# Some Results on Evolutionary 2 x 2 Asymmetric Games


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9th Workshop on Markov Processes and Related Topics
Er-Mei Mountain, Chen-Du, July 6-10, 2013

# Battle of Sex Game

- 2 Players = { Husband, Wife}.
  2 Strategies = { Football, Ballet }.
- Payoffs (2 $\times$ 2 asymmetric game ):

| | *Football* | *Ballet* |
|---|---|---|
| *Football* | *2, 1* | *-1, -1* |
| *Ballet* | *-1, -1* | *1 , 2* |

- 3 Nash Equilibria:
  { Football, Football}, { Ballet, Ballet}. Unlikely in general.
- 1 mixed strategy:
  Husband: Prob( Football ) $= \frac{3}{5}$, Prob( Ballet ) $= \frac{2}{5}$.
  Wife: Prob( Football ) $= \frac{2}{5}$, Prob( Ballet ) $= \frac{3}{5}$.
- Expected payoff of (Husband, Wife) $= (\frac{1}{4}, \frac{1}{4})$.
- Is it reasonable?
- J. H. Wang, The Theory of Games, Oxford U. Press, 1988.

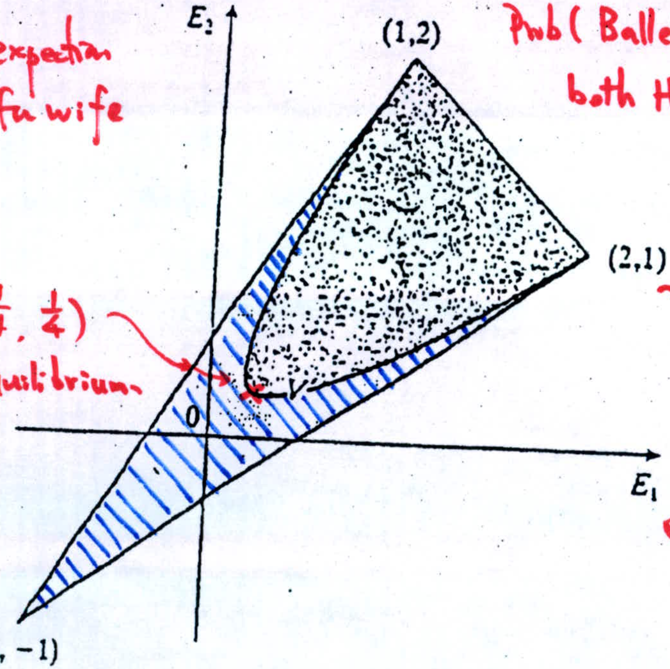


**Figure 4.1** From J. H. Wang's The Theory of Games, 1988

Prob ( Football ) = { 1 / 0 } fu Husband

& Prob ( Football ) = { 0 / 1 } fu wife

# Prisoner's Dilemma Game

2 isolated prisoners in cell, waiting to be sentenced.

- Strategy set { Defect, Cooperation }. Like spin $\{\pm\}$.
- Defect $=$ confess to be guilty.
- Payoffs ($2 \times 2$ symmetric game ):

| | *D* | *C* |
|---|---|---|
| *D* | *6 years, 6 years* | *3 months, 10 years* |
| *C* | *10 years, 3 months* | *1 year, 1 year* |

- D: low list price. C: higher list price for a certain product.
- Unique Nash Equilibrium is (D, D).
- Payoff for (C, C) is better. Yet, no communication allowed.
- Payoff for strategy D $>$ payoff for strategy C.
- Egoist ( for strategy $D$) vs. Altruist ( for strategy $C$).
- Any way out of the dilemma?

