Derrida's Random Energy Model and Large Deviations

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The Sherrington-Kirkpatrick Model

The Sherrington-Kirkpatrick model is a model for spin glass in physics. Given $n \geq 1$, let

$$\Sigma_n = \{-1, 1\}^n$$

be the configuration space of spin vectors.

For each $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma_n$, the *Hamiltonian* of the SK model is

$$H_n(\sigma) = \sum_{\text{distinct } i,j=1}^{n} g_{ij}\sigma_i\sigma_j$$

where g_{ij}, describing the random long-range interaction, are iid normal random variables with mean zero and variance $\frac{r^2}{n}$. One object of interests is the study of the limit of

$$\max_{\sigma \in \Sigma_n} H_n(\sigma)$$

as n tends to infinity.

More specifically, the limit

$$\lim_{n\to\infty} \frac{1}{n}\mathsf{E}[\max H_n(\sigma)].$$

This is equivalent to the study of

$$\lim_{n\to\infty} \frac{1}{n\beta}\mathsf{E}[\log Z_n(\beta)]$$

where β is the inverse temperature parameter and

$$Z_n(\beta) = \sum_{\sigma\in\mathbf{\Sigma_n}} \exp\{\beta H_n(\sigma)\}$$

is the *partition function*.

In fact,

$$\begin{aligned} \frac{1}{n}\mathsf{E}[\max H_n(\sigma)] &\leq \frac{1}{n\beta}\mathsf{E}[\log Z_n(\beta)] \\ &\leq \frac{\log 2}{\beta} + \frac{1}{n}\mathsf{E}[\max H_n(\sigma)]. \end{aligned}$$

The quantity

$$F_n(\sigma) = \frac{1}{n}\mathsf{E}[\log Z_n(\beta)]$$

is called the *free energy* and the limit $\lim_{n\to\infty} F_n(\beta)$, if exists, is denoted by $F(\beta)$. Clearly

$$\lim_{n\to\infty} \frac{1}{n}\mathsf{E}[\max H_n(\sigma)] = \lim_{\beta\to\infty} \frac{F(\beta)}{\beta}.$$

The free energy is closely related to the following Gibbs measure

$$Q_n(\sigma) = \frac{\exp\{\beta H_n(\sigma)\}}{Z_n(\beta)}$$

in the sense that the dominant support of the Gibbs measure is around the maximum $\max_{\sigma \in \Sigma_n} H_n(\sigma)$ when n is large.

p-Spin Systems

For any $p \geq 2$, let $\{g_{i_1,\cdots,i_p} : 1 \leq i_1, \ldots, i_p \leq n, \text{distinct}\}$ be a family of iid normal random variables with mean zero and variance $\frac{r^2 p!}{2n^{p-1}}$. Then the p-spin system has the following Hamiltonian

$$H_{n,p}(\sigma) = \sum_{\text{distinct } i_1,\cdots,i_p=1}^{n} g_{i_1,\cdots,i_p} \sigma_{i_1} \cdots \sigma_{i_p}$$

Derrida's Random Energy Model

When p tends to infinity, the Hamiltonian of p-spin systems becomes a family of normal randoms variables.

Definition: For any $n \geq 1$, Derrida's *random energy model* is a system having 2^n energy levels $\{E_\sigma : \sigma \in \Sigma_n\}$ such that

(1) for each σ, E_σ is normal with mean zero and variance $\frac{nr^2}{2}$

(2) All energy levels are independent.

The independent property is an additional simplification. Derrida also introduced a generalized random energy model handling dependency.

Below we write N for 2^n so that $n = \frac{\log N}{\log 2}$.

For $\beta = -\frac{1}{T}$, consider the Gibbs measure

$$Z^{-1}\exp\{\eta E_\sigma\},\ Z = \sum_{\sigma\in\Sigma_n}\exp\{\beta E_\sigma\}.$$

It is known that a phase transition in terms of the average number of energy levels occurs at a critical temperature

$$T_c = \sqrt{\frac{2}{\log 2}}\, r.$$

Set $\alpha = \frac{T}{T_c}$. Then the Gibbs measure associated with the random energy model can be represented as

$$\frac{\exp\{-\alpha^{-1}\sqrt{2\log N}X_i\}}{\sum_{j=1}^{N}\exp\{-\alpha^{-1}\sqrt{2\log N}X_j\}}$$

where $X_1, X_2, \ldots$ are iid standard normal random variables.

Question: What happens when N tends to infinity?

Thermodynamic Limit

Let X be a standard normal random variable independent of the iid sequence $X_1, X_2, \ldots$. Consider a_N such that

$$\mathbb{P}\{X > a_N\} = \frac{1}{N}.$$

It is known that

$$\lim_{N\to\infty} = \frac{a_N}{\sqrt{2\log N}} = 1.$$

A classical result states that

$$\mathbb{P}\{\sqrt{2\log N}(\max_{1\le i\le N} X_i - a_N) \le x\} \to e^{-e^{-x}},$$

which is the Gumbel distribution.

Let $X_{N,1},\ldots,X_{N,N}$ be the descending order statistics of $X_1,\ldots,X_N$ and

$$Y_{N,k} = \sqrt{2\log N}(X_{N,k} - a_N).$$

By the extreme value theorem, $(Y_{N,1},\ldots,Y_{N,N})$ converges in distribution to $(Y_1, Y_2, \ldots)$ where

$$Y_k = -\log W_k,\ W_k = V_1 + \ldots + V_k,$$

and $V_1, V_2, \ldots$ are iid exponential with parameter one.

