Ergodicity of the 2D Navier-Stokes equations with Degenerate Multiplicative Noise[☆]

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Abstract

Consider the two-dimensional, incompressible Navier-Stokes equations on the torus $T^2 = [-\pi, \pi]^2$ driven by a degenerate noise

$$dw_t = \nu\Delta w_t dt + B(\mathcal{K}w_t, w_t)dt + \sum_{i=1}^{m} q_i(w_t)e_i dB_i(t). \tag{1}$$

We prove that the semigroup P_t generated by the solutions to (1) has asymptotically strong Feller property. Moreover, we also prove that semigroup $\{P_t\}_{t\geq 0}$ is exponentially ergodic in some sense.

Keywords: Stochastic Navier-Stokes equation; Asymptotically strong Feller property; Ergodicity

1. Introduction and Notations

This work is motivated by the paper [5], in which, Martin Hairer and Jonathan C. Mattingly considered the following two-dimensional, incompressible Navier-Stokes equations on the torus $T^2 = [-\pi, \pi]^2$ driven by a additive degenerate noise

$$dw_t = \nu\Delta w_t dt + B(\mathcal{K}w_t, w_t)dt + QdB(t). \tag{2}$$

With the asymptotically strong Feller property that they discovered, they proved the uniqueness and existence of the invariant measure for the semigroup generated by the solution to (2). In this article, we consider stochastic Navier-Stokes equations as follows

$$dw_t = \nu\Delta w_t dt + B(\mathcal{K}w_t, w_t)dt + Q(w_t)dB(t), \quad w_t|_{t=0} = w_0. \tag{3}$$

Recall that the Navier-Stokes equations are given by

$$\partial_t u + (u \cdot \nabla)u = \nu\triangle u - \nabla p + \xi, \quad div(u) = 0.$$

where $\xi(x, t)$ is the external force field acting on the fluid. Denote $H = L_0^2$, the space of real-valued square-integrable functions on the torus with vanishing mean. The vorticity w is defined by $w = \nabla \wedge u = \partial_2 u_1 - \partial_1 u_2$. $B(u, \omega) = -(u \cdot \nabla)\omega$. For $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, $k^\perp = (k_2, -k_1)$, $\omega_k = \langle \omega, (2\pi)^{-1} e^{(ik\cdot x)} \rangle_H$. The operator $\mathcal{K}$ is defined in Fourier space by

$$(\mathcal{K}\omega)_k = \langle \mathcal{K}\omega, (2\pi)^{-1} e^{(ik\cdot x)} \rangle = -i\omega_k k^\perp / \|k\|^2.$$

We write $\mathbb{Z}^2 \setminus \{(0, 0)\} = \mathbb{Z}^2_+ \cup \mathbb{Z}^2_-$, where

$$\mathbb{Z}^2_+ = \{(k_1, k_2) \in \mathbb{Z}^2 : \ k_2 > 0\} \cup \{(k_1, 0) \in \mathbb{Z}^2 : \ k_1 > 0\},$$
$$\mathbb{Z}^2_- = \{(k_1, k_2) \in \mathbb{Z}^2 : \ -(k_1, k_2) \in \mathbb{Z}^2_+\},$$

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and set, for $k \in \mathbb{Z}^2 \backslash \{(0,0)\}$,

$$f_k(x) = \begin{cases} sin(k \cdot x) & k \in \mathbb{Z}^2_+, \\ cos(k \cdot x) & k \in \mathbb{Z}^2_-. \end{cases} \tag{4}$$

We also fix a set $\mathcal{Z}_0 = \{k_i : i = 1, \cdots, m\} \subseteq \mathbb{Z}^2 \setminus \{(0,0)\}$, and let $e_i = f_{k_i}, k_i \in \mathcal{Z}_0$. We denote by $\{\beta_i\}_{i=1}^m$ the canonical basis of $\mathbb{R}^m$. In this article, the linear map $Q(w_t) : \mathbb{R}^m \to H$ is given by $Q(w_t)\beta_i = q_i(w_t)e_i,\ 1 \le i \le m.\ q_i : H \to \mathbb{R}, i = 1, \cdots, m$ are some functions. $B(t) = (B_1(t), \cdots, B_m(t))$ is a standard m-dimension Brownian motion. We denote by P_t the semigroup generated by the the stochastic differential equation given by (3).

Assume $\Phi_t(B, w_0) : C([0,t]; \mathbb{R}^m) \times H \to H$ be the map such that $w_t = \Phi_t(B, w_0)$ for initial condition w_0 and noise realization B. Given a $v \in L^2_{loc}(\mathbb{R}^+, \mathbb{R}^m)$, the Malliavin derivative of the H-valued random variable w_t in the direction v, denoted by $\mathcal{D}^v w_t$ is defined by

$$\mathcal{D}^v \omega_t = \lim_{\varepsilon \to 0} \frac{\Phi_t(B + \varepsilon V, w_0) - \Phi_t(B, w_0)}{\varepsilon},$$

where the limit holds almost surely with respect to Wiener measure and $V(t) = \int_0^t v(s)ds$.

Let $\{J_{s,t}\}_{s \le t}$ be the derivative flow between times s and t, i.e for every $\xi \in H$, $J_{s,t}\xi$ is the solution of

$$\begin{cases} dJ_{s,t}\xi = \nu \triangle J_{s,t}\xi dt + \tilde{B}(\omega_t, J_{s,t}\xi)dt + DQ(\omega_t)J_{s,t}\xi dB_t, \\ J_{s,s}\xi = \xi, \end{cases} \tag{5}$$

where $\tilde{B}(\omega_t, J_{s,t}\xi) = B(\mathcal{K}\omega_t, J_{s,t}\xi) + B(\omega_t, \mathcal{K}J_{s,t}\xi)$. $J_{0,t}\xi$ is the effect on w_t of an infinitesimal perturbation of the initial condition in the direction ξ. DQ is Fréchet derivation of Q.

Observe that $\mathcal{D}^v \omega_t = A_{0,t}v$, where $A_{s,t} : L^2([s,t], \mathbb{R}^m) \to H$:

$$A_{s,t}v = \int_s^t J_{r,t}Q(w_r)v(r)dr. \tag{6}$$

Denote by H_N be the space spanned by $\left\{f_k : k \in \mathbb{Z}^2 \setminus \{(0,0)\}, |k| \le N\right\}$.

For $\alpha \in \mathbb{R}$ and a smooth function w on $[-\pi, \pi]^2$ with mean 0, denote $\|w\|_\alpha$ by

$$\|w\|^2_\alpha = \sum_{k \in \mathcal{Z}^2 \backslash \{(0,0)\}} |k|^{2\alpha}|w_k|^2, \ \ \omega_k = \langle \omega, (2\pi)^{-1}e^{(ik \cdot x)} \rangle_H, \tag{7}$$

and $\|w\| := \|w\|_0$. If $A : \mathbb{R}^m \to H$ is a linear map, then

$$\|A\|^2 := \sum_{i=1}^m \|A\beta_i\|^2,$$

for example, $\|Q(w_t)\|^2 = \sum_{i=1}^m q_i(w_t)^2\|e_i\|^2$.

If $B(u,v) = (u \cdot \nabla)v, \mathcal{S} = \left\{(s_1, s_2, s_3) \in \mathbb{R}^3_+ : \sum s_i \ge 1, s \ne (1,0,0), (0,1,0), (0,0,1)\right\}$. Then the following relations are useful. The proof can see [2] or [5] .

$$\langle B(u,v), w \rangle = -\langle B(u,w), v \rangle, \qquad \text{if } \nabla \cdot u = 0, \tag{8}$$

$$|\langle B(u,v), w \rangle| \le C\|u\|_{s_1}\|v\|_{1+s_2}\|w\|_{s_3}, \quad (s_1, s_2, s_3) \in \mathcal{S}, \tag{9}$$

$$\|\mathcal{K}w\|_\alpha = \|w\|_{\alpha-1}, \tag{10}$$

$$\|w\|^2_{\frac{1}{2}} \le \|w\|_1\|w\|. \tag{11}$$

In section 2, we will give some estimations of the solution to (3). In section 3, we proved that the semigroup generated by the solutions to (1) has asymptotically strong feller property under some conditions, that is there exists constants $\eta > 0, \gamma > 0$ and $\tilde{C}$ such that for every t

$$|\nabla P_t\varphi(w)| \le \tilde{C} \exp(\frac{5\eta}{\nu}\|w\|^2)\left(\|\varphi\|_\infty + e^{-\gamma t}\|\nabla\varphi\|_\infty\right). \tag{12}$$

To prove (12), in [5], they constructed a deterministic equation. But we consider multiplicative noise, so we need to construct a stochastic partial differential equation. It is more complicated. In section 4, under some conditions, we prove that semigroup P_t generated by the solution to (1) exists unique invariant measure and the semigroup $\{P_t\}_{t\geq 0}$ is exponentially ergodic in some sense.

2. Some Properties For Solution

We first give a Lemma which comes from the Lemma A.1 in [9].

Lemma 2.1 *Let $M(s)$ be a continuous martingale with quadratic variation $[M,M](s)$ such that $\mathbb{E}[M,M] < \infty$. Define the semi-martingale $N(s) = -\frac{\alpha}{2}[M,M](s) + M(s)$ for any $\alpha > 0$. If $\gamma \geq 0$, then for any $\beta \geq 0$ and $T > \frac{1}{\beta}$*

$$\mathbb{P}\left\{\sup_{t\in[T-\frac{1}{\beta},T]} \int_0^t e^{-\gamma(t-s)} dN(s) > \frac{e^{\frac{\gamma}{\beta}}}{\alpha} K\right\} < e^{-K}.$$

Specially,

$$\mathbb{P}\left\{\sup_t N(t) > \frac{1}{\alpha} K\right\} < e^{-K}.$$

Theorem 2.1 *Assume $|q_i(w_t)| \leq C, 1 \leq i \leq m$, then there exists a constant $C_1 = C_1(\eta, \nu, m)$ such that*

$$\mathbb{E}\exp\{\sup_{t\geq 0}(\eta\|w_t\|^2)\} \leq C_1 e^{\eta\|w_0\|^2}, \tag{13}$$

holds for every $\eta \leq \eta_0 = \frac{\nu}{8C^2\pi^2}$. Thus the invariant measure for P_t exists.