# Prisoner's Dilemma Game continued...

More generally, the payoffs, with $b > d > a > c$, are

| | $D$ | $C$ |
|---|---|---|
| $D$ | $a, a$ | $b, c$ |
| $C$ | $c, b$ | $d, d$ |

- Nash Equilibrium is (D, D). But (C, C) is better.
- Payoff for strategy D $>$ payoff for strategy C.
- Definition. (s, t) is called a Nash equilibrium if

$$\text{payoff at } (s, t) \geq \text{ payoff at } (s, t') \quad \forall t' \in S;$$

$$\text{payoff at } (s, t) \geq \text{ payoff at } (s', t) \quad \forall s' \in S.$$

  No player gains by changing his present strategy alone.
- No under-table deal. No side-payment. No talk.

# Prisoner's Dilemma Game continued...

New models: play many times: 1-time codebook unbreakable, many players, local structure.

- Key features : strategy-revision dynamics.
  Energy in the physical models. Variety in social study.
- 2 players with repeated games.
  Like eye for eye.
  Fictitious play. Cf. Hofbauer & Sandholm (2002).
- $\infty$ many players. Continous time.
  Lotka-Volterra differential equation. Global interaction
  State $x \in [0, 1]$ = the population proportion playing strategy $C$.
  Reaction-diffusion equation: Local interaction
- Our setup: $N \geq 5$ players, discrete time, local interaction.
- Similar to interacting particle systems.
- Goal: long-run behavior.

# Dynamics I. Strategy revision by imitation

By inertia, each player imagines to play the above PD game once with each of their two neighbors, according to their present strategies.

$$s_{i-1} \Leftarrow s_i \Rightarrow s_{i+1}$$

Let $z_i(\vec{s}) =$ player $i$'s total expected payoff if none changes the present strategy.

- Imitating-best-player among his neighbors and himself: the rational choice for player $i$ at time $t+1$ is

$$r_i(\vec{s}) \in M_i(\vec{s}) \stackrel{\text{def}}{=} \{s_j : z_j(\vec{s}) = \max z_k(\vec{s}) \text{ for } k \in N_i \cup \{i\}\}.$$

# Dynamics I. continued...

- Imitating-best-strategy: each player $i$ will imitate the most successful action yielding the highest average payoff which was adopted among his neighbors and himself at time $t$. Let $\delta$ be the Kronecker notation. Then

$$a_i^E(\vec{s}) = \begin{cases} \frac{\sum_{k \in N_i \cup \{i\}} z_k(\vec{s}) \cdot \delta_{E,s_k}}{\sum_{k \in N_i \cup \{i\}} \delta_{E,s_k}}, & \text{if } E \in \{s_{i-1}, s_i, s_{i+1}\}, \\ -\infty, & \text{if } E \neq s_{i-1} = s_i = s_{i+1}, \end{cases}$$

means the average payoff for strategy $E \in \{C, D\}$ among player $i$ and his neighbors. Therefore, player $i$'s next-period rational choice $r_i(\vec{s})$ satisfies

$$r_i(\vec{s}) \in \bar{M}_i(\vec{s}) \stackrel{\text{def}}{=} \{E \in \{C, D\} : a_i^E(\vec{s}) = \max(a_i^C(\vec{s}), a_i^D(\vec{s}))\}.$$

# Dynamics I. continued...

- The computation of $M_i(\vec{s})$ and $\bar{M}_i(\vec{s})$ for player $i$ involves

$$(s_{i-2}, s_{i-1}, s_i, s_{i+1}, s_{i+2})$$

14 out of 32 cases need to be considered. E.g.

$$r_i(\vec{s}) = s_i \text{ if } s_{i-1} = s_i = s_{i+1}.$$

- For brevity, $r(s_{i-2}, s_{i-1}, s_i, s_{i+1}, s_{i+2}) \stackrel{\text{def}}{=} r_i(\vec{s})$.
- Strict rule : In case, $\{C,\ D\} = M_i(\vec{s})$ ( or $\bar{M}_i(\vec{s})$),

$$r_i(\vec{s}) = s_i \text{ by inertia.} \quad \text{Deterministic process.}$$

- Essentially the same results for the loose rule. Random.
- A time-homogeneous Markov chain on $S = \{C,\ D\}^n$ with transition probability matrix $Q_0(\vec{s}, \vec{u}) = 1$ iff $\vec{u} = \vec{r}(\vec{s})$, where the rational choice $\vec{r}(\vec{s}) = (r_1(\vec{s}), r_2(\vec{s}), \ldots, r_n(\vec{s}))$ is uniquely determined for state $\vec{s} \in S$ by the strict rule.

