Estimates of the blowup time for a stochastic semilinear wave equation with white noise

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In view of equation, our main aims are analytic properties of solution:

- Well-posedness
- Blowup
- Longtime behavior
- Regularity
- etc.

Main problems of blowup are

- Blow up?
- Blowup criterion
- Estimates of the blowup time
- Blowup points
- After the blowup
- etc.

Present ways of blowup

- Transformation, $cudW$
- Probability, $cudW$
- Branch process

1. Equation

Consider

$$\begin{cases} \partial_t^2 u + \alpha \partial_t u - \phi(x)\nabla^2 u - f(u) = \sigma(u, \partial_t u, \nabla u, x, t)\partial_t W(t,x), \ t > 0, \ x \in \mathbb{R}^N, \\ u(t,x)|_{t=0} = u_0(x), \ \partial_t u(t,x)|_{t=0} = u_1(x), \ x \in \mathbb{R}^N. \end{cases} \tag{0.1}$$

Here $\alpha \geq 0$ is a constant; $\{W(t,x), \ t \geq 0, \ x \in \mathbb{R}^N\}$ is a Wiener random field with respect to $(\Omega, \ \mathscr{F}, \ \{\mathscr{F}_t\}_{t\geq 0};\mathbb{P})$ with mean zero, which will be defined precisely later; $\{u(t,x), \ t \geq 0, \ x \in \mathbb{R}^N\}$ is a continuous $\mathscr{F}_t$-adapted random field; $f : \mathbb{R}^N \to \mathbb{R}$; $\sigma : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty) \to \mathbb{R}$; $\phi : \mathbb{R}^N \to (0,\infty)$; $u_0, \ u_1 : \mathbb{R}^N \times \Omega \to \mathbb{R}$ are $\mathscr{F}_0$-measurable; ∇ is the gradient operator; ∇^2 is the Laplace operator; $\sigma(u, \partial_t u, \nabla u, x, t))\partial_t W(t,x)$ is white noise of Itô type.

2. Physical backgroud

The deterministic models of (0.1) are of interest in applications in various areas of mathematical physics, as well as in geophysics and ocean acoustics, where, for example, the coefficient $\phi(x)$ represents the speed of sound at the point $x \in \mathbb{R}^N$.

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Throughout the talk, $N \geq 3$ and

(**A1**) $g(x) := (\phi(x))^{-1}$ *and* $g \in C^\delta(\mathbb{R}^N) \cap L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *for some* $\delta \in (0, 1)$.

3. Mathematical backgroud

For $\alpha = 0$, $\phi = 1$, $N \leq 3$, $f(u) = \lambda u^3 - \gamma^2 u$ ($\lambda > 0$) and the additive noise term $\sigma(x,t)\partial_t W(t,x)$, in 2002, P.L.Chow proved the existence of explosive solution in mean L^2-norm for the wave equation under the assumption that $\int_0^t \sigma(x,s) d_t W(s,x)$ is a zero-mean $\{\mathscr{F}_t\}_{t\geq 0}$-martingale, where $d_t W(t,x)$ is the Itô differential of $W(t,x)$ on t.

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In 2009, P.L.Chow was concerned with the problem of explosive solutions for the semilinear stochastic wave equations (0.1) with $\alpha = 0$, $\phi = 1$ and the noise term $\sigma(u, \nabla u, x, t)\partial_t W(t, x)$ in a domain $D \subset \mathbb{R}^N$ for $N \leq 3$. Under the assumption (A3) in this paper, Chow thought that $\int_0^t \sigma(u(s), \nabla u(s), x, s) d_t W(s, x)$ is a zero-mean $\{\mathscr{F}_t\}_{t\geq 0}$-martingale. So, Chow obtained the L^2- norm of the solution will blow up at a finite time in the mean-square sense under appropriate conditions on the initial data and the nonlinear term.

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But, if u blows up before the time t with positive probability, then $\int_0^t \sigma(u(s), \nabla u(s), x, s) d_t W(s, x)$ is meaningless.

Main difficulties:

- $\mathbb{E}\int_0^t \sigma(u(s), \nabla u(s), x, s) d_t W(s,x) = 0$?
- Do better?

4. An idea

Two prior times T and T_1 are presented by (nonrandom) initial data and parameters of a stochastic differential equation, where $T_1 > T > 0$. Under the assumption that the solution exists almost surely after the time T_1, a pair of contradiction can be found at the time T, which implies that the solution blows up before or at the time T_1 with positive probability.

5. Functional Spaces

Define two spaces $L_g^2(\mathbb{R}^N)$, $\mathscr{D}^{1,2}(\mathbb{R}^N)$ and their product space $\mathscr{D}^{1,2}(\mathbb{R}^N) \times L_g^2(\mathbb{R}^N)$.