The Gibbs measure in descending order can be written as

$$\frac{\exp\{-\alpha^{-1}\sqrt{2\log N}X_{N,i}\}}{\sum_{j=1}^{N}\exp\{-\alpha^{-1}\sqrt{2\log N}X_j\}}$$
$$= \frac{\exp\{-\alpha^{-1}\sqrt{2\log N}Y_{N,i}\}}{\sum_{j=1}^{N}\exp\{-\alpha^{-1}\sqrt{2\log N}Y_{N,j}\}}$$
$$\Rightarrow \frac{Y_i^{-1/\alpha}}{\sum_{i=1}^{\infty}Y_j^{-1/\alpha}}, \quad N\to\infty.$$

Consider the stable subordinator ρ_t with Lévy measure

$$\Lambda(du) = \frac{\alpha}{\Gamma(1-\alpha)} u^{-(1+\alpha)} d\,u.$$

Denote the descending jump sizes of ρ_t over $[0,1]$ by $J_1 > J_2 > \cdots$. Set

$$P_i(\alpha, 0) = \frac{J_i}{\rho_1},\ \ i = 1, 2, \ldots.$$

The law of $(P_1(\alpha,0), P_2(\alpha,0), \ldots)$, denoted by $PD(\alpha, 0)$, is called the *Poisson-Dirichlet distribution* with parameter $(\alpha, 0)$.

Let

$$\tilde{W}_i = \phi(J_i) = \frac{J_i^{-\alpha}}{\Gamma(1-\alpha)}, i = 1, 2, \ldots .$$

It follows from the mapping theorem for Poisson process that $\tilde{W}_1 > \tilde{W}_2 > \cdots$ are the points of a Poisson process with mean measure

$$\mu(d\,u) = \Lambda \circ \phi^{-1}(d\,u) = d\,u.$$

Thus

$$\begin{aligned} \tilde{W}_1 &\stackrel{d}{=} V_1 \\ \tilde{W}_k - \tilde{W}_{k-1} &\stackrel{d}{=} V_k \\ \tilde{W}_i &\stackrel{d}{=} W_i \end{aligned}$$

This implies that $(\frac{J_1}{\rho_1}, \frac{J_2}{\rho_1}, \ldots)$ and

$$(\frac{Y_1^{-1/\alpha}}{\sum_{i=1}^{\infty} Y_j^{-1/\alpha}}, \frac{Y_2^{-1/\alpha}}{\sum_{i=1}^{\infty} Y_j^{-1/\alpha}}, \ldots)$$

have the same distribution.

In other words, the law of

$$(\frac{Y_1^{-1/\alpha}}{\sum_{i=1}^{\infty} Y_j^{-1/\alpha}}, \frac{Y_2^{-1/\alpha}}{\sum_{i=1}^{\infty} Y_j^{-1/\alpha}}, \ldots)$$

is the Poisson-Dirichlet distribution $PD(\alpha, 0)$.

Conclusion:The thermodynamic limit of the ordered Gibbs measure of Derrida's random energy model is a Poison-Dirichlet distribution!

At low temperature, we have

$$0 < \alpha = \frac{T}{T_c} < 1.$$

Large Deviations

Let

$$\nabla = \{(p_1, ..., p_n, ...) : p_1 \geq p_2 \geq \cdots \geq 0, \ \sum_{i=1}^{\infty} p_i = 1\}$$

$$\nabla_n = \{(p_1, ..., p_n, 0, 0, ...) \in \nabla : \sum_{i=1}^{n} p_i = 1\}, \ n \geq 1$$

and

$$\nabla_\infty = \bigcup_{n=1}^{\infty} \nabla_n.$$

Theorem 1. (F and Gao (10)) *The family* $\{PD(\alpha,0) : 0 < \alpha < 1\}$ *on space* ∇ *satisfies an LDP as* α *approaches zero with speed* $-\log\alpha$ *and rate function*

$$I(\mathbf{p}) = \begin{cases} 0, & \mathbf{p} \in \nabla_1 \\ n-1, & \mathbf{p} \in \nabla_n, p_n > 0, n \geq 2 \\ \infty, & \mathbf{p} \notin \nabla_\infty \end{cases}$$

Remarks:

1 The limit $\alpha \to 0$ corresponds to temperature tending to zero.

2 At the moment when the temperature moves from zero, the energy moves along a finite number of levels even though the total number of energy levels is infinite for any positive temperature.

Let $(P_1(\alpha,0), P_2(\alpha,0),\ldots)$ have the $PD(\alpha,0)$ distribution. By direct calculation,

$$\mathsf{E}[\sum_{i=1}^{\infty} P_i^2(\alpha,0)] = 1-\alpha.$$

Therefore

$$(P_1(\alpha,0), P_2(\alpha,0),\ldots) \to (0,0,\ldots), \alpha \to 1.$$

Theorem 2. (Zhou(13)) *The family* $\{PD(\alpha,0) : 0 < \alpha < 1\}$ *on space* $\bar{\nabla}$ *satisfies an LDP as* α *approaches one with speed* $\log\Gamma(1-\alpha)$ *and rate function*

$$I(\mathbf{p}) = \max\{n \geq 1 : p_n > 0\}$$

where

$$\bar{\nabla} = \{(p_1,...,p_n,...) : p_1 \geq p_2 \geq \cdots \geq 0, \ \sum_{i=1}^{\infty} p_i \leq 1\}.$$

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