# Dynamics II. Mutation

Players will simultaneously, but independently alter their rational choices $\{r_i(\vec{s})\}$ with identical probability $\epsilon > 0$.
Mutation : an important factor in biology evolution.
Rationality may not be good always.
Greedy algorithm. Monkey forever.
A learning process. People make less mistakes as time $\to \infty$.
Here $\epsilon$ is fixed but small.
If $\epsilon = \epsilon(t)$ then that leads to simulated annealing.

- Kirkpatrick, Gebatt and Vecchi, Optimization by simulated annealing, *Science* **220** (1983), 671-680.
- Geman and Geman, Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images, *IEEE Trans. Pattern Analysis and Machine Intelligence* **6** (1984), 721-741.

All together, it is a Markov chain $\{X_t : t = 0, 1, ...\}$ on $S$.
Its transition matrix $Q_\epsilon$, a perturbation of $Q_0$, given by

$$Q_\epsilon(\vec{s}, \vec{u}) = \epsilon^{d(\vec{r}(\vec{s}),\ \vec{u})} \cdot (1 - \epsilon)^{n - d(\vec{r}(\vec{s}),\ \vec{u})} \text{ for all } \vec{s}, \vec{u} \in S.$$

Here, $d(\vec{r}(\vec{s}), \vec{u}) = |\{i \in N : r_i(\vec{s}) \neq u_i\}|$
$= \#$ of mismatches between the next truly-adopted strategy $\vec{u}$ and the revised rational choice $\vec{r}(\vec{s})$ at state $\vec{s}$.

- $Q_\epsilon(\vec{s}, \vec{u}) \approx \epsilon^{U(\vec{s},\vec{u})}$ for $\epsilon << 1$.
- Here $U(\vec{s}, \vec{u}) = d(\vec{r}(\vec{s}), \vec{u})$ means the cost from $\vec{s}$ to $\vec{u}$.

# Dynamics II. continued...

- $Q_\epsilon(\vec{s}, \ \vec{u}) > 0$ for all $\vec{s}, \ \vec{u} \in S$.
- Mutation makes our dynamic process $\{X_t\}$ ergodic.
- The unique invariant distribution $\mu_\epsilon$ is characterized by $\mu_\epsilon = \mu_\epsilon \cdot Q_\epsilon$.
- Method of Ventcel-Freidlin can be applied.
  $\mu_\epsilon$ is specified in terms of spanning-trees.
  Freidlin & Wentzell, Random Perturbations of Dynamical Systems. 1984.
- Goal: to find $\mu_* \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} \mu_\epsilon$ and its support $S_*$.
- In particular, whether $\mu_\epsilon(\vec{C}) = 1$. Or

$$(C, C, C, ..., C) \stackrel{\text{def}}{=} \vec{C} \in S_* \stackrel{\text{def}}{=} \{\vec{s} \in S \ : \ \mu_*(\vec{s}) > 0\}?$$

  I.e. whether all-cooperation is possible in the long run?
- Elements in $S_*$ are called the Long Run Equilibria.