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Let $h_1,\ h_2:\ \mathbb{R}^N \to \mathbb{R}$. The inner product of h_1 and h_2 is denoted by

$$(h_1, h_2)_{L^2_g} = \int_{\mathbb{R}^N} g h_1 h_2 dx.$$

The space $L^2_g(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the norm $\|h\|_{L^2_g} = (h, h)^{\frac{1}{2}}_{L^2_g}$.

Define

$$\mathscr{D}^{1,2}(\mathbb{R}^N) = \{h \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : \nabla h \in (L^2(\mathbb{R}^N))^N\}$$

which is as the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the norm

$$\|h\|_{\mathscr{D}^{1,2}} = \left(\int_{\mathbb{R}^N} |\nabla h|^2 dx\right)^{\frac{1}{2}}$$

which can induced by the inner product

$$(h_1, h_2)_{\mathscr{D}^{1,2}} = \left(\int_{\mathbb{R}^N} \nabla h_1 \cdot \nabla h_2 dx\right)^{\frac{1}{2}}$$

for $h_1,\ h_2 \in \mathscr{D}^{1,2}(\mathbb{R}^N)$.

Let $H = \mathscr{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$ be the product space of $\mathscr{D}^{1,2}(\mathbb{R}^N)$ and $L^2_g(\mathbb{R}^N)$, whose norm

$$\|\varphi\|_H = (\|\varphi_1\|^2_{\mathscr{D}^{1,2}} + \|\varphi_2\|^2_{L^2_g})^{\frac{1}{2}}$$

and inner product

$$(\varphi, \psi)_H = (\varphi_1, \psi_1)_{\mathscr{D}^{1,2}} + (\varphi_2, \psi_2)_{L^2_g}, \quad \varphi = (\varphi_1, \varphi_2)^T, \psi = (\psi_1, \psi_2)^T \in H.$$

There exists a complete system of eigensolutions $\{\phi_n, \mu_n\}$ in L_g^2 with the following properties

$$\begin{cases} -\phi\nabla^2\phi_j = \mu_j\phi_j, \ j = 1, 2, \cdots, \phi_j \in \mathscr{D}^{1,2}(\mathbb{R}^N), \\ 0 < \mu_1 \le \mu_2 \le \cdots, \quad \mu_j \to \infty, \ as \ j \to \infty. \end{cases}$$

6. Wiener field

Define $W(t,x)$ as

$$W(t,x)=\sum_{j=1}^{\infty}\gamma_j W_j(t)\phi_j(x)$$

where γ_j is nonrandom, $\{W_j(t):\ t\geq 0\}$ is a sequence of independent standard Wiener processes. The spatial correlation function $r(x,y)=\sum_{j=1}^{\infty}\gamma_j^2\phi_j(x)\phi_j(y)$ is assumed to be integrable on $\mathbb{R}^N\times\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} g(x)r(x,x)dx=\sum_{j=1}^{\infty}\gamma_j^2\int_{\mathbb{R}^N} g(x)\phi_j^2(x)dx<\infty.$$

Suppose $\{y(t,x),\ t \geq 0,\ x \in \mathbb{R}^N\}$ is a continuous $\mathscr{F}_t$-adapted random field satisfies

$$d_t y(t,x) = b(t, y(t,x))dt + \mathbf{a}(t, y(t,x))d_t W(t,x)$$

for $t > t_0 \geq 0$ and $x \in \mathbb{R}^N$.

Lemma 0.1

Suppose that $h(t,y) \in C^{1,2}$. Then,

$$\begin{aligned} d_t h(t, y(t,x)) = &\Big(\frac{\partial h}{\partial t}(t, y(t,x)) + (b\frac{\partial h}{\partial y})(t, y(t,x)) \\ &+ r(x,x)(\frac{\mathbf{a}^2}{2}\frac{\partial^2 h}{\partial y^2})(t, y(t,x))\Big) dt \\ &+ \left(\frac{\partial h}{\partial y}\mathbf{a}\right)(t, y(t,x))d_t W(t,x) \end{aligned}$$

for $t > t_0$.