Proof: The exists and uniqueness of the solution to (3) can see Appendix C. From Itô formula,

$$d\eta\|w_t\|^2 + 2\eta\nu\|w_t\|_1^2 dt = 2\eta\langle w_t, Q(w_t)dB_t\rangle + \eta\|Q\|^2 dt. \tag{14}$$

Using the fact that $\|w_t\| \leq \|w_t\|_1$, $\|Q\|^2 \leq C_2^2 := 4\pi^2 C^2 m$,

$$d\eta\|w_t\|^2 + \nu\eta\|w_t\|^2 dt \leq 2\eta\langle w_t, , Q(w_t)dB_t\rangle + C_2^2\eta dt - \nu\eta\|w_t\|^2 dt,$$

that is,

$$\eta d(\|w_t\|^2 e^{\nu t}) \leq 2\eta e^{\nu t}\langle w_t, Q(w_t)dB_t\rangle + C_2^2\eta e^{\nu t}dt - \nu\eta e^{\nu t}\|w_t\|^2 dt.$$

So,

$$\eta\|w_t\|^2 - \eta e^{-\nu t}\|w_0\|^2 - \eta\frac{C_2^2}{\nu} \leq 2\eta\int_0^t e^{-\nu(t-s)}\langle w_s, Q(w_s)dB_s\rangle - \eta\nu\int_0^t e^{-\nu(t-s)}\|w_s\|^2 ds.$$

From Lemma 2.1, if $\eta \leq \eta_0 = \frac{\nu}{8C^2\pi^2}$, we get

$$\mathbb{E}\exp\left\{\sup_{t\geq 0}\left(\eta\|w_t\|^2 - \eta e^{-\nu t}\|w_0\|^2 - \eta\frac{C_2^2}{\nu}\right)\right\} \leq 2, \tag{15}$$

here we use the fact that if a random variable X satisfies $\mathbb{P}(X \geq C) \leq \frac{1}{C^2}$ for all $C \geq 0$, then $\mathbb{E}X \leq 2$. So this Theorem 2.1 follows from (15). □

Theorem 2.2 *Assume $|q_i(w_t)| \leq C, 1 \leq i \leq m$, then exists $\tilde{C} = \tilde{C}(\eta, \nu, m)$ such that*

$$\mathbb{E}\exp\left(\eta\sup_{t\geq 0}(\|w_t\|^2 + \nu\int_0^t \|w_r\|_1^2 dr - 4\pi^2 C^2 mt)\right) \leq \tilde{C}\exp(\eta\|w_0\|^2), \tag{16}$$

holds for every $\eta \leq \eta_0 = \frac{\nu}{8C^2\pi^2}$.

Proof: From (14) and the fact $||w_t|| \leq ||w_t||_1$, we have

$$\eta||w_t||^2 + \eta\nu\int_0^t ||w_r||_1^2 dr - \eta\int_0^t ||Q(w_r)||^2 dr - \eta||w_0||^2 \leq 2\eta\int_0^t \langle w_r, Q(w_r)dB_r\rangle - \eta\nu\int_0^t ||w_r||^2 dr.$$

For $|q_i| \leq C$,

$$\eta||w_t||^2 + \eta\nu\int_0^t ||w_r||_1^2 dr - 4\pi^2\cdot\eta\cdot m\cdot C^2 t - \eta||w_0||^2 \leq 2\eta\int_0^t \langle w_r, Q(w_r)dB_r\rangle - \eta\nu\int_0^t ||w_r||^2 dr.$$

From the Lemma 2.1 and condition $\eta \leq \eta_0 = \frac{\nu}{8\pi^2 C^2}$,

$$\mathbb{E}\exp\left(\eta\sup_{t\geq 0}\left(||w_t||^2 + \nu\int_0^t ||w_r||_1^2 dr - 4\pi^2 mC^2 t - ||w_0||^2\right)\right) \leq 2,$$

from which the theorem 2.2 follows. □

In order to introduce Theorem 2.3, we first give some definitions. Let $(X_t)_{0\leq t\leq T}$ be a continuous stochastic process take values in an open interval $I \subseteq R$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a filtration on this space. Let $C_x([u,v], I) = \{f \in C([u,v], I), f(u) = x\}$. As usual, we equip the space $C_x([u,v], I), x \in I$ with uniform topologies.

We say a process $\{X_t, t \geq 0\}$ has conditional full support(CFS) with respect to the filtration $\mathcal{F}_t$, or briefly F-CFS, if
(a) X is adapted to F,
(b) for all $t \in [0, T)$ and $\mathbb{P}-$almost all $\omega \in \Omega$,

$$\text{supp}\left(\text{Law}[(X_u)_{u\in[t,T]}|\mathcal{F}_t](\omega)\right) = C_{X_t(\omega)}([t,T], I).$$

The next theorem comes from Theorem 3.12 in [11].

Lemma 2.2 *$(X_t)_{0\leq t\leq T}$ be a continuous stochastic process. $(W_t)_{t\in[0,T]}$ is a Brownian motion, ϕ and ψ progressive measurable function from $[0,T]\times C([0,T])^2$ to $\mathbb{R}$, and ξ is a random variable. Define*

$$h_t := \phi(t, W, X),\ \ k_t := \psi(t, W, X),\ \ \mathcal{F}_t := \sigma\{\xi, W_s, X_s : s \in [0,t]\},\ t \in [0,T].$$

If W_t is a Brownian motion with respect $\{\mathcal{F}_t\}_{t\in[0,T]}$,

$$\mathbb{E}\left[e^{\lambda\int_0^T k_s^{-2} ds}\right] < \infty\ \ \forall\lambda > 0,\ \ \mathbb{E}\left[e^{2\int_0^T k_s^{-2} h_s^2 ds}\right] < \infty,\ \textit{and} \tag{17}$$

$$\int_0^T k_s^2 ds \leq \bar{K}\ \ \textit{a.s for some constant}\ \ 0 < \bar{K} < \infty, \tag{18}$$

then the process

$$Z_t := \xi + \int_0^t h_s ds + \int_0^t k_s dW_s,\ t \in [0,T].$$

has CFS.

The next Lemma comes from Lemma 3.1 in [3]. Since we consider Multiplicative Noise, the proof needs little changes. For the convenience of the reader, we prove it again.

Theorem 2.3 *Assume there exists a constant $\tilde{K}$ such that $0 < \tilde{K} \leq |q_i(w_t)|, 1 \leq i \leq m$. w_t is the solution to equation (1). Fix $C_0 > 0, C_1 > 0$, Let $\mathcal{B}_0 = \mathcal{B}(C_0)$ and $\mathcal{B}_1 = \mathcal{B}(C_1)$ be two arbitrary balls about the origin $0 \in H$ and h be some positive constant. Then there exists a constant $T_0 = T_0(C_0, C_1) > 0$, so that for any $T \geq T_0$, there is a constant $p^* = p^*(T, h, C_0, C_1) > 0$ with*

$$\inf_{w_0' \in\mathcal{B}_0} \mathbb{P}_{w_0'}\{\omega(t) \in \mathcal{B}_1\ \textit{for all}\ t \in [T, T+h]\} \geq p^* > 0.$$

Proof: Let $v(t) = \omega(t) - \hat{f}(t)$, where $\hat{f}(t) = \int_0^t Q(\omega_r) dB_r$. Using (3), we see that $v(t)$ satisfies

$$\frac{\partial v}{\partial t} = \nu \triangle (v + \hat{f}) - (u \cdot \nabla)(v + \hat{f}), \text{ here } u_t = \mathcal{K} w_t.$$

Taking the L^2-inner product with v on the both side of this equation produces,

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 = -\nu\|\nabla v\|^2 - \int_{[-\pi,\pi]^2} v(x)(u(x)\cdot\nabla)\hat{f}(x)dx - \int_{[-\pi,\pi]^2} v(x) \triangle \hat{f}(x)dx.$$

By standard estimate on the nonlinear term (see [2]) and the fact we are on the torus, we have

$$\left|\int v(x)(u(x)\cdot\nabla)\hat{f}(x)dx\right| \le C_3\|v\|\,\|\nabla u\|\,\|\triangle\hat{f}\|.$$

Since $\|\nabla u\| = \|\omega\|$ and $\omega = v + \hat{f}$, the above estimate gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|^2 &\le -\nu\|\nabla v\|^2 + C_3\|v\|\cdot\|v+\hat{f}\|\cdot\|\triangle\hat{f}\| + \|v\|\cdot\|\triangle\hat{f}\| \\ &\le -\nu\|\nabla v\|^2 + C_3\|v\|^2\|\triangle\hat{f}\| + C_3\|v\|\,\|\hat{f}\|\,\|\triangle\hat{f}\| + \|v\|\cdot\|\triangle\hat{f}\|. \end{aligned}$$

Use the *Poincaré* inequality, $\|v\|^2 \le \|\nabla v\|^2$, we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \le -\left(\frac{\nu}{2} - C_3\|\triangle\hat{f}\|\right)\|v\|^2 + \frac{4C_3^2}{\nu}\|\hat{f}\|^2\|\triangle\hat{f}\|^2 + \frac{4}{\nu}\|\triangle\hat{f}\|^2.$$

Fix any $\delta > 0$ and define for any $T > 0$

$$\Omega'(\delta, T) = \left\{g \in C([0, T+h]; H) : \sup_{s\in[0,T+h]} \|\triangle g(s)\| \le \min\{\delta, \frac{\nu}{4C_3}\}\right\}.$$

