W is space time standard Brownian sheet, and $\dot{W} = \frac{\partial^2}{\partial t \partial y} W$.

It is known that $u(t, y)$ is Hölder continuous of exponent $\frac{1}{4}$- for time parameter and $\frac{1}{2}$- for space parameter. Namely, for any $\alpha < 1/4$ and any $\beta < 1/2$, there is a random constant $C_{\alpha,\beta}$ such that

$$|u(t, y) - u(s, x)| \le C_{\alpha,\beta} \left(|t - s|^{\alpha} + |x - y|^{\beta} \right) .$$

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stochastic partial differential equations

For every α in $[0, 1/4]$, there is an integrable random constant C_α such that for all $(s, x), (t, y)$ in $[0, 1]^2$

$$\begin{aligned} & |u(t,y) - u(t,x) - u(s,y) + u(s,x)| \\ \leq \quad & C_\alpha |t-s|^{1/4-\alpha} |x-y|^{2\alpha} \sqrt{|\log(|t-s||x-y|)|} \end{aligned}$$

Thank You