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7. Assumptions

Assume, in addition to (**A1**) that

(**A2**) $|f(\xi)| \leq c_1 \left(1 + |\xi|^\beta\right)$, $|f(\xi_1) - f(\xi_2)| \leq c_1 \left(|\xi_1|^{\beta-1} + |\xi_2|^{\beta-1}\right) |\xi_1 - \xi_2|$, $\xi, \xi_1, \xi_2 \in \mathbb{R}$, *where* $\beta \in [1, \frac{N+2}{N-2}]$ *and* $c_1 > 0$ *are constants.*

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(**A3**) *For any constant* $t_0 > 0$ *and* $t \in [0, t_0]$,

$$\int_{\mathbb{R}^N} g(x)\sigma^2(\xi,\eta,\zeta,x,t)r(x,x)dx \le c_2\left(1+\|\xi\|^{2\gamma}_{L^{2\gamma}_g}+\|\eta\|^2_{L^2_g}+\|\zeta\|^2_{L^2_g}\right),$$

$$\int_{\mathbb{R}^N} g(x)(\sigma(\xi_1,\eta_1,\zeta_1,x,t)-\sigma(\xi_2,\eta_2,\zeta_2,x,t))^2 r(x,x)dx$$
$$\le c_2\left(\|\xi_1-\xi_2\|^2_{L^2_g}\left(\|\xi_1\|^{2(\gamma-1)}_{L^{2(\gamma-1)}_g}+\|\xi_2\|^{2(\gamma-1)}_{L^{2(\gamma-1)}_g}\right)+\|\eta_1-\eta_2\|^2_{L^2_g}+\|\zeta_1-\zeta_2\|^2_{L^2_g}\right)$$

for (ξ,η,ζ), $(\xi_i,\eta_i,\zeta_i) \in L^{2\gamma}_g \times L^2_g \times (L^2_g)^N$, $i = 1,\ 2$; $\gamma \in [1, \frac{N}{N-2}]$ *and* $c_2 = c_2(t_0) > 0$ *are constants.*

8. Solutions

Theorem 0.1

Under the assumptions (**A1**)-(**A3**)*, there exists a unique mild solution* $(u, \partial_t u) \in C([0, \tau); H)$ *of* (0.1) *for any* $\mathscr{F}_0$*-measurable* $(u_0, u_1) \in H$*, where* $\tau > 0$ *is a stopping time and* $[0, \tau)$ *is the maximal existing interval of* $(u, \partial_t u)$.

Definition 0.1

A stopping time $\tau > 0$ *is said to be a finite blowup time of the solution* $u(t, x)$ *to* (0.1)*, if the following three conditions are fulfilled:*

- $\mathbb{P}(\tau < \infty) > 0$,
- *For any* $t < \tau$, $\|u(t)\|^2_{L^2_g} + \|\partial_t u(t)\|^2_{L^2_g} < \infty$,
- $\lim_{t \to \tau-} (\|u(t)\|^2_{L^2_g} + \|\partial_t u(t)\|^2_{L^2_g}) = \infty$ *in* $\{\tau < \infty\}$.

Definition 0.2

$T_{bc} = \inf\{t > 0 : \mathbb{P}(\tau \leq t) > 0\}$ *is the blowup critical time of* (0.1), *where* τ *is the blowup time of the solution* $u(t,x)$ *to* (0.1).

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$T_{mbc} = \sup\{t > 0 : \mathbb{E}(\|u(t)\|^2_{L^2_g} + \|\partial_t u(t)\|^2_{L^2_g}) < \infty\}$ *is the m.s. blowup critical time of* (0.1).

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Obviously, $T_{mbc} \leq T_{bc}$ if T_{bc} exists.

9. Main results

For the nonlinear power f in (0.1), throughout the paper, the following assumption is made: there exists some constant $\epsilon > 0$ such that

$$f(s)s \geq (2+\epsilon)\mathcal{F}(s) \tag{0.2}$$

for any $s \in \mathbb{R}$, where $\mathcal{F}(s) = \int_0^s f(\eta)d\eta$.

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for any $s \in \mathbb{R}$, where $\mathcal{F}(s) = \int_0^s f(\eta)d\eta$. The typical form of nonlinear power is as

$$f(s) = |s|^{\beta-1}s - m^2 s, \quad s \in \mathbb{R}, \quad 1 < \beta < \frac{N+2}{N-2}.$$

Let $\sigma(s) = \sigma(u(s), \partial_t u(s), \nabla u(s), x, s)$ and τ be always as in Theorem 0.1. The assumptions (**A1**)-(**A3**) are always satisfied.