If $\hat{f} \in \Omega'$, then exists a constant C_4 which is independent of h, T such that

$$\|v(t)\|^2 \le \|v(0)\|^2 \exp(-\frac{\nu}{2}t) + \frac{C_4}{\nu^2}\left[\min\left(\delta, \frac{\nu}{4C_3}\right)^4 + \min\left(\delta, \frac{\nu}{4C_3}\right)^2\right].$$

Hence if $\|w_0'\| < C_0$, then given any $C_1 > 0$ there exists a T and a δ such that $\|v(T)\| < C_1/2$. By possible decreasing of δ , if $f \in \Omega'$, we can assume $\|v(t)\| < C_1/2$ for $t \in [T, T+h]$. Putting everything together, we have that for approximate T and δ ,

$$\|\omega_0'\| \le C_0, \hat{f} \in \Omega'(\delta, T) \Rightarrow \|\omega(t)\| \le \|v(t)\| + \|f(t)\| \le C_1 \quad \forall t \in [T, T+h].$$

Since for any $T > 0$ and $\delta > 0$, $\Omega'(T, \delta)$ is an open set in the supremum topology, we know

$$\mathbb{P}\{\hat{f}, \hat{f} \in \Omega'(T, \delta)\} > 0, \tag{19}$$

(19) comes from Lemma 2.2. So this theorem has been proved. □

3. Asymptotically Strong Feller Property

Let π_l be the orthogonal projection onto the span of H_N and $\pi_h = 1 - \pi_l$. In the below of this article, we will assuming $H_N \subseteq Range(Q)$ and exists positive constant $\tilde{K}$ such that $0 < \tilde{K} \le \inf_{i:e_i\in H_N} |q_i(w_t)|$ almost surely. Thus $Q_l \stackrel{def}{=} \pi_l Q$ is invertible and Q_l^{-1} bounded on H_N. Assume the span of $\{e_1, \cdots, e_m\}$ contains H_N (that is assume $\left\{k : k \in \mathbb{Z}^2 \setminus \{(0,0)\}, |k| \le N\right\} \subseteq \mathcal{Z}_0$), so Q_l^{-1} is a well defined and bounded operator from H_N to $\mathbb{R}^m$.

Denote $\hat{C} = \max\left\{\sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{2\pi}\frac{1}{|k|^{\frac{3}{2}}},\quad 2^{\frac{3}{8}} \cdot (2\pi)^{\frac{1}{20}} \cdot \left(\sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{|k|^{10}}\right)^{10} \cdot (\sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{|k|^{\frac{4}{3}}})^{\frac{3}{8}}\right\}$.

Theorem 3.1 *Assume $H_N \subseteq span\{e_1, \cdots, e_m\}$, and π_l is the orthogonal projection onto the span of H_N. Assume $\|Dq_i(w_t)\| \le C$, $0 < \tilde{K} \le \inf_{i:e_i \in H_N} |q_i(w_t)|$ and $q_i(\theta) = q_i(\pi_l \theta)$ for any $\theta \in H$, $1 \le i \le m$. Assume there exists positive constant a, $a \le \frac{N^2}{m}$. N is big enough such that $\max\{\frac{\sqrt{24}\hat{C}^2}{\eta\nu}, \sqrt{\frac{32C^2}{\nu}}\} \le N$ for $\eta = \min\{\frac{\nu^2}{40\pi^2C^2}, \frac{\nu^2 a}{80\pi^2C^2}\}$. Then there exists constants $\tilde{C} = \tilde{C}(\eta, N, \nu, \bar{K}, \hat{C}, m, C)$ and $\gamma = \gamma(N, \nu) > 0$ such that*

$$|\nabla P_t\varphi(w_0)| \le \tilde{C} \exp(\frac{5\eta}{\nu}\|w_0\|^2)\left(\|\varphi\|_\infty + e^{-\gamma t}\|\nabla\varphi\|_\infty\right). \tag{20}$$

Remark 3.1 *In the proving process we will not use the condition $0 < a \le \frac{N^2}{m}$, $\eta = \min\{\frac{\nu^2}{40\pi^2C^2}, \frac{\nu^2 a}{80\pi^2C^2}\}$, $\max\{\frac{\sqrt{24}\hat{C}^2}{\eta\nu}, \sqrt{\frac{32C^2}{\nu}}\} \le N$. Instead, we will use the condition : assume η, N, m satisfy $\eta \le \min\{\frac{\nu^2}{40\pi^2C^2}, \frac{\nu^2N^2}{80\pi^2C^2m}\}$ and $\max\{\frac{\sqrt{24}\hat{C}^2}{\eta\nu}, \sqrt{\frac{32C^2}{\nu}}\} \le N$. The condition $\eta \le \frac{\nu^2N^2}{80\pi^2C^2m}$ seems unreasonable, but it is necessary in additive case. In Proposition 4.11, [5], it needs $\mathcal{E}_0 = tr(QQ^*) = \sum_{k=1}^m |q_k|^2$ is smaller than a constant which is independent of N.*

There exists some situations which the conditions of Theorem 3.1 are satisfied. For example, $m = 4N^2 + 4N$, $span\{e_1, \cdots, e_m\} \supseteq H_N, a = 1/8$, $\max\{\frac{\sqrt{24}\hat{C}^2}{\eta\nu}, \sqrt{\frac{32C^2}{\nu}}\} \le N$.

Proof: Denote

$$\zeta_t^l = \pi_l\zeta_t,\ \zeta_t^h = \pi_h\zeta_t,\ \tilde{B}(w,u) = B(\mathcal{K}w,u) + B(\mathcal{K}u,w).$$

Set $\rho_t = J_{0,t}\xi - \mathcal{D}^v w_t = J_{0,t}\xi - A_{0,t}v_t$, where the definition of $J_{0,t}\xi$ and $A_{0,t}v_{0,t}$ see (5)(6), then ρ_t satisfies the following equation

$$d\rho_t = \nu\Delta\rho_t dt + \tilde{B}(w_t,\rho_t)dt + DQ(w_t)\rho_t dB_t - Q(w_t)v_t dt. \tag{21}$$

For any $\xi \in H$ with $\|\xi\| = 1$, define ζ_t by

$$\begin{cases} d\zeta_t = -\frac{\zeta_t^l}{2\|\zeta_t^l\|}dt + DQ(w_t)\zeta_t dB_t + 4\pi^2C^2m\Delta\zeta_t^l dt + \pi_h\tilde{B}(w_t,\zeta_t)dt + \nu\Delta\zeta_t^h dt - 8\pi^2C^2m\zeta_t^l\|\zeta_t^l\|^2 dt \\ \zeta_0 = \xi \end{cases}, \tag{22}$$

with the convention $0/0 = 0$. We set the infinitesimal perturbation v by

$$v_t = Q_l^{-1}F_t, F_t = \frac{\zeta_t^l}{2\|\zeta_t^l\|} + \pi_l\tilde{B}(w_t,\zeta_t) - 4\pi^2C^2m\Delta\zeta_t^l + \nu\Delta\zeta_t^l + 8\pi^2C^2m\zeta_t^l\|\zeta_t^l\|^2. \tag{23}$$

From Appendix A and Appendix B, there exists a unique solution to equation (24) and (29). So equation (22) exists a unique solution. Also from Appendix A, we know that (23) is meaningful. It clear from (21) and (22) that ρ_t and ζ_t satisfy the same equation, and thus $\rho_t = \zeta_t$. Since ζ_t^l satisfies

$$d\zeta_t^l = -\frac{\zeta_t^l}{2\|\zeta_t^l\|}dt + \pi_l DQ(w_t)\zeta_t dB_t + 4\pi^2C^2m\Delta\zeta_t^l dt - 8\pi^2C^2m^2\zeta_t^l\|\zeta_t^l\|^2 dt,\ \ \zeta_0^l = \xi^l := \pi_l\xi. \tag{24}$$

from Itô formula,

$$d\|\zeta_t^l\|^2 = -\|\zeta_t^l\|dt + \|\pi_l DQ(w_t)\zeta_t\|^2 dt - 8\pi^2C^2m\|\zeta_t^l\|_1^2 dt + 2\langle\zeta_t^l, \pi_l DQ(w_t)\zeta_t dB_t\rangle - 16\pi^2C^2m\|\zeta_t^l\|^4 dt. \tag{25}$$

From

$$\begin{aligned} \|\pi_l DQ(w_t)\zeta_t\|^2 &= \sum_{i=1}^m \|(Dq_i(w_t)\zeta_t)e_i\|^2 = \sum_{i=1}^m \|(Dq_i(w_t)\zeta_t^l)e_i\|^2 \\ &\le \sum_{i=1}^m (2\pi)^2C^2\|\zeta_t^l\|^2 = 4\pi^2C^2m\|\zeta_t^l\|^2, \end{aligned} \tag{26}$$

and (25), we have

$$d\mathbb{E}\|\zeta_t^l\|^2 \leq -\mathbb{E}\|\zeta_t^l\|dt,$$

which is

$$\mathbb{E}\|\zeta_t^l\|^2 - \mathbb{E}\|\zeta_0^l\|^2 \leq -\int_0^t \mathbb{E}\|\zeta_s^l\|ds.$$

Since $(\mathbb{E}\|\zeta_t^l\|)^2 \leq \mathbb{E}\|\zeta_t^l\|^2, \mathbb{E}\|\zeta_0^l\|^2 = (\mathbb{E}\|\zeta_0^l\|)^2$,

$$(\mathbb{E}\|\zeta_t^l\|)^2 - (\mathbb{E}\|\zeta_0^l\|)^2 \leq -\int_0^t \mathbb{E}\|\zeta_s^l\|ds. \tag{27}$$

From the above inequality, we known that as $t \geq 2$,

$$\mathbb{E}\|\zeta_t^l\| = 0. \tag{28}$$

Since ζ_t^h satisfy the following equation

$$d\zeta_t^h = \nu\Delta\zeta_t^h dt + \pi_h\tilde{B}(w_t, \zeta_t)dt + \pi_h DQ(w_t)dB_t. \tag{29}$$

by Lemma 3.4, if $\max\{\frac{\sqrt{24}\hat{C}^2}{\eta\nu}, \sqrt{\frac{32C^2}{\nu}}\} \leq N$, $\eta \leq \min\{\frac{\nu^2}{40\pi^2C^2}, \frac{\nu^2N^2}{80\pi^2C^2m}\}$, then there exists constants $\hat{C}_2, \gamma$ such that

$$\mathbb{E}\|\zeta_t^h\| \leq (\mathbb{E}\|\zeta_t^h\|^2)^{\frac{1}{2}} \leq \hat{C}_2 e^{\frac{5\eta}{2\nu}\|w_0\|^2}e^{-\gamma t}, \ \forall t > 0. \tag{30}$$

We next need to get control over the size of perturbation ν. Since v is adapted to the Wiener path,

$$\left(\mathbb{E}\left|\int_0^t v(s)dB(s)\right|\right)^2 \leq \int_0^t \mathbb{E}\|v(s)\|^2 ds \leq C_1\int_0^t \mathbb{E}\|F_s\|^2 ds. \tag{31}$$

From the definition of F_t, $\|\pi_l\tilde{B}(u,w)\| \leq C_0 \cdot \|u\| \cdot \|w\|$ for some constant C_0 (see [4], Lemma A.4), Lemma 3.3 and (23)(28), we have (the constant C_0 may different from line to line)

$$\mathbb{E}\|F_s\|^2 \leq C_0(1_{\{s\leq 2\}} + \mathbb{E}\|w_s\|^2\|\zeta_s^h\|^2). \tag{32}$$

From Lemma 3.4, when $\eta \leq \min\{\frac{\nu^2}{40\pi^2C^2}, \frac{\nu^2N^2}{80\pi^2C^2m}\}$ and $\max\{\frac{\sqrt{24}\hat{C}^2}{\eta\nu}, \sqrt{\frac{32C^2}{\nu}}\} \leq N$.

$$\begin{aligned}
\mathbb{E}(\|w_s\|^2\|\zeta_s^h\|^2) &= \mathbb{E}\left(\|w_s\|^2 e^{\frac{1}{4}\nu N^2 t - \int_0^t \frac{5}{2}\eta\|w_r\|_1^2 dr} \cdot e^{-\frac{1}{4}\nu N^2 t + \int_0^t \frac{5}{2}\eta\|w_r\|_1^2 dr}\|\zeta_s^h\|^2\right) \\
&\leq \left(\mathbb{E}\|w_s\|^4 e^{-\frac{1}{2}\nu N^2 s + \int_0^s 5\eta\|w_r\|_1^2 dr}\right)^{\frac{1}{2}} \left(\mathbb{E}e^{\frac{1}{2}\nu N^2 s - \int_0^s 5\eta\|w_r\|_1^2 dr}\|\zeta_s^h\|^4\right)^{\frac{1}{2}} \\
&\leq C_0 e^{\frac{5\eta}{2\nu}\|w_0\|^2}\left(\mathbb{E}\|w_s\|^4 e^{-\frac{1}{2}\nu N^2 s + \int_0^s 5\eta\|w_r\|_1^2 dr}\right)^{\frac{1}{2}}.
\end{aligned}$$

By Theorem 2.2, when $\eta \leq \min\{\frac{\nu^2}{40\pi^2C^2}, \frac{\nu^2N^2}{80\pi^2C^2m}\}$, there exists a constant C_3 such that

$$\begin{aligned}
\mathbb{E}\|w_s\|^4 e^{-\frac{1}{2}\nu N^2 s + \int_0^s 5\eta\|w_r\|_1^2 dr} &= e^{-\frac{1}{4}\nu N^2 s}\mathbb{E}\|w_s\|^4 e^{-\frac{1}{4}\nu N^2 s + \int_0^s 5\eta\|w_r\|_1^2 dr} \\
&\leq C_3 e^{-\frac{1}{4}\nu N^2 s} e^{\frac{5\eta}{\nu}\|w_0\|^2}.
\end{aligned}$$

Thus, there exists constant C_4 such that

$$\mathbb{E}(\|w_s\|^2\|\zeta_s^h\|^2) \leq C_4 e^{-\frac{1}{8}\nu N^2 s} e^{\frac{5\eta}{\nu}\|w_0\|^2}. \tag{33}$$

Therefore, there exists constant C_5 such that if $\eta \leq \min\{\frac{\nu^2}{40\pi^2C^2}, \frac{\nu^2N^2}{80\pi^2C^2m}\}$,

$$\mathbb{E}\left|\int_0^\infty v(s)dB(s)\right| \leq C_5 \exp\left(\frac{5\eta}{2\nu}\|w_0\|^2\right). \tag{34}$$

Since

$$\begin{aligned}\langle \nabla P_t\varphi(\omega_0),\xi\rangle &= \mathbb{E}_{\omega_0}(\langle \nabla\varphi(\omega_t),\xi\rangle) = \mathbb{E}_{\omega_0}((\nabla\varphi)(\omega_t)J_{0,t}\xi)\\ &= \mathbb{E}_{\omega_0}((\nabla\varphi)(\omega_t)\mathcal{D}^v w_t) + \mathbb{E}_{\omega_0}((\nabla\varphi)(\omega_t)\rho_t)\\ &= \mathbb{E}_{\omega_0}(\mathcal{D}^v\varphi(\omega_t)) + \mathbb{E}_{\omega_0}((\nabla\varphi)(\omega_t)\rho_t)\\ &\leq \|\varphi\|_\infty \mathbb{E}_{\omega_0}|\int_0^t v(s)dB(s)| + \|\nabla\varphi\|_\infty \mathbb{E}_{\omega_0}\|\rho_t\|,\end{aligned}$$

so from (28), (30), (34), we proved this theorem.

□

Lemma 3.1 *u, v, w are smooth functions belong to H, then*

$$|\langle B(u,v),w\rangle| \leq \hat{C}\|u\|_1\|v\|_1\|w\|_{1/2}. \tag{35}$$

Proof: Set $s_1 = \frac{4}{5},\ s_2 = 0,\ s_3 = \frac{1}{4},\ q_1 = 10,\ q_2 = 2,\ q_3 = \frac{8}{3},\ q_4 = 40$, then $\sum_{i=1}^4(1/q_i) = 1$. Set $\Omega = T^2 = [-\pi,\pi]^2$, from Holder inequality

$$\begin{aligned}\int_\Omega u(x)\nabla v(x)w(x)dx &\leq \|u\|_{L^{q_1}}\|\nabla v(x)\|_{L^{q_2}}\|w\|_{L^{q_3}}\|1\|_{L^{q_4}}\\ &\leq (2\pi)^{\frac{1}{20}}\|u\|_{L^{q_1}}\|\nabla v(x)\|_{L^2}\|w\|_{L^{q_3}}.\end{aligned} \tag{36}$$

From sobolev embedding theorem, we have

$$\begin{aligned}\|u\|_{L^{q_1}(\Omega)} &\leq C_{s_1}\|u\|_{s_1} \leq C_1\|u\|_1,\\ \|w\|_{L^{q_3}} &\leq C_{s_3}\|w\|_{s_3} \leq C_2\|w\|_{\frac{1}{2}}.\end{aligned}$$

So

$$\int_\Omega u(x)\nabla v(x)w(x)dx \leq (2\pi)^{\frac{1}{20}}C_1C_2\|u\|_1\|v(x)\|_1\|w\|_{\frac{1}{2}}.$$

We next calculate C_1, C_2. Set $\{e_k^{'}, k\in\mathbb{Z}^2\backslash\{(0,0)\}$ is the orthonormal basis for H, that is $e_k^{'} = \frac{1}{2\pi}e^{ik\cdot x}, k\in\mathbb{Z}^2\setminus\{(0,0)\}$. For the calculating of C_1, we assume $u = \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} u_k e_k^{'}$ and $b^2 = \|u\|_1^2 = \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}}|k|^2|u_k|^2$. Thus $|u_k| \leq \frac{b}{|k|}$ and

$$\|u\|_{L^{q_1}}^{q_1} \leq 2^9\sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}}\int_\Omega |u_k e_k^{'}|^{q_1}dx \leq 2^9(2\pi)^{-8}\sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}}|\frac{b}{|k|}|^{10}.$$

From the above inequality, we have

$$\|u\|_{L^{q_1}} \leq b\left(\sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}}\frac{1}{|k|^{10}}\right)^{10},$$

thus we get

$$C_1 = \left(\sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}}\frac{1}{|k|^{10}}\right)^{10}.$$

For the calculating of C_2, we assume $w = \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}}\omega_k e_k^{'}$ and $b^2 = \|w\|_{\frac{1}{2}}^2 = \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}}|k||\omega_k|^2$. Then $|w_k| \leq \frac{b}{\sqrt{|k|}}$ and

$$\|w\|_{L^{q_3}}^{q_3} \leq 2^2\sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}}\int_\Omega |\omega_k e_k^{'}|^{q_3}dx \leq 2^2\cdot(2\pi)^2\cdot(2\pi)^{-\frac{8}{3}}\sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}}|\frac{b}{\sqrt{|k|}}|^{\frac{8}{3}}.$$

From the above inequality,

$$\begin{aligned}\|w\|_{L^{q_3}} &\leq 2^{\frac{3}{8}} b(\sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{|k|^{\frac{4}{3}}})^{\frac{3}{8}} \\ &\leq 2^{\frac{3}{8}}(\sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{|k|^{\frac{4}{3}}})^{\frac{3}{8}}\|w\|_{\frac{1}{2}}.\end{aligned}$$

thus we get

$$C_2 = 2^{\frac{3}{8}}(\sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{|k|^{\frac{4}{3}}})^{\frac{3}{8}}.$$