# Results for PD games

- $\{\vec{C}, \vec{D}\} \subseteq S_0$. Let $M \stackrel{\text{def}}{=} S_0 \setminus \{\vec{C}, \vec{D}\}$ the set of mixed stationary states at $\epsilon = 0$, which means cooperators and defectors coexist peacefully.
- For $\vec{s} \in M \neq \emptyset$ can be expressed as follows:

$$\cdots \underbrace{D \cdots D}_{d_k} \underbrace{C \cdots C}_{c_k} \underbrace{D \cdots D}_{d_1} \underbrace{C \cdots C}_{c_1} \underbrace{D \cdots D}_{d_2} \underbrace{C \cdots C}_{c_2} \cdots$$

  $d_i =$ length of the $i$th $D$-string,
  $c_j =$ length of the $j$th $C$-string starting from a certain player.
- For positive integers $m$ and $\ell$, define

$$M_{\geq m,\, \geq \ell} \stackrel{\text{def}}{=} \{\vec{s} \in S \ : \ \text{all } d_i \geq m,\ c_j \geq \ell\}$$

$$M_{m,\, \ell} \stackrel{\text{def}}{=} \{\vec{s} \in S \ : \ \text{all } d_i = m,\ c_j = \ell\}.$$

H.C. Chen and Y. Chow, *Adv. Applied Probab.*, **41** (2009), 154-176.
**Theorem 1.** For Imitating-Best-Player dynamics, $S_* = \{\vec{D}\}$ and $E_\epsilon(T) \approx \epsilon^{-1}$ *as* $\epsilon \downarrow 0$.
Here $T$ = waiting time to hit $S_*$.

- All-defection $\vec{D}$ is the unique LRE of the IBP dynamics.
- Because
$$r(*, C, D, C, *) = D$$
and
$$r(*, D, C, D, *) = D,$$
which shows the strength of $D$ against $C$.

# Results continued...

**Theorem 2.** Assume the Imitating-Best-Strategy dynamics.
(i) If $a+b > \frac{c+3d}{2}$, $S_0 = \{\vec{C}, \vec{D}\}$, $S_* = \{\vec{D}\}$ and $E_\epsilon(T) \approx \epsilon^{-1}$.
(ii) If $a+b \leq \frac{c+3d}{2}$ and $\frac{3a+b}{2} < c+d$, then $S_0 = \{\vec{C}, \vec{D}\} \cup M$, where the mixed stationary states in $M$ has all $d_i \in \{1,\ 2,\ 3\}$ and, besides $c_i \geq 3$,

$$c_i \geq 5 \text{ if } (d_i, d_{i+1}) = (1,1);\ c_i \geq 4 \text{ if } (d_i, d_{i+1}) = (1,2) \text{ or } (2,1).$$

$$\begin{cases} S_* & = \{\vec{D}\} \text{ and } E_\epsilon(T) \approx \epsilon^{-1} \text{ for } n = 5, \\ S_* & = \{\vec{D}\} \text{ and } E_\epsilon(T) \approx \epsilon^{-\lceil \frac{n}{10} \rceil} \text{ for } 6 \leq n \leq 20, \\ S_* & = S_0 \text{ and } E_\epsilon(T) \approx \epsilon^{0} \text{ for } 21 \leq n < 30 \text{ but } n \neq 25, \\ S_* & = S_0 \setminus M_{2,\,3} \text{ and } E_\epsilon(T) \approx \epsilon^{-1} \text{ for } n = 25 \text{ or } 30, \\ S_* & = (S_0 \setminus M_{2,\,3}) \setminus \{\vec{D}\} \text{ and } E_\epsilon(T) \approx \epsilon^{-3} \text{ for } n \geq 31. \end{cases}$$

(iii) If $a+b \leq \frac{c+3d}{2}$ and $\frac{3a+b}{2} \geq c+d$, then
$S_0 = \{\vec{C}, \vec{D}\} \cup M_{\geq 2,\, \geq 3}$, $S_* = \{\vec{D}\}$ and $E_\epsilon(T) \approx \epsilon^{-1}$.