We always let r_0, r_1 and $E(0)$ be constants in this section, where

$$r_0 = \int_{\mathbb{R}^{\mathbb{N}}} g|u_0|^2 dx, \quad r_1 = \int_{\mathbb{R}^{\mathbb{N}}} g u_0 u_1 dx$$

$$E(t) = \frac{1}{2}\|\partial_t u(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^2_{\mathscr{D}^{1,2}} - (\mathcal{F}(u(t)), 1)_{L^2_g} + \alpha \int_0^t \|\partial_t u(s)\|^2_{L^2_g} ds.$$

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Suppose

$$\sigma_1(x,t) = \sup_{\xi,\eta,\zeta} \sigma(\xi,\eta,\zeta,x,t) \text{ exists}, \quad \sigma_\infty = \tfrac{1}{2}\int_0^\infty (\sigma_1^2(s), r)_{L^2_g} ds < \infty. \tag{0.3}$$

Theorem 0.2

Let

$$T_1 = -\frac{(2(1+\alpha r_0)+\epsilon(|r_1|-r_1))^2 - 2\epsilon^2 r_0(E(0)+\sigma_\infty)}{\epsilon^2(E(0)+\sigma_\infty)(2+\epsilon|r_1|)}.$$

Suppose (0.3) *holds and* $E(0)+\sigma_\infty < 0$. *Then,* $T_{bc} \leq T_1$.

Introduce a process as follows

$$I(t) = \|u(t)\|^2_{\mathscr{D}^{1,2}} - (f(u(t)), u(t))_{L^2_g}, \quad t \geq 0$$

which is important in the following two theorems.

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Theorem 0.3

Set

$$T_2 = \frac{8(2+\epsilon)(\alpha r_0 + 1)^2 + 2\epsilon^3 \mu_1 r_0^2}{\epsilon^3 \mu_1 r_0 (2 + \epsilon r_1)}.$$

Suppose (0.3) *holds,* $r_0 > 0$, $r_1 \geq 0$ *and* $E(0) + \sigma_\infty = 0$. *Then,* $T_{bc} \leq T_2$.

Theorem 0.4

Define

$$T_3 = \frac{8(2+\epsilon)(1+\alpha r_0)^2 + 2\epsilon^2 r_0(\epsilon\mu_1 r_0 - 2(2+\epsilon)(E(0)+\sigma_\infty))}{\epsilon^2(\epsilon\mu_1 r_0 - 2(2+\epsilon)(E(0)+\sigma_\infty))(2+\epsilon r_1)}$$

If (0.3) *holds,* $I(0) < 0$, $r_1 \geq 0$ *and* $r_0 > \frac{2(2+\epsilon)}{\epsilon\mu_1}(E(0)+\sigma_\infty) > 0$, *then* $T_{bc} \leq T_3$.

10. Other results

Suppose

$$\frac{1}{2}\sigma^2(\xi,\eta,\zeta,x,t)r(x,x) \le \alpha|\eta|^2, \quad t > 0 \tag{0.4}$$

for any $\xi, \eta \in \mathbb{R}$ and $\zeta,\ x \in \mathbb{R}^N$.

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$$\frac{1}{2}\sigma^2(\xi,\eta,\zeta,x,t)r(x,x) \le \alpha|\eta|^2, \quad t>0 \tag{0.4}$$

for any $\xi,\eta \in \mathbb{R}$ and $\zeta,\ x \in \mathbb{R}^N$.Define

$$\mathscr{E}(t) = \frac{1}{2}\|\partial_t u(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^2_{\mathscr{D}^{1,2}} - (\mathcal{F}(u(t)),1)_{L^2_g}.$$

The following three theorems deal with other assumption on σ.

Theorem 0.5

Set

$$T_4 = \frac{(1+|r_1|-r_1)^2 - 2r_0\mathscr{E}(0)}{-\epsilon\mathscr{E}(0)(1+|r_1|)}.$$

Suppose (0.4) *holds and* $\mathscr{E}(0) < 0$. *Then,* $T_{bc} \leq T_4$.

Theorem 0.6

Let

$$T_5 = \frac{2(2+\epsilon+\epsilon\mu_1 r_0^2)}{\epsilon^2\mu_1 r_0(1+r_1)}.$$

If (0.4) *holds,* $r_0 > 0$, $r_1 \geq 0$ *and* $\mathscr{E}(0) = 0$, *then* $T_{bc} \leq T_5$.

Theorem 0.7

Define

$$T_6 = \frac{2(2+\epsilon+\epsilon\mu_1 r_0^2 - 2(2+\epsilon)\mathscr{E}(0)r_0)}{\epsilon(\epsilon\mu_1 r_0 - 2(2+\epsilon)\mathscr{E}(0))(1+r_1)}$$

Suppose (0.4) *holds,* $I(0) < 0$*,* $r_1 \geq 0$ *and* $r_0 > \frac{2(2+\epsilon)}{\epsilon\mu_1}\mathscr{E}(0) > 0$*. Then,* $T_{bc} \leq T_6$.

Remark 0.1

After some light modification, blowup problems for other stochastic semilinear wave equations with white noises can also be solved, e.g. with strong damping, in bounded domains, or time white noises, etc.

11. Further research

- $f(u) = 0$, $\sigma(u)dW$?

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