□

Lemma 3.2 *u, v, w are smooth functions belong to H, then*

$$|\langle B(u,v), w\rangle| \leq \hat{C}\|u\|_{\frac{3}{2}}\|v\|_1\|w\|.$$

Proof: From Hölder inequality,

$$\begin{aligned}\int_\Omega u(x)\nabla v(x)w(x)dx &\leq \|u\|_{L^\infty}\|\nabla v(x)\|_{L^2}\|w\|_{L^2} \\ &\leq C_3\|u\|_{\frac{3}{2}}\|\nabla v(x)\|_{L^2}\|w\|_{L^2}\end{aligned}$$

The constant C_3 is calculated as follows. Set $\{e_k^{'}, k \in \mathbb{Z}^2\setminus\{(0,0)\}$ is the orthonormal basis for H and assume $u = \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} u_k e_k^{'}$ and $b^2 = \|u\|_{\frac{3}{2}}^2 = \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} |k|^3|u_k|^2$. Then $|u_k| \leq \frac{b}{|k|^{\frac{3}{2}}}$ and

$$\|u\|_{L^\infty} \leq \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{2\pi}\frac{1}{|k|^{\frac{3}{2}}}\cdot b,$$

thus we know that $C_3 = \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{2\pi}\frac{1}{|k|^{\frac{3}{2}}}$. □

Lemma 3.3 *Assume the conditions of Theorem* 3.1 *hold and ζ_t^l is the solution to* (24)*, then* $\mathbb{E}(\sup_{0\leq s\leq 2}\|\zeta_s^l\|^2)^6 < \infty$.

Proof: From (25)(26), we have

$$d\|\zeta_t^l\|^2 + \|\zeta_t^l\|dt \leq 2\langle\zeta_t^l, \pi_l DQ(w_t)\zeta_t dB_t\rangle - 16\pi^2C^2m\|\zeta_t^l\|^4dt.$$

For

$$\begin{aligned}|\langle\zeta_t^l, \pi_l DQ(w_t)\zeta_t\rangle|^2 &= |\langle\zeta_t^l, \pi_l DQ(w_t)\zeta_t^l\rangle|^2 \\ &= \left|\sum_{i=1}^m\langle\zeta_t^l, e_i\rangle Dq_i(w_t)\zeta_t^l\right|^2 \\ &\leq 4\pi^2C^2m\|\zeta_t^l\|^4.\end{aligned}$$

Thus by using (26) and Lemma 2.1,

$$\begin{aligned}&\mathbb{P}\{\sup_{0\leq t\leq 2}\|\zeta_t^l\|^2 - \|\zeta_0^l\|^2 \geq K\} \\ &\leq \mathbb{P}\left\{\sup_{0\leq t\leq 2}\int_0^t 2\langle\zeta_s^l, \pi_l DQ(w_s)\zeta_s dB_s\rangle - \int_0^t 16\pi^2C^2m\|\zeta_s^l\|^4ds \geq K\right\} \\ &\leq \mathbb{P}\left\{\sup_{0\leq t\leq 2}\int_0^t 2\langle\zeta_s^l, \pi_l DQ(w_s)\zeta_s dB_s\rangle - \int_0^t 4|\langle\zeta_s^l, \pi_l DQ(w_s)\zeta_s\rangle|^2ds \geq K\right\} \\ &\leq \exp(-K).\end{aligned}$$

So this Lemma has been proved. □

Lemma 3.4 *Assume the conditions of Theorem* 3.1 *hold. If* $\max\{\frac{\sqrt{24}\hat{C}^2}{\eta\nu}, \sqrt{\frac{32C^2}{\nu}}\} \leq N$, *then for any* $\eta \leq \min\{\frac{\nu^2}{40\pi^2C^2}, \frac{\nu^2N^2}{80\pi^2C^2m}\}$, *there exists constants* $\hat{C}_2 = \hat{C}_2(m, N, \eta, \nu, \hat{C}, C)$, $C(N, m, \eta, \hat{C}, C, \nu)$ *and* $\gamma = \gamma(\nu, N) > 0$ *such that*

$$
\begin{aligned}
\mathbb{E}\|\zeta_t^h\|^2 &\leq \hat{C}_2 e^{\frac{5\eta}{\nu}\|w_0\|^2} e^{-\gamma t}, \quad \forall t \geq 0, \\
\mathbb{E}\|\zeta_t^h\|^4 e^{\frac{1}{2}\nu N^2 t - \int_0^t 5\eta\|w_r\|_1^2 dr} &\leq C(N, m, \eta, \hat{C}, C, \nu) e^{\frac{5\eta}{\nu}\|w_0\|^2}, \quad \forall t \geq 0. \qquad (37)
\end{aligned}
$$

Proof: First, we give some estimations. From Lemma 3.1, Lemma 3.2, (8), (10) and (11), for any $\eta > 0$,

$$
\begin{aligned}
|\langle \zeta_t^l, B(\mathcal{K}w_t, \zeta_t^h)\rangle| &\leq \hat{C}\|\zeta_t^h\|_1\|w_t\|_{\frac{1}{2}}\|\zeta_t^l\| && (38) \\
&\leq \frac{\nu}{6}\|\zeta_t^h\|_1^2 + \frac{6\hat{C}^2}{\nu}\|w_t\|_1^2\|\zeta_t^l\|^2. && (39) \\
\langle w_t, B(\mathcal{K}\zeta_t^l, \zeta_t^h)\rangle| &\leq \hat{C}\|\zeta_t^h\|_1\|w_t\|_{\frac{1}{2}}\|\zeta_t^l\| && (40) \\
&\leq \frac{\nu}{6}\|\zeta_t^h\|_1^2 + \frac{6\hat{C}^2}{\nu}\|w_t\|_1^2\|\zeta_t^l\|^2. && (41)
\end{aligned}
$$

$$
\begin{aligned}
|\langle \zeta_t^h, B(\mathcal{K}\zeta_t^h, w_t)\rangle| &= |\langle w_t, B(\mathcal{K}\zeta_t^h, \zeta_t^h\rangle)| \\
&\leq \hat{C}\|\zeta_t^h\|_{\frac{1}{2}}\|w_t\|_1\|\zeta_t^h\| && (42) \\
&\leq \eta\|w_t\|_1^2\|\zeta_t^h\|^2 + \frac{\hat{C}^2}{\eta}\|\zeta_t^h\|_{\frac{1}{2}}^2 \\
&\leq \eta\|w_t\|_1^2\|\zeta_t^h\|^2 + \frac{\hat{C}^2}{\eta}\|\zeta_t^h\|_1\|\zeta_t^h\| \\
&\leq \eta\|w_t\|_1^2\|\zeta_t^h\|^2 + \frac{\hat{C}^2}{\eta}\left[\frac{\eta\nu}{6\hat{C}^2}\|\zeta_t^h\|_1^2 + \frac{6\hat{C}^2}{\eta\nu}\|\zeta_t^h\|^2\right] \\
&= \eta\|w_t\|_1^2\|\zeta_t^h\|^2 + \frac{\nu}{6}\|\zeta_t^h\|_1^2 + \frac{6\hat{C}^4}{\eta^2\nu}\|\zeta_t^h\|^2. && (43)
\end{aligned}
$$

For any $\theta \in H, i = 1, \cdots, m, q_i(\theta) = q_i(\pi_l\theta)$, thus

$$
\begin{aligned}
\|\pi_h DQ(w_t)\zeta_t\|^2 &\leq \sum_{i=1}^m \|Dq_i(w_t)\zeta_t\|^2\|e_i\|^2 \\
&= \sum_{i=1}^m \|Dq_i(w_t)\zeta_t^l\|^2\|e_i\|^2 \\
&\leq 4\pi^2 mC^2\|\zeta_t^l\|^2.
\end{aligned}
$$

When $N \geq \frac{\sqrt{24}\hat{C}^2}{\eta\nu}$,

$$
\begin{aligned}
d\|\zeta_t^h\|^2 &= -2\nu\|\zeta_t^h\|_1^2 dt + 2\langle \zeta_t^h, \tilde{B}(w_t, \zeta_t)\rangle dt + 2\langle \zeta_t^h, DQ(w_t)\zeta_t dB_t\rangle + \|\pi_h DQ(w_t)\zeta_t\|^2 dt \\
&= -2\nu\|\zeta_t^h\|_1^2 dt - 2\langle \zeta_t^l, B(\mathcal{K}w_t, \zeta_t^h)\rangle dt - 2\langle w_t, B(\mathcal{K}\zeta_t, \zeta_t^h)\rangle dt \\
&\quad +2\langle \zeta_t^h, DQ(w_t)\zeta_t dB_t\rangle + \|\pi_h DQ(w_t)\zeta_t\|^2 dt \\
&\leq -2\nu\|\zeta_t^h\|_1^2 dt + 4\hat{C}\|w_t\|_{\frac{1}{2}}\|\zeta_t^h\|_1\|\zeta_t^l\| dt + 2\hat{C}\|w_t\|_1\|\zeta_t^h\|_{\frac{1}{2}}\|\zeta_t^h\| dt \\
&\quad +2\langle \zeta_t^h, DQ(w_t)\zeta_t dB_t\rangle + 4\pi^2 mC^2\|\zeta_t^l\|^2 dt \\
&\leq -\nu\|\zeta_t^h\|_1^2 dt + \frac{24\hat{C}^2}{\nu}\|w_t\|_1^2\|\zeta_t^l\|^2 dt + 2\eta\|w_t\|_1^2\|\zeta_t^h\|^2 dt + \frac{12\hat{C}^4}{\eta^2\nu}\|\zeta_t^h\|^2 dt \\
&\quad +2\langle \zeta_t^h, DQ(w_t)\zeta_t dB_t\rangle + 4\pi^2 mC^2\|\zeta_t^l\|^2 dt \\
&\leq (-\frac{1}{2}\nu N^2 + 2\eta\|w_t\|_1^2)\|\zeta_t^h\|^2 dt + \frac{24\hat{C}^2}{\nu}\|w_t\|_1^2\|\zeta_t^l\|^2 dt \\
&\quad +2\langle \zeta_t^h, DQ(w_t)\zeta_t dB_t\rangle + 4\pi^2 mC^2\|\zeta_t^l\|^2 dt, && (44)
\end{aligned}
$$

here the second equality, we have used (8). In the first inequality, we have used (38)(40)(42). In the second inequality we have used (39)(41)(43). In the last inequality, we have used the fact $\|\zeta_t^h\|_1 \geq N\|\zeta_t^h\|$.