# Coordination Games

- 2 players and 2 strategies $\{A,\ B\}$.
- Payoffs ($2 \times 2$ symmetric game ):

| | $A$ | $B$ |
|---|---|---|
| $A$ | $a$ , $a$ | $b$ , $c$ |
| $B$ | $c$, $b$ | $d$ , $d$ |

- Assume $a > c,\ d > b,\ d > a$, and $a + b > c + d$.
- 2 Nash Equilibria are $(B, B)$ and $(A, A)$.
- $d > a \Rightarrow$ strategy $B$ is Pareto efficient.
- $a + b > c + d \Rightarrow$ strategy $A$ is risk dominant.
- LRE under the evolutionary dynamics can be obtained.
- By scaling, we may set $c = 0$ and $d = 1$.
  So $a + b > 1$, $0 < a < 1$ and $0 < b < 1$.

# Results for Coordination Games

H.C. Chen, Y. Chow and L.C. Wu, *Economics Bulletin* **32** (2012) and *Intern. J. Game Theory* (2013), to appear.

**Theorem 3.** For Imitating-Best-Player dynamics,
$S_* = \{\vec{B}\}$ except the following two cases:
(i) When $b > 1/2$, we have $S_* = \{\vec{A}\}$ *if* $5 \le n \le 6$,
$S_* = \{\vec{A}, \vec{B}\} \cup M_{1\ \ge 3}$ if $7 \le n \le 12$,
and $S_* = \{\vec{B}\} \cup M_{1\ \ge 3}$ *if* $n \ge 13$.
(ii) When $b = 1/2$, we have $S_* = \{\vec{A}, \vec{B}\}$ if $5 \le n \le 6$,
and $S_* = \{\vec{B}\}$ if $n \ge 7$.

**Theorem 4** Assume Imitating-Best-Strategy dynamics.
(a) If $\frac{3a+b}{2} \ge 1$ then $S_* = \{\vec{A}\}$.

# Results for Coordination Games continued...

(b) If $\frac{3a+b}{2} < 1$ and $b \le \frac{3}{4}$, then

$$\begin{cases} S_* = \{\vec{A}\} & \textit{for } 5 \le n \le 14, \\ S_* = \{\vec{A},\ \vec{B}\} \cup M_{\ge 3,\ \le 2} & \textit{for } 15 \le n \le 21, \\ S_* = \{\vec{B}\} \cup M_{\ge 3,\ \le 2} & \textit{for } n \ge 22. \end{cases}$$

(c) If $\frac{3a+b}{2} < 1$ and $b > \frac{3}{4}$, then

$$\begin{cases} S_* = \{\vec{A}\} & \textit{for } n = 5, \\ S_* = \{\vec{A}\} & \textit{for } 6 \le n \le 20, \\ S_* = \{\vec{A},\ \vec{B}\} \cup \tilde{M} & \textit{for } 21 \le n < 30,\ n \ne 25, \\ S_* = (\{\vec{A},\ \vec{B}\} \cup \tilde{M}) \backslash M_{3,\ 2} & \textit{for } n = 25 \textit{ or } 30, \\ S_* = (\{\vec{B}\} \cup \tilde{M}) \backslash M_{3,\ 2} & \textit{for } n \ge 31. \end{cases}$$

$$\cdots \underbrace{A\cdots A}_{a_k}\underbrace{B\cdots B}_{b_k}\underbrace{A\cdots A}_{a_1}\underbrace{B\cdots B}_{b_1}\underbrace{A\cdots A}_{a_2}\underbrace{B\cdots B}_{b_2}\cdots . \qquad (*)$$

Here $a_i$ and $b_i$ are the lengths of its $i$-th $A$-string and $B$-string.

$$M_{m,\,p} \stackrel{\text{def}}{=} \{\vec{s} \in S \;:\; \text{all } a_i = m,\ b_j = p \text{ in } (*)\}.$$

$$M_{\leq m,\,\geq p} \stackrel{\text{def}}{=} \{\vec{s} \in S \;:\; \text{all } a_i \leq m,\ b_j \geq p \text{ in } (*)\}.$$

Furthermore,

$$\tilde{M} = \{\vec{s} \in M_{\leq 3,\,\geq 3} \;:\; b_i \geq 4 \text{ if } (a_i,\ a_{i+1}) = (1,2) \text{ or } (2,1), \text{ and } b_i \geq 5 \text{ if } (a_i,\ a_{i+1}) = (1,1) \text{ in } (*)\}.$$

# Battle of Sex Game

- 2 Players $=$ { Husband, Wife}.
  2 Strategies $=$ { Football, Ballet }.
- Payoffs (2 $\times$ 2 asymmetric game ):

| | *Football* | *Ballet* |
|---|---|---|
| *Football* | *2, 1* | *-1, -1* |
| *Ballet* | *-1, -1* | *1 , 2* |

- 3 Nash Equilibria:
  { Football, Football}, { Ballet, Ballet}. Unlikely in general.
- 1 mixed strategy:
  Husband: Prob( Football ) $= \frac{3}{5}$, Prob( Ballet ) $= \frac{2}{5}$.
  Wife: Prob( Football ) $= \frac{2}{5}$, Prob( Ballet ) $= \frac{3}{5}$.
- Expected payoff of (Husband, Wife) $= (\frac{1}{4}, \frac{1}{4})$.
- Is it reasonable?
- Play repeatedly for 2 players?
  N-person BOS game with different payoff functions.
- Try the evolutionary approach.

# Battle of Sex Game continued

- Payoffs ($2 \times 2$ asymmetric game ):

| | *Football* | *Ballet* |
|---|---|---|
| *Football* | *a*, *b* | *0*, *0* |
| *Ballet* | *c*, *c* | *b* , *a* |

- Here $a > b > 0 \geq c$.
- 3 Nash Equilibria:
  $\{$ Football, Football$\}$, $\{$ Ballet, Ballet$\}$. Unlikely in general.
  1 mixed strategy:
  Husband: P ( Football ) $= \frac{a-c}{a+d-c}$, P ( Ballet ) $= \frac{d}{a+d-c}$.
  Wife: P ( Football ) $= \frac{d}{a+d-c}$, P ( Ballet ) $= \frac{a-c}{a+d-c}$.
- Goal: Expected payoff of (Husband, Wife) $= (\frac{a+d}{2}, \frac{a+d}{2})$.
- 2n Players sit around a circle. H- and W-types alernating .
  2 Strategies $= \{$ Football, Ballet $\}$ for each player.

# Battle of Sex Game continued

For imitating best player dynamics:

- Singleton can hold only if it is

$$\mathrm{FBFBF} \text{ or } \mathrm{BFBFB}$$

- Any $\mathrm{F}$ string of length $\geq 2$ can hold. So does $\mathrm{H}$ string.
- $S_0 = \{\vec{F}, \vec{H}\} \cup M_{\geq 2, \geq 2} \cup \{FBFBFBFB....FB\}$
- $S_* = \{\vec{F}, \vec{H}\}$ and each with probability $\frac{1}{2}$.
- Goal achieved.
  Expected payoff of (Husband, Wife) $= (\frac{a+b}{2}, \frac{a+b}{2})$ under $\mu_*$.

# Battle of Sex Game continued

For imitating best strategy dynamics:

- Singleton can hold under $Q_0$ only if it is FBFBF or BFBFB.
- If player $i$ is Husband-type, then $*\mathrm{BBF}*$ can hold under $Q_0$.
- FFFBF can hold, but BFFBB cannot.
- FFFBB can hold iff $a \leq 2b$.
- BFFBF can hold iff $a + b + c \geq 0$.
- Any B string of length $\geq 3$ starting and ending with Husband-type players can hold.
- $S_* = \{\vec{F}, \vec{H}\}$ and each with probability $\frac{1}{2}$.
- Goal achieved.

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