By the same argument above, we have when $N \geq \sqrt{\frac{32C^2}{\nu}}$,

$$\begin{aligned} d\|\zeta_t^h\|^4 &= 2\|\zeta_t^h\|^2 d\|\zeta_t^h\|^2 + 4|\langle\zeta_t^h, DQ(w_t)\zeta_t\rangle|^2 dt \\ &\leq (-\nu N^2 + 4\eta\|w_t\|_1^2)\|\zeta_t^h\|^4 dt + \frac{48\hat{C}^2}{\nu}\|w_t\|_1^2\|\zeta_t^l\|^2\|\zeta_t^h\|^2 dt \\ &\quad + 8\pi^2 mC^2\|\zeta_t^l\|^2\|\zeta_t^h\|^2 dt + 16\pi^2 C^2\|\zeta_t^l\|^2\|\zeta_t^h\|^2 dt + h(t)dB(t) \\ &\leq (-\nu N^2 + 4\eta\|w_t\|_1^2)\|\zeta_t^h\|^4 dt + \eta\|w_t\|_1^2\|\zeta_t^h\|^4 dt + \frac{48^2\hat{C}^4}{\eta\nu^2}\|w_t\|_1^2\|\zeta_t^l\|^4 dt \\ &\quad + \frac{1}{4}\nu N^2\|\zeta_t^h\|^4 dt + \frac{4\cdot(8\pi^2 mC^2)^2}{\nu N^2}\|\zeta_t^l\|^4 dt + 2C^2\cdot(2\pi)^4\|\zeta_t^l\|^4 dt + 8C^2\|\zeta_t^h\|^4 dt + h(t)dB(t) \\ &\leq (-\frac{1}{2}\nu N^2 + 5\eta\|w_t\|_1^2)\|\zeta_t^h\|^4 dt + a\|w_t\|_1^2\|\zeta_t^l\|^2 dt + b\|\zeta_t^l\|^4 dt + h(t)dB(t), \end{aligned} \tag{45}$$

where $a = \frac{48^2\hat{C}^4}{\eta\nu^2}$, $b = 2C^2\cdot(2\pi)^4 + \frac{4\cdot(8\pi^2 mC^2)^2}{\nu N^2}$, $DQ(w_t)\zeta_t = \sum_{i=1}^m \left(Dq_i(w_t)\zeta_t^l\right)e_i$ and $h(t)$ is some process.

From (45), we can obtain

$$\begin{aligned} \mathbb{E}\|\zeta_t^h\|^4 e^{\frac{1}{2}\nu N^2 t - \int_0^t 5\eta\|w_r\|_1^2 dr} &\leq \|\zeta_0^h\|^4 + \mathbb{E}\int_0^t e^{(\frac{1}{2}\nu N^2 s - \int_0^s 5\eta\|w_r\|_1^2 dr)}(a\|w_s\|_1^2\|\zeta_s^l\|^2 + b\|\zeta_s\|^4)ds \\ &\leq \|\zeta_0^h\|^4 + \mathbb{E}\int_0^2 e^{(\frac{1}{2}\nu N^2 s - \int_0^s 5\eta\|w_r\|_1^2 dr)}(a\|w_s\|_1^2\|\zeta_s^l\|^2 + b\|\zeta_s\|^4)ds \\ &\leq \|\zeta_0^h\|^4 + \tilde{C}_1\mathbb{E}\left(\int_0^2 \|w_s\|_1^2 ds \sup_{s\in[0,2]}\|\zeta_s^l\|^2\right) + \tilde{C}_2\mathbb{E}\sup_{s\in[0,2]}\|\zeta_s^l\|^4, \\ &\leq C(N, m, \eta, \hat{C}, C, \nu)e^{\frac{5\eta}{\nu}\|w_0\|^2}, \end{aligned} \tag{46}$$

In the last inequality, we have used Theorem 2.2, Lemma 3.3.

Thus

$$\begin{aligned} \mathbb{E}\|\zeta_t^h\|^2 &= \mathbb{E}\|\zeta_t^h\|^2 e^{\frac{1}{4}\nu N^2 t - \int_0^t \frac{5}{2}\eta\|w_r\|_1^2 dr}\cdot e^{-\frac{1}{4}\nu N^2 t + \int_0^t \frac{5}{2}\eta\|w_r\|_1^2 dr} \\ &\leq \left(\mathbb{E}\|\zeta_t^h\|^4 e^{\frac{1}{2}\nu N^2 t - \int_0^t 5\eta\|w_r\|_1^2 dr}\right)^{\frac{1}{2}}\left(\mathbb{E}e^{-\frac{1}{2}\nu N^2 t + \int_0^t 5\eta\|w_r\|_1^2 dr}\right)^{\frac{1}{2}} \\ &= \left(\mathbb{E}\|\zeta_t^h\|^4 e^{\frac{1}{2}\nu N^2 t - \int_0^t 5\eta\|w_r\|_1^2 dr}\right)^{\frac{1}{2}}\left(\mathbb{E}e^{-\frac{1}{4}\nu N^2 t + \int_0^t 5\eta\|w_r\|_1^2 dr}\right)^{\frac{1}{2}} e^{-\frac{\nu N^2 t}{8}}. \end{aligned}$$

From Theorem 2.2, (46), we know that when $\eta \leq \min\{\frac{\nu^2}{40\pi^2 C^2}, \frac{\nu^2 N^2}{80\pi^2 C^2 m}\}$,

$$\begin{aligned} \mathbb{E}\|\zeta_t^h\|^2 &\leq C(N, m, \eta, \hat{C}, C, \nu)e^{\frac{5\eta}{2\nu}\|w_0\|^2}\cdot\tilde{C}e^{(\frac{5\eta}{2\nu}\|w_0\|^2)}e^{-\frac{\nu N^2 t}{8}} \\ &= C(N, m, \eta, \hat{C}, C, \nu)e^{\frac{5\eta}{\nu}\|w_0\|^2}e^{-\gamma t}. \end{aligned}$$

□

4. Ergodicity

For getting the exponential convergence, we using the methods in [6]. In the Assumption 4.1, 4.2, 4.3 and Theorem 4.1 below, we assume that we are given a random flow Φ_t on a Banach space H. We will assume that the map $x \mapsto \Phi_t(\omega, x)$ is C^1 for almost every element ω of the underlying probability space. We will denote by $D\Phi_t$ the Fréchet derivative of $\Phi_t(\omega, x)$ with respect to x.

Let $C(\mu_1, \mu_2)$ for the set of all measures Γ on $H \times H$ such that $\Gamma(A \times H) = \mu_1(A)$ and $\Gamma(H \times A) = \mu_2(A)$ for every Borel set $A \subset H$. The following three assumptions are from [6].

Assumption 4.1 *There exists a function $V : H \to [1,\infty)$ with the following properties:*

1. *There exists two strictly increasing continuous functions V^* and V_* from $[0,\infty) \to [1,\infty)$ such that*
$$V_*(\|x\|) \le V(x) \le V^*(\|x\|) \tag{47}$$
for all $x \in H$ and such that $\lim_{a\to\infty} V_*(a) = \infty$.
2. *There exists constants C and $\kappa \ge 1$ such that*
$$aV^*(a) \le CV_*^{\kappa}(a), \tag{48}$$
for every $a > 0$.
3. *There exists a positive constants C, $r_0 < 1$, a decreasing function $\xi : [0,1] \to [0,1]$ with $\xi(1) < 1$ such that for every $h \in H$ with $\|h\| = 1$*
$$\mathbb{E}V^r(\Phi_t(x))(1 + \|D\Phi_t(x)h\|) \le CV^{r\xi(t)}(x), \tag{49}$$
for every $x \in H$, every $r \in [r_0, \kappa]$, and every $t \in [0,1]$.

Assumption 4.2 *There exists a $C_1 > 0$ and $p \in [0,1)$ so that for every $\alpha \in (0,1)$ there exists positive $T(\alpha)$ and $C(\alpha)$ with*
$$\|DP_t\varphi(x)\| \le C_1 V^p(x)\left(C(\alpha)\sqrt{(P_t|\varphi|^2)(x)} + \alpha\sqrt{(P_t\|D\varphi\|^2)(x)}\right), \tag{50}$$
for every $x \in H$ and $t \ge T(\alpha)$.

Assumption 4.3 *Given any $C > 0$, $r \in (0,1)$ and $\delta > 0$, there exists a T_0 so that for any $T \ge T_0$ there exists an $a > 0$ so that*
$$\inf_{|x|,|y|\le C} \sup_{\Gamma\in\mathcal{C}(P_T^*\delta_x, P_T^*\delta_y)} \Gamma\left\{(x', y') \in H\times H : \rho_r(x', y' < \delta)\right\} \ge a. \tag{51}$$

If Assumption 4.1 is satisfied, then for every Fréchet differentiable function $\varphi : H \to R$, we introduce the following norm
$$\|\varphi\|_V = \sup_{x\in H} \frac{|\varphi(x)| + \|D\varphi(x)\|}{V(x)},$$
and for $r \in (0,1]$, a family of distance ρ_r on H is defined by
$$\rho_r = \inf_{\gamma} \int_0^1 V^r(\gamma(t))\|\dot\gamma(t)\|dt,$$
where the infimum runs over all paths γ such that $\gamma(0) = x$ and $\gamma(1) = y$. For simple, we will write ρ for ρ_1.

If the setting of the semigroup P_t possesses an invariant measure μ_*, we define
$$\|\varphi\|_\rho = \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{\rho(x,y)} + \left|\int_H \varphi(x)\mu_*(dx)\right|. \tag{52}$$

The next Theorem comes from Theorem 3.6, Corollary 3.5 and Theorem 4.5 in [6].

Theorem 4.1 *Let Φ_t be a stochastic flow on a Banach space H which is almost surely C^1 and satisfy Assumption 4.1. Denote by P_t the corresponding Markov semigroup and assume that it satisfies Assumption 4.2 and 4.3. Then there exists a unique invariant probability measure μ_* for P_t and exists constants $\gamma > 0$ and $C > 0$ such that*
$$\begin{aligned}\|P_t\varphi - \mu_*\varphi\|_\rho &\le Ce^{-\gamma t}\|\varphi - \mu_*\varphi\|_\rho,\\ \|P_t\varphi - \mu_*\varphi\|_V &\le Ce^{-\gamma t}\|\varphi - \mu_*\varphi\|_V,\end{aligned}$$
for every Fréchet differentiable function $\varphi : H \to R$ and every $t > 0$.

The next Lemma comes from Lemma 5.1 in [6].

Lemma 4.1 *Let U be a real-valued semi-martingale*

$$dU(t,\omega) = F(t,\omega)dt + G(t,\omega)dB(t,\omega),$$

where B is a standard Brownian motion. Assume that there exists a process Z and positive constants b_1, b_2, b_3, with $b_2 > b_3$, such that $F(t,\omega) \le b_1 - b_2 Z(t,\omega)$, $U(t,\omega) \le Z(t,\omega)$, and $G(t,\omega)^2 \le b_3 Z(t,\omega)$ almost surely. Then the bound

$$\mathbb{E}\exp\left(U(t) + \frac{b_2 e^{-b_2 t/4}}{4}\int_0^t Z(s)ds\right) \le \frac{b_2\exp(\frac{2b_1}{b_2})}{b_2 - b_3}\exp\left(U(0)e^{-\frac{b_2}{2}t}\right),$$

holds for every $t \ge 0$.

Theorem 4.2 *Assume the conditions of Theorem* 3.1 *is satisfied, then Assumption* 4.1 *is satisfied for*

$$V(x) = e^{\eta_0\|x\|^2},\ where\ \eta_0 = \frac{\nu}{16C^2(2\pi)^2}.$$

Moreover, there exists a unique invariant probability measure μ_* for P_t and constants $\gamma > 0$ and $C > 0$ such that

$$\|P_t\varphi - \mu_*\varphi\|_V \le Ce^{-\gamma t}\|\varphi - \mu_*\varphi\|_V,$$

for every Fréchet differentiable function $\varphi : H \to \mathbb{R}$ and every $t > 0$.

Proof: In order to prove this Theorem, by Theorem 4.1, we only to confirm w_t satisfy Assumption 4.1 and P_t satisfy Assumption 4.2, 4.3. From Itô formula,

$$d\eta\|w_t\|^2 + 2\eta\nu\|w_t\|_1^2 dt = 2\eta\langle w_t, Q(w_t)dB_t\rangle + \eta\sum_{i=1}^m |q_i(w_t)|^2\|e_i\|^2 dt.$$

From Lemma 4.1,

$$\mathbb{E}\exp\left(U(t) + \frac{b_2 e^{-b_2 t/4}}{4}\int_0^t Z(s)ds\right) \le \frac{b_2\exp(\frac{2b_1}{b_2})}{b_2 - b_3}\exp\left(U(0)e^{-\frac{b_2}{2}t}\right).$$

where $U(t) = \eta\|w_t\|^2$, $Z(t) = \eta\|w_t\|_1^2$, $b_1 = 4\pi^2\eta mC^2$, $b_2 = 2\nu$, $b_3 = 4(2\pi)^2\eta C^2$. Therefore when $\eta \le \frac{\nu}{4C^2(2\pi)^2}$,

$$\mathbb{E}\exp\left(\eta\|w_t\|^2 + \frac{2\nu e^{-2\nu t/4}}{4}\int_0^t \eta\|w_s\|_1^2 ds\right) \le 2\exp(\frac{(2\pi)^2\eta mC^2}{\nu})\exp\left(\eta\|w_0\|^2 e^{-\frac{2\nu}{2}t}\right). \tag{53}$$

For $\|\xi\| = 1$,denote $\xi_t = J_t\xi = Dw_t^x\xi$, where x is the initial value and D is the differential operator with x. So ξ_t satisfy the following equation

$$d\xi_t = \nu\Delta\xi_t dt + \tilde{B}(w_t,\xi_t)dt + \sum_{i=1}^m (Dq_i(w_t)\xi_t)e_i dB_i(t). \tag{54}$$

and thus

$$d\|\xi_t\|^2 \le -2\nu\|\xi_t\|_1^2 dt + 2\langle B(\mathcal{K}\xi_t, w_t),\xi_t\rangle dt + \sum_{i=1}^m C^2 m\|\xi_t\|^2\|e_i\|^2 dt + h_t dB_t.$$

By the the similar method to get (43), we can obtain

$$2\langle B(\mathcal{K}\xi_t, w_t),\xi_t\rangle \le \eta\|w_t\|_1^2\|\xi_t\|^2 + \nu\|\xi_t\|_1^2 + \frac{16\hat{C}^4}{\eta^2\nu}\|\xi_t\|^2,$$

and

$$d\|\xi_t\|^2 \le \eta\|w_t\|_1^2\|\xi_t\|^2 dt + (\frac{16\hat{C}^4}{\eta^2\nu} + 4\pi^2 C^2 m)\|\xi_t\|^2 dt + h_t dB_t.$$

Define the function $h(\eta) = (\frac{16\hat{C}^4}{\eta^2\nu} + 4\pi^2 C^2 m)$, from the above inequality we have

$$\mathbb{E}\|\xi_t\|^2 \exp(-h(\eta)t - \int_0^t \eta\|w_s\|_1^2 ds) \le 1, \forall \eta > 0. \tag{55}$$

Set $b = e^{-\frac{\nu}{2}}$, $\eta \le \frac{\nu}{8C^2(2\pi)^2}$, $t \in [0,1]$. From (53) and (55),

$$\begin{aligned}
\mathbb{E}\exp(\eta\|w_t\|^2)\|\xi_t\| &= \mathbb{E}\left(\exp(\eta\|w_t\|^2)\exp(\frac{b\eta\nu}{2}\int_0^t \|w_s\|_1^2 ds)\right)\cdot\left(\|\xi_t\|\exp(-\frac{b\eta\nu}{2}\int_0^t \|w_s\|_1^2 ds)\right)\\
&\le \left(\mathbb{E}\exp(2\eta\|w_t\|^2 + b\eta\nu\int_0^t \|w_s\|_1^2 ds)\right)^{\frac{1}{2}}\left(\mathbb{E}\|\xi_t\|^2\exp(-b\eta\nu\int_0^t \|w_s\|_1^2 ds)\right)^{\frac{1}{2}}\\
&\le \left(2\exp(\frac{2\eta\cdot 4\pi^2 mC^2}{\nu})\exp\left(2\eta\|w_0\|^2 e^{-\frac{2\nu}{2}t}\right)\right)^{\frac{1}{2}}\exp(\frac{h(b\eta\nu)}{2}t)\\
&= \exp\left(\eta\|w_0\|^2 e^{-\frac{2\nu}{4}t}\right)\left(2\exp(\frac{2\eta\cdot 4\pi^2 mC^2}{\nu})\right)^{\frac{1}{2}}\exp(\frac{h(b\eta\nu)}{2}t).
\end{aligned}$$

Set $\eta_0 = \frac{\nu}{16C^2(2\pi)^2}$, we know that the above inequality is satisfied for all $\eta \in [0, 2\eta_0]$. So w_t satisfies Assumption 4.1 for $V(x) = e^{\eta_0\|x\|^2}$, $\kappa = 2$, $r_0 = \frac{1}{2}$ and $V_*(a) = V^*(a) = e^{\eta_0 a^2}$. From Theorem 2.3, we know P_t satisfy Assumption 4.3. Since

$$\begin{aligned}
\langle\nabla P_t\varphi(\omega_0), \xi\rangle &= \mathbb{E}_{\omega_0}(\mathcal{D}^v\varphi(\omega_t)) + \mathbb{E}_{\omega_0}((\nabla\varphi)(\omega_t)\rho_t)\\
&\le \sqrt{(P_t|\varphi|^2)(x)}\left(\mathbb{E}_{\omega_0}|\int_0^t v(s)dB(s)|^2\right)^{\frac{1}{2}} + \sqrt{(P_t\|\nabla\varphi\|^2)(x)}\left(\mathbb{E}_{\omega_0}\|\rho_t\|^2\right)^{\frac{1}{2}},
\end{aligned}$$

by (28), (30), (31), (32), (33), Lemma 3.4 and the fact $\rho_t = \zeta_t$, we know Assumption 4.2 is satisfied.

□

Appendix A. The existence and uniqueness of strong solution to equation (24)

Under the conditions of Theorem 3.1, the strong solution to equation (24) has and only has one solution.
Proof: Because $q_i(\theta) = q_i(\pi_l\theta)$, $1 \le i \le m$, for any $\theta \in H$, so the equation (24) is the same as the following equation

$$d\zeta_t^l = -\frac{\zeta_t^l}{2\|\zeta_t^l\|}dt + \sum_{i=1}^m \pi_l D(q_i(w_t)e_i)\zeta_t^l dB_i(t) + C^2 m\Delta\zeta_t^l dt - 2C^2 m^2\zeta_t^l\|\zeta_t^l\|^2 dt, \ \ \zeta_0^l = \xi^l. \tag{A.1}$$

This equation is essentially a finite dimension stochastic differential equation. If $\|\zeta_0^l\| = 0$, then zero is the solution to $(A.1)$(for 0/0=0). If $\|\zeta_0^l\| > 0$, define the following stopping times

$$\begin{aligned}
\tau_n &= \inf\left\{t > 0, \ \|\zeta_t^l\| \le 1/n\right\}, \ \ \forall n \in \mathbb{Z}^+,\\
\tau &= \lim_{n\to\infty}\tau_n.
\end{aligned}$$

When $t \le \tau_n$, there exists a unique solution x_t^n to equation $(A.1)$. So

$$\zeta_t^l = \begin{cases} x_t^n & t \le \tau_n, \text{ for some } n,\\ 0 & t \ge \tau, \end{cases}$$

is a strong solution to $(A.1)$. In the following, we will prove that if $\zeta_0^l = 0$, then any solution to equation $(A.1)$ will be zero. Let X_t be the solution to equation $(A.1)$ with $X_0 = 0$. Then,

$$d\|X_t\|^2 \le -\|X_t\|dt + \sum_{i=1}^{m} \|Dq_i(w_t)X_t\|^2 dt - 2C^2 m\|X_t\|_1^2 dt - 4C^2 m^2 \|X_t\|^4 dt + h(t)dB_t,$$

for some process h_t. For $\|Dq_i(w_t)\| \le C$, so

$$\mathbb{E}\|X_t\|^2 \le \|X_0\|^2 = 0,$$

then $X_t = 0$, almost everywhere, $\forall t > 0$. Then the uniqueness follows.

□

Appendix B. The existence and uniqueness of solution to equation (29)

In this Appendix, we will prove that under the conditions of Theorem 3.1, equation (29) has and only has one solution.

Proof: Because $q_i(\theta) = q_i(\pi_l\theta)$, $1 \le i \le m$, for any $\theta \in H$, so the equation (29) is the same as the following equation

$$\begin{aligned} dY_t &= \nu\Delta Y_t dt + \pi_h B(\mathcal{K}w_t, Y_t)dt + \pi_h B(\mathcal{K}Y_t, w_t)dt + \sum_{i=1}^{m} \pi_h D\left(q_i(w_t)e_i\right)\zeta_t^l dB_t \\ &\quad + \pi_h B(\mathcal{K}w_t, \zeta_t^l)dt + \pi_h B(\mathcal{K}\zeta_t^l, w_t)dt. \end{aligned} \tag{B.1}$$

We mainly use Theorem 1.2 in [1], so we need to check conditions (H1), (H2), (H3), (H4) in [1]. Recall that operator π_h is defined in section 3, and the space H is defined in section 1. The spaces $\mathcal{H}, V$ in conditions (H1), (H2), (H3), (H4) is defined as follows:

$$V = \left\{ w \in H_0^{1,2}([-\pi,\pi]^2, \mathbb{R}) \cap \pi_h H : \int_{[-\pi,\pi]^2} w(x)dx = 0 \right\}, \quad \|w\|_V^2 = \int_{[-\pi,\pi]^2} |\nabla w|^2 dx, \tag{B.2}$$

and

$$\mathcal{H} = \left\{ w \in L^2([-\pi,\pi]^2, \mathbb{R}) \cap \pi_h H : \int_{[-\pi,\pi]^2} w(x)dx = 0 \right\}, \quad \|w\|_{\mathcal{H}}^2 = \int_{[-\pi,\pi]^2} |w|^2 dx. \tag{B.3}$$

From (7) and $\|w\| := \|w\|_0$, we know that for any $\varphi \in \mathcal{H}$

$$\|\varphi\|_{\mathcal{H}} = \|\varphi\|_{L^2} = \|\varphi\| = \|\varphi\|_H,$$

and for any $\varphi \in V$

$$\|\varphi\|_V = \|\varphi\|_1.$$

Define stopping times

$$\tau_n = \inf\{t \ge 0, \ \ \|w_t\|_H \ge n\}.$$

For (H2), when $t \le \tau_n$, by inequality $ab \le (1/p)a^p + (1/q)b^q$ $(1/p + 1/q = 1)$ and (11), for any $\epsilon > 0$,

$$\begin{aligned} {}_{V^*}\langle \pi_h B(\mathcal{K}w_t, v_1 - v_2), v_1 - v_2\rangle_V &= 0, \\ {}_{V^*}\langle \pi_h B(\mathcal{K}(v_1 - v_2), w_t), v_1 - v_2\rangle_V &\le C\|\mathcal{K}(v_1 - v_2)\|_{\frac{3}{2}} \cdot \|w_t\| \cdot \|v_1 - v_2\|_1 \\ &= C\|(v_1 - v_2)\|_{\frac{1}{2}} \cdot \|w_t\| \cdot \|v_1 - v_2\|_1 \\ &\le C\|(v_1 - v_2)\|^{\frac{1}{2}} \cdot \|w_t\| \cdot \|v_1 - v_2\|_1^{\frac{3}{2}} \\ &\le \epsilon\|v_1 - v_2\|_1^2 + C(\epsilon)\|w_t\|^4\|v_1 - v_2\|^2. \end{aligned} \tag{B.4}$$

Set $\epsilon = \frac{\nu}{2}$, we know that (H2) is satisfied for $t \le \tau_n$.

For (H3), set $\epsilon = \frac{\nu}{2}$ in $(B.4)$, we know that exists constant C such that

$${}_{V^*}\langle \pi_h B(\mathcal{K}(v), w_t), v\rangle_V \le \frac{\nu}{2}\|v_1 - v_2\|_1^2 + C\|w_t\|^4\|v_1 - v_2\|^2,$$

then we can know that when $t \le \tau_n$, $(H3)$ is satisfied for $\alpha = 2, \ \theta = \frac{\nu}{2}$. □

Appendix C. The existence and uniqueness of solution to equation (1)

In this Appendix, we will prove that under the conditions of Theorem 2.1, equation (1) has and only has one solution.

Proof: We mainly use Theorem 1.2 in [1], so we need to check conditions (H1), (H2), (H3), (H4) in [1]. In this Appendix, The spaces $\mathcal{H}, V$ in conditions (H1), (H2), (H3), (H4), is defined as follows:

$$V = \left\{ w \in H_0^{1,2}([-\pi,\pi]^2, \mathbb{R}) : \int_{[-\pi,\pi]^2} w(x)dx = 0 \right\}, \quad \|w\|_V^2 = \int_{[-\pi,\pi]^2} |\nabla w|^2 dx, \tag{C.1}$$

and

$$\mathcal{H} = \left\{ w \in L^2([-\pi,\pi]^2, \mathbb{R}) : \int_{[-\pi,\pi]^2} w(x)dx = 0 \right\}, \quad \|w\|_{\mathcal{H}}^2 = \int_{[-\pi,\pi]^2} |w|^2 dx. \tag{C.2}$$

For (H2), by using inequality $ab \leq (1/p)a^p + (1/q)b^q$ $(1/p + 1/q = 1)$ and (11), for any $\epsilon > 0$,

$$\begin{aligned}
{}_{V^*}\langle B(\mathcal{K}v_1, v_1) - B(\mathcal{K}v_2, v_2), v_1 - v_2\rangle_V &= -{}_{V^*}\langle B(\mathcal{K}v_1, v_1), v_2\rangle_V - {}_{V^*}\langle B(\mathcal{K}v_2, v_2), v_1\rangle_V \\
&= -{}_{V^*}\langle B(\mathcal{K}v_1, v_1 - v_2), v_2\rangle_V + {}_{V^*}\langle B(\mathcal{K}v_2, v_1 - v_2), v_1\rangle_V \\
&= -{}_{V^*}\langle B(\mathcal{K}v_1, v_1 - v_2), v_2\rangle_V + {}_{V^*}\langle B(\mathcal{K}v_2, v_1 - v_2), v_2\rangle_V \\
&= -{}_{V^*}\langle B(\mathcal{K}v_1 - \mathcal{K}v_2, v_1 - v_2), v_2\rangle_V \\
&= -{}_{V^*}\langle B(\mathcal{K}v_1 - \mathcal{K}v_2, v_2), v_1 - v_2\rangle_V,
\end{aligned}$$

then by the similar way in obtaining $(B.4)$, for $\epsilon = \frac{\nu}{2}$, exists a constant $C(\epsilon)$ such that

$$ {}_{V^*}\langle B(\mathcal{K}v_1 - \mathcal{K}v_2, v_2), v_1 - v_2\rangle_V \leq \epsilon\|v_1 - v_2\|_1^2 + C(\epsilon)\|v_2\|^4\|v_1 - v_2\|^2.$$

So (H2) is satisfied for $\rho(v) = C(\epsilon)\|v\|_{\mathcal{H}}^4$.

For (H4), by hölder inequality,

$$|{}_{V^*}\langle B(\mathcal{K}v, v), w\rangle_V| \leq \sqrt{2}\|\mathcal{K}v\|_{L^4}\|v\|_V\|w\|_{L^4}. \tag{C.3}$$

For any smooth function φ, (see Lemma 2.1 in [8] for example)

$$\|\varphi\|_{L^4}^4 \leq 2\|\varphi\|_{L^2}^2 \cdot \|\nabla\varphi\|_{L^2}^2. \tag{C.4}$$

So, from $(C.3)$, $(C.4)$

$$|{}_{V^*}\langle B(\mathcal{K}v, v), w\rangle_V|^2 \leq 2\sqrt{2}\|\mathcal{K}v\|_{L^2}\|\nabla(\mathcal{K}v)\|_{L^2}\|v\|_V^2 \cdot \sqrt{2}\|w\|_{L^2}\|\nabla w\|_{L^2},$$

so,

$$|B(\mathcal{K}v, v)|_{V^*} \leq \|v\|_{L^2}\|v\|_V. \tag{C.5}$$

From $(C.5)$, it is not difficult to check that (H4) is satisfied for $\alpha = 2$, $\beta = 2$